Math 147, Homework 8 Solutions
Due: June 5, 2012

You are encouraged to collaborate on the homework problems. Each student must understand and write up his or her own clear and legible solutions.

1. In class we showed that if $v : M \to \mathbb{R}^k$ has a non-degenerate zero at $z_0 \in M$, then the index of v at z_0 is $+1$ or -1. Is it true that if the index of v at an isolated zero is $+1$ or -1 then that zero is non-degenerate?

   Solution.

   No. Consider the vector field $v\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $v(x, y) = (x^2 + y^2)(x, y)$. Note that $(0, 0)$ is an isolated zero of v. Vector field v acts as the identity map on any small circle about the origin, and hence the index of v at $(0, 0)$ is $+1$. However, we have

   $$dv_{(0,0)} = \begin{bmatrix} 3x^2 + y^2 & 2xy \\ 2xy & x^2 + 3y^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{(0,0)},$$

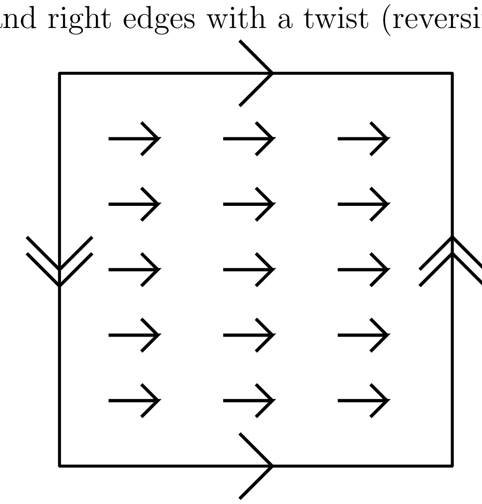
   and hence $(0, 0)$ is a degenerate zero.

2. Find vector fields with isolated zeros on the real projective plane $\mathbb{R}P^2$ and the Klein bottle and compute their indices.

   Solution.

   A vector field $v\colon S^2 \to \mathbb{R}^2$ on the sphere induces a vector field on $\mathbb{R}P^2$ if $v(-x) = -v(x)$ for all $x \in S^2$. One vector field that satisfies this property is given by $v(x, y, z) = (-y, x, 0)$. This vector field on S^2 has two isolated zeros at the antipodal points $\pm(0, 0, 1)$, which are both circulations. A circulation has index one (see page 133 of Guillemin and Pollack). Hence the induced vector field on $\mathbb{R}P^2$ has a single isolated zero of index one. This agrees with the Poincare–Hopf Theorem because the Euler characteristic of $\mathbb{R}P^2$ is one.

   Note the Klein bottle is diffeomorphic to the manifold obtained from a flat rectangle by identifying the top and bottom edges (preserving direction) and by identifying the left and right edges with a twist (reversing direction).



The non-vanishing horizontal vector field drawn above agrees with these identifications, and hence gives a non-vanishing vector field on the Klein bottle. This agrees with the Poincare–Hopf Theorem because the Euler characteristic of the Klein bottle is zero.

3. In this question, you construct will explicit example of a compact two manifold in $\mathbb{R}^3$ without boundary whose genus is g. Fix points $(x_1, y_1), \dots (x_g, y_g) \in \mathbb{R}^2$ and a radius r so that the closed balls of radius r about (x_i, y_i) are disjoint and contained in the open unit disk.

   (a) Let $f_0(x, y) = x^2+y^2-1$ and $f_i(x, y) = (x-x_i)^2+(y-y_i)^2-r^2$ for $1 \leq i \leq g$. Show that:

$$S_g = \big\{(x, y, z) : z^2 + f_0(x, y)f_1(x, y) \dots f_g(x, y) = 0\big\}$$

   is a compact smooth 2-manifold without boundary. [Hint: Where is $f_0(x, y) \dots f_g(x, y)$ negative?]

   Solution.

   Define $h\colon \mathbb{R}^3 \to \mathbb{R}$ by $h(x, y, z) = z^2 + f_0(x, y)f_1(x, y) \dots f_g(x, y)$. We want to show that 0 is a critical value of h. Suppose for a contradiction that we have a critical point (x', y', z') with $h(x', y', z') = 0$. Since $\partial_z h = 2z$, necessarily $z' = 0$. Hence $f_0(x', y')f_1(x', y') \dots f_g(x', y') = 0$ and so $f_i(x', y') = 0$ for some i. Given this, we calculate that

