CSU Math 570
Fall 2017

Homework 4

Due Friday, September 15 at the beginning of class

Reading.
Read pages 209–210 and 213–214 of Chapter 7. Read Chapter 4 (we'll only discuss selections thereof).

Problems.

1. Prove that if a coproduct exists in a category, then it is unique up to isomorphism. That is, prove that if $(S', (\iota'_\alpha))$ and $(S'', (\iota''_\alpha))$ are both coproducts of the family of objects $(X_\alpha)_{\alpha\in A}$, then S' and S'' are isomorphic.

   Remark: We did a similar proof in class for products. Using the statement for products, a correct proof of the statement for coproducts would be the following.

   > *Let C be a category; we must show coproducts are unique in C. Since products are unique in the category C^{op}, it follows that the dual statement is true in the dual category C, i.e. coproducts are unique in C.*

   See `https://en.wikipedia.org/wiki/Dual_(category_theory)` *for more details. You can give me multiple proofs if you like, but give me at least one proof that is independent of our knowledge of unique products.*

2. Let $(X_\alpha)_{\alpha\in A}$ be a family of topological spaces, and equip $\coprod_{\alpha\in A} X_\alpha$ with the disjoint union topology (page 64). Prove that $\coprod_{\alpha\in A} X_\alpha$ is the coproduct of $(X_\alpha)_{\alpha\in A}$ in the category of topological spaces (pages 213–214) as follows.

   (a) Define the maps $\iota_\alpha \colon X_\alpha \to \coprod_{\alpha\in A} X_\alpha$

   (b) Prove that $(\coprod_{\alpha\in A} X_\alpha, (\iota_\alpha))$ satisfies the necessary universal property ("Given any topological space W and morphisms $f_\alpha \colon X_\alpha \to W$, there exists ...").

3. Let $X = (\mathbb{R}\times\{0\}) \cup (\mathbb{R}\times\{1\}) \subseteq \mathbb{R}^2$ (equipped with the standard topology, i.e. its topology as a subset of $\mathbb{R}^2$, or equivalently its topology as a metric space). Define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ if $x \neq 0$. The quotient space $X/\sim$ is called the *line with two origins.*

   (a) Show that $X/\sim$ is not Hausdorff (and hence not a manifold).

   (b) Show that $X/\sim$ is locally Euclidean.
   Remark: If at some point you say "I have shown that this set $U \subseteq X/\sim$ is open and I have defined a map $f \colon U \to V$ that is clearly a homeomorphism to an open subset V of $\mathbb{R}$", then I will not care if you prove f is a homeomorphism or not.

4. Let X and Y be topological spaces, and let $f\colon X \to Y$. Suppose that $X = \cup_{\alpha \in A} U_\alpha$, that U_α is open in X for all α, and that $f|_{U_\alpha}\colon U_\alpha \to Y$ is continuous for all α. Prove that $f\colon X \to Y$ is continuous.

5. Make a Möbius band out of a strip of paper, and then cut it along its central circle (don't turn this in). Now, draw a picture to show that identifying diametrically opposite points on one of the boundary circles of a cylinder creates a Möbius band. That is, draw a picture to show that if you take a cylinder $S^1 \times [0,1] = \{(x,y,z) \mid x^2 + y^2 = 1 \text{ and } 0 \leq z \leq 1\}$ and identify each $(x,y,1)$ with $(-x,-y,1)$, then you get a Möbius band.

   Remark: As I'll write up in my solutions or explain in class, this can be used to show that if you take the 2-sphere S^2, cut out a disk, and then glue in a Möbius band along its boundary circle, then you get the projective plane $\mathbb{R}\mathrm{P}^2$.