Vietoris-Rips Thickenings of Spheres

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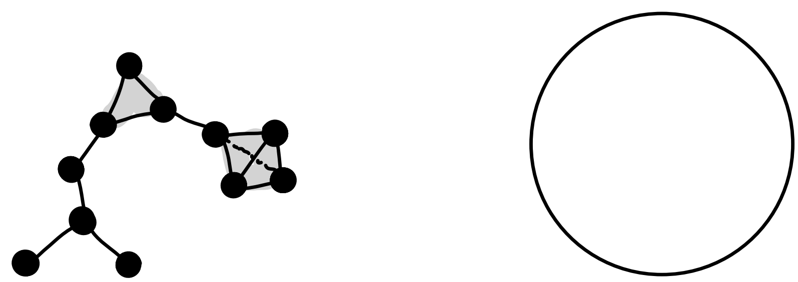
X metric space, $r \geq 0$.

Def The Vietoris-Rips simplicial complex $VR(X,r)$ has

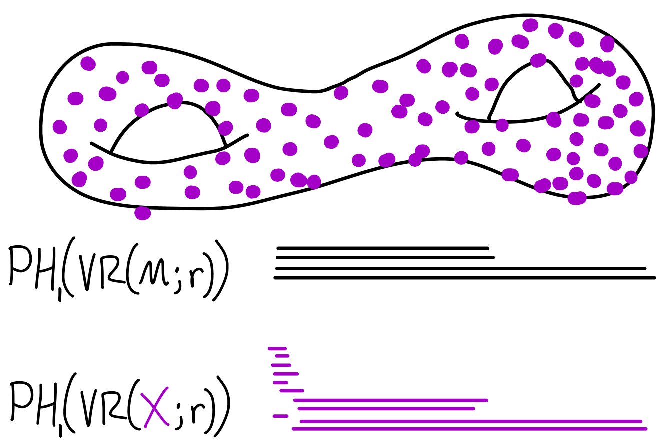
- vertex set X
- finite simplex $\sigma \subseteq X$ when $\text{diameter}(\sigma) \leq r$.



History

- Cohomology theory for metric spaces
- Geometric group theory
- Applied topology

Stability



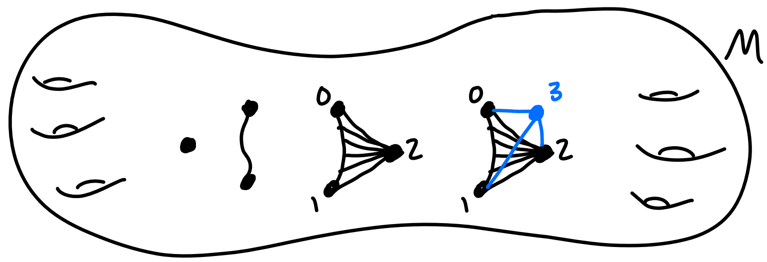
Chazal, de Silva, Oudot, 2014
Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, 2009

Thm (Hausmann 1995)

M compact Riemannian manifold.
Then $\exists\ r_0 > 0$ such that $VR(M;r) \simeq M \ \forall r < r_0$.

Proof Sketch

$$\begin{array}{c} VR(M;r) \\ \downarrow \\ M \end{array}$$

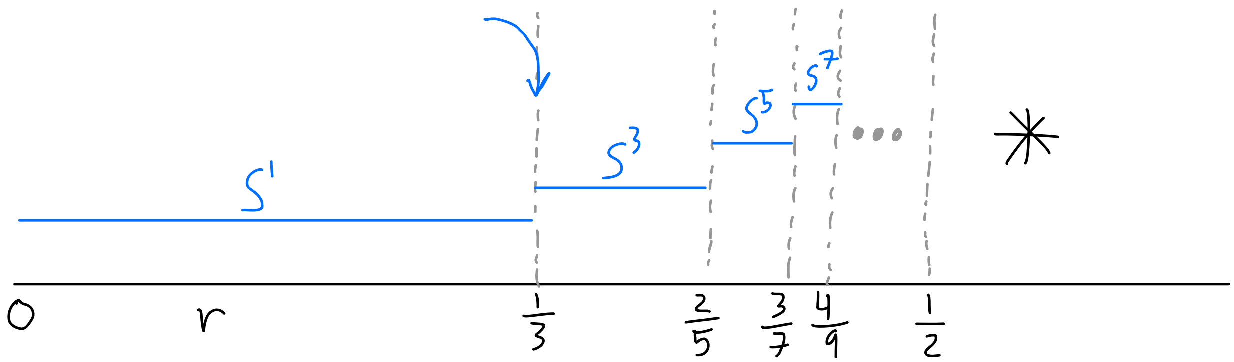


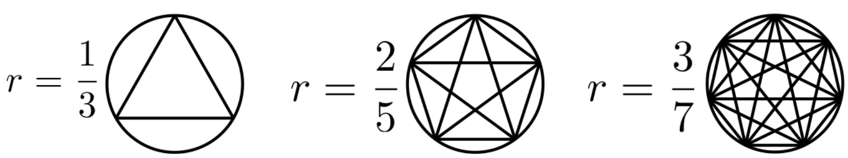
- Not canonical
- $M \hookrightarrow VR(M;r)$ not continuous.