$$\partial_x h = 2(x - x_i) \prod_{j \neq i} f_j(x', y')$$

   and

$$\partial_y h = 2(y - y_i) \prod_{j \neq i} f_j(x', y').$$

   So either $f_j(x', y') = 0$ for some $j \neq i$ or $(x', y') = (x_i, y_i)$. The former is not possible because the closed balls of radius r are disjoint, and hence we cannot have $f_i(x', y') = 0$ and $f_j(x', y') = 0$ for some $j \neq i$. The latter is not possible because if $(x', y') = (x_i, y_i)$, then $f_i(x', y') = f_i(x_i, y_i) = -r^2 \neq 0$. Hence we have reached a contradiction, and so 0 is a critical value of h. By Lemma 1 on page 11 of Milnor, $S_g = h^{-1}(0)$ is a smooth manifold of dimension $3 - 1 = 2$ without boundary.

   To see that S_g is closed, note that it is the preimage of the closed set $\{0\}$ under the continuous map h. Next we show that S_g is bounded. If $(x, y, z) \in S_g$ then $x^2+y^2 \leq 1$ since otherwise $f_0(x, y) \dots f_g(x, y)$ is positive. Hence $|f_i(x, y)| \leq 2^2 + 2^2 + r^2 = 8 + r^2$ for each i. It follows that

$$|z| = \sqrt{f_0(x, y)f_1(x, y) \dots f_g(x, y)} \leq (8 + r^2)^{g/2}.$$

   So S_g is also bounded and hence compact.

(b) Show that the genus of S_g is at least g. (It is true that the genus of S_g is g, but that is harder to prove).

Solution.

Let $C_i = \{(x, y, 0) \mid f_i(x, y) = 0\}$. Each C_i is a circle in S_g, and the circles are disjoint because the closed balls of radius r about each (x_i, y_i) are disjoint. Since $S_g \setminus \coprod_i C_i$ is connected, this shows that the genus of S_g is at least g.

(c) Determine the Euler characteristic of S_g. You do not need to do this rigorously; you can use triangulations or construct a vector field with non-degenerate zeros and compute its index.

Solution.

As in problem 4 on homework 7, let $\pi_{(1,0,0)} : S_g \to \mathbb{R}^3$ be the vector field whose value at x is the orthogonal projection of $(1, 0, 0)$ onto $T(S_g)_x$. This vector field has a source at $(1, 0, 0)$, a sink at $(-1, 0, 0)$, and a saddle at each of the points $(x_i + r, y_i)$ and $(x_i - r, y_i)$ for $1 \leq i \leq g$. These are the only zeros of the vector field. Since each source and sink has index 1 and each saddle has index -1, the index of the vector field is $2-2g$. By the Poincare-Hopf Theorem, the Euler characteristic of S_g is $2 - 2g$.

4. The *Grassmanian* $Gr(m, k)$ is the set of rank m orthogonal projection matrices in $M_k(\mathbb{R})$. The correspondence sending $A \in Gr(m, k)$ to its image $L_A = A(\mathbb{R}^k)$ gives a bijection between $Gr(m, k)$ and the set of m-dimensional linear subspaces of $\mathbb{R}^k$ (you may assume this although it is not hard to prove). In this question, you will show that $Gr(m, k)$ is a smooth manifold of dimension $m(k - m)$. Throughout this problem, fix $A \in Gr(m, k)$ and L_A its image.

   (a) Suppose the columns of the $k \times m$-matrix B form an orthonormal basis for L_A. Show that $A = BB^T$.

   Solution.

   Let $b_1, b_2, \ldots, b_m$ be the columns of B. We can complete this to an orthonormal basis, $\{b_1, \ldots, b_m, v_{m+1}, \ldots, v_k\}$ for all of $\mathbb{R}^k$. Note that for $1 \leq i \leq m$ we have

$$\begin{aligned} BB^T(b_i) &= B(e_i) \\ &= b_i \\ &= A(e_i) \\ &= A^2(e_i) \quad \text{since } A \text{ is a projection matrix} \\ &= A(b_i). \end{aligned}$$

   Note that for $m + 1 \leq i \leq k$ we have

$$BB^T(v_i) = B(0) = 0 = A(v_i).$$

   Hence A and BB^T act the same on a basis for $\mathbb{R}^k$, so $A = BB^T$.

(b) Show that the set $U_{L_A} = \left\{B \in Gr(m,k) : L_B \cap L_A^\perp = \{0\}\right\}$ is an open neighborhood of A in $Gr(m,k)$.

Solution.

