CSU Math 570
Fall 2017

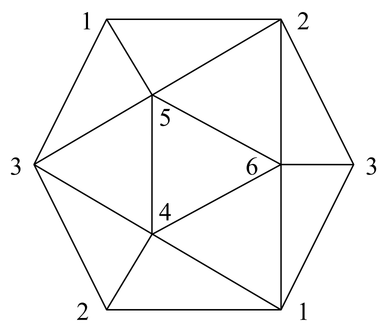
Homework 12

Due Friday, December 1 at the beginning of class

Reading. Read Chapter 13 pages 339-351, 355-356 (statement of Mayer-Vietoris Theorem), and 364-368 (Homology of Spheres and Degree Theory for Spheres).

Problems.

1. Let X be the 2-dimensional simplicial complex with ten 2-simplices drawn below; recall that X is homeomorphic to the projective plane $\mathbb{RP}^2$.



Show that the 2-dimensional simplicial homology group of X is $H_2(X) \cong 0$.

2. Prove Proposition 13.5 in our book: If X is a space, $\{X_\alpha\}_{\alpha \in X}$ is the set of path components of X, and $\iota_\alpha \colon X_\alpha \hookrightarrow X$ is the corresponding inclusion, then for each $p \geq 0$ the map $\oplus_{\alpha \in A} H_p(X_\alpha) \to H_p(X)$ whose restriction to singular homology group $H_p(X_\alpha)$ is $(\iota_\alpha)_* \colon H_p(X_\alpha) \to H_p(X)$ is an isomorphim. Proceed by the following steps.

   (a) Show the maps $(\iota_\alpha)_\# \colon C_p(X_\alpha) \to C_p(X)$ give an isomorphism $g_C \colon \oplus_{\alpha \in A} C_p(X_\alpha) \to C_p(X)$, defined by $g_C((c_\alpha)_{\alpha \in A}) = \sum_\alpha (\iota_\alpha)_\#(c_\alpha)$, where $c_\alpha \in C_p(X_\alpha)$. Injectivity is clear, and the first sentence of the proof in our book implies surjectivity.

   (b) Show that restricting g_C gives an isomorphism $g_Z \colon \oplus_{\alpha \in A} Z_p(X_\alpha) \to Z_p(X)$. The injectivity of g_Z follows from that of g_C; you need to show that g_Z is well-defined and surjective.

   (c) Show that restricting g_C gives an isomorphism $g_B \colon \oplus_{\alpha \in A} B_p(X_\alpha) \to B_p(X)$. The injectivity of g_B follows from that of g_C; you need to show that g_B is well-defined and surjective.

   (d) Deduce that g_C induces an isomorphism $\oplus_{\alpha \in A} H_p(X_\alpha) \to H_p(X)$.

3. Prove Proposition 13.6: For any topological space X, the singular homology group $H_0(X)$ is a free abelian group with basis consisting of an arbitrary point in each path component.

   Remark: Our book contains a detailed proof; you can learn and use this proof!

4. Use the Mayer-Vietoris Theorem to prove Theorem 13.23: For $n \geq 1$, the singular homology groups of the sphere S^n are

$$H_p(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ or } n \\ 0 & \text{if } 0 < p < n \text{ or } p > n. \end{cases}$$

   Remark: Our book contains a detailed proof; you can learn and use this proof!