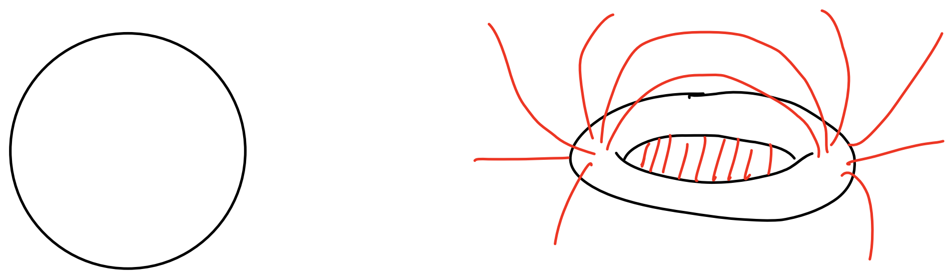
A. Adamaszek, "The Vietoris-Rips complexes of a circle", 2017

S^1 is circle with geodesic metric, unit circumference.

<u>Thm</u> $\quad VR(S^1;r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{if } r = \frac{k}{2k+1} \end{cases} \qquad k \in \mathbb{N}$

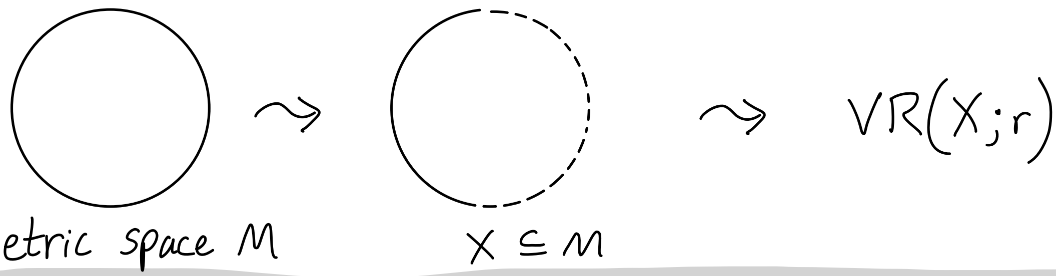






Metric Reconstruction

A simplicial complex whose vertex set is a metric space should often be equipped with an optimal transport metric (instead of the simplicial complex topology).

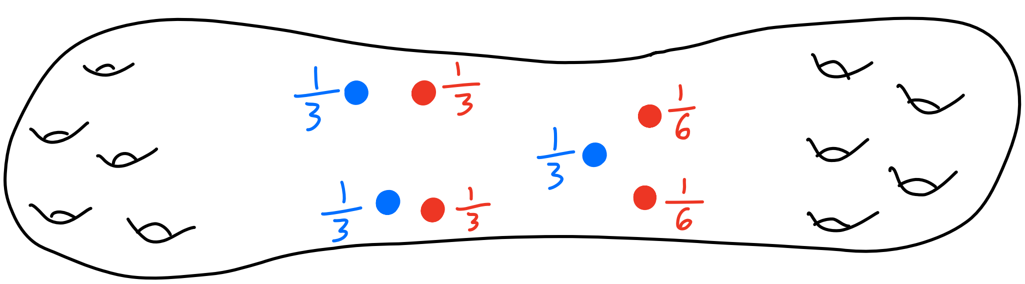


Adamaszek, A, Frick, 2018, "Metric reconstruction via optimal transport"

Def X metric space, $r \geq 0$.
The Vietoris-Rips metric thickening is

$$VR(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; \begin{array}{l} x_i \in X, \quad \mathrm{diam}(\{x_0, \dots, x_k\}) \leq r, \\ \lambda_i \geq 0, \quad \sum \lambda_i = 1 \end{array} \right\},$$

equipped with the optimal transport metric.