We will show that $Gr(m,k) \setminus U_{L_A}$ is closed in $Gr(m,k)$. Suppose $\{B_i\}$ is a sequence in $Gr(m,k) \setminus U_{L_A}$ converging to matrix B in $Gr(m,k)$. Then for each i there exists some vector $x_i \neq 0$ with $x_i \in L_{B_i} \cap L_A^\perp$. Since $x_i \neq 0$, we may assume that $\|x_i\| = 1$. Moreover, since B_i is a projection matrix and $x_i \in L_{B_i}$, we have $B_i x_i = x_i$. As a sequence in the compact sphere, $\{x_i\}$ has a convergent subsequence $\{x_{i_j}\}$ with limit point $0 \neq x \in L_A^\perp$. Hence $\{B_{i_j} x_{i_j}\} = \{x_{i_j}\}$ converges to $Bx = x$ with $0 \neq x \in L_A^\perp$. This shows that B is in $Gr(m,k) \setminus U_{L_A}$, and hence $Gr(m,k) \setminus U_{L_A}$ is closed and U_{L_A} is open in $Gr(m,k)$.

(c) For any $B \in U_{L_A}$, show that L_B is the graph of a linear map from L_A to $L_A^\perp$.

Solution.

Let $\pi_{L_A} \colon \mathbb{R}^k \to L_A$ be the projection map. Let $\pi_{L_A}|_{L_B} \colon L_B \to L_A$ be the restriction of this projection map to L_B. Since the null space of π_{L_A} is $L_A^\perp$, since $L_B \cap L_A^\perp = \{0\}$, and since the dimensions of L_A and L_B are equal, we have that $\pi_{L_A}|_{L_B} \colon L_B \to L_A$ is a linear isomorphism. Let $\pi_{L_A^\perp} \colon \mathbb{R}^k \to L_A^\perp$ be the projection map. Consider the linear map $f = \pi_{L_A}^\perp \circ (\pi_{L_A}|_{L_B})^{-1} \colon L_A \to L_A^\perp$. We claim that L_B is the graph of this linear map f.

If z is in L_B, then $z = (\pi_{L_A}|_{L_B})^{-1}(x)$ for some $x \in L_A$. So

$$\begin{aligned}
x + f(x) &= x + \pi_{L_A}^\perp((\pi_{L_A}|_{L_B})^{-1}(x)) \\
&= x + \pi_{L_A}^\perp(z) \\
&= \pi_{L_A|_{L_B}}(z) + \pi_{L_A}^\perp(z) \\
&= \pi_{L_A}(z) + \pi_{L_A}^\perp(z) \\
&= z.
\end{aligned}$$

This shows that z is in the graph of f. Conversely, if z is in the graph of f, then $z = x + f(x)$ for some $x \in L_A$. Note that $\pi_{L_A}((\pi_{L_A}|_{L_B})^{-1}(x)) = x$. Hence we have

$$\begin{aligned}
z &= x + f(x) \\
&= x + \pi_{L_A}^\perp((\pi_{L_A}|_{L_B})^{-1}(x)) \\
&= \pi_{L_A}((\pi_{L_A}|_{L_B})^{-1}(x)) + \pi_{L_A}^\perp((\pi_{L_A}|_{L_B})^{-1}(x)) \\
&= (\pi_{L_A}|_{L_B})^{-1}(x)
\end{aligned}$$

which is in L_B. We have shown that z is in L_B if and only if it is in the graph of the linear map f.

(d) Show that U_{L_A} is diffeomorphic to $\mathbb{R}^{m(k-m)}$.

Solution.

First we will show that U_{L_A} is diffeomorphic to the set of linear maps from L_A to $L_A^\perp$. The forward map is given in part (c). It's inverse is given as follows. Fix a basis for L_A. Now, suppose we are given a linear map from L_A to $L_A^\perp$. It's graph Γ is a m-dimensional linear subspace of $\mathbb{R}^k$ satisfying $\Gamma \cap L_A^\perp = \{0\}$. The map $(\pi_{L_A}|_\Gamma)^{-1} \colon L_A \to \Gamma$ smoothly maps our basis for L_A to a basis for Γ. We apply Gram-Schmidt to get a orthonormal basis for Γ. Let B be the $k \times m$-matrix whose columns are the vectors in this basis. By part (a), BB^T is in $Gr(m,k)$, and this is the image of the linear map under the inverse. Since Gram-Schmidt depends smoothly on its inputs, the inverse is smooth. Hence U_{L_A} is diffeomorphic to the set of linear maps from L_A to $L_A^\perp$, which is diffeomorphic to the set of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^{k-m}$, which is equal to the set of matrices of size $m \times (k-m)$, which is equal to $\mathbb{R}^{m(k-m)}$.