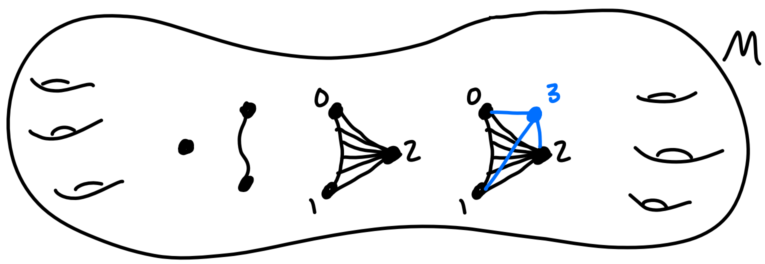


Thm (Hausmann 1995)

M compact Riemannian manifold.
Then $\exists\ r_0 > 0$ such that $VR(M;r) \simeq M\ \forall r < r_0$.

Proof Sketch

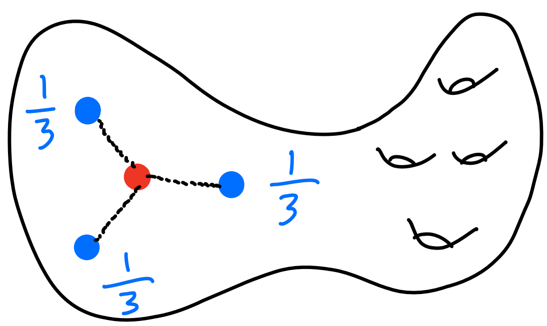
$$VR(M;r) \to M$$



- Not canonical
- $M \hookrightarrow VR(M;r)$ not continuous.

Our Proof Sketch

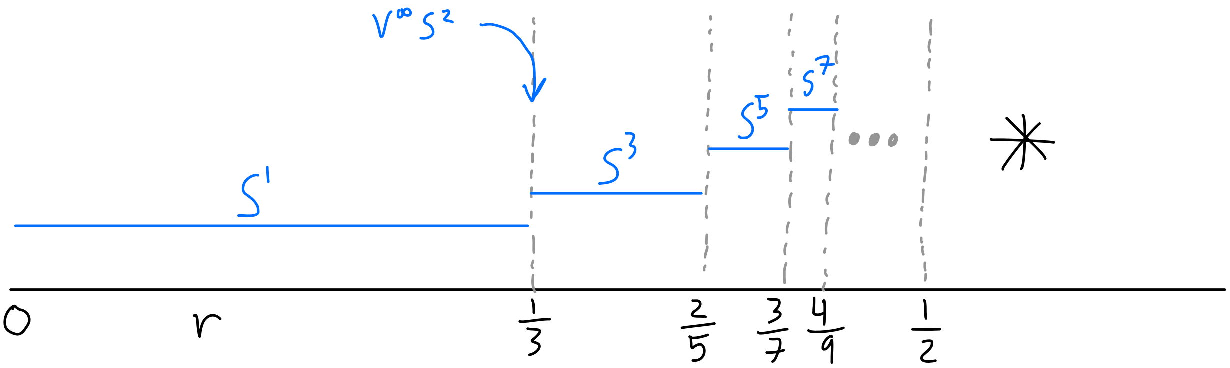
$$VR^m(M;r) \rightleftarrows M \qquad \sum \lambda_i \delta_{x_i} \mapsto \text{Karcher or Frechét mean}$$

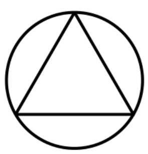
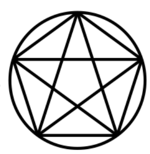
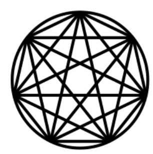


A, Adamaszek, "The Vietoris-Rips complexes of a circle", 2017

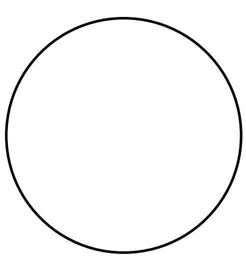
S^1 is circle with geodesic metric, unit circumference.

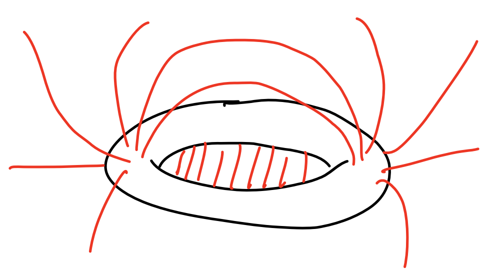
<u>Thm</u> $VR(S^1;r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ \bigvee^{\infty} S^{2k} & \text{if } r = \frac{k}{2k+1} \end{cases}$ $k \in \mathbb{N}$



$r = \frac{1}{3}$  $r = \frac{2}{5}$  $r = \frac{3}{7}$ 

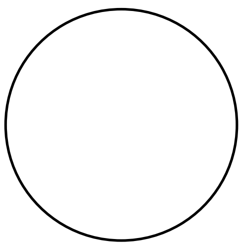
By contrast, $VR^m(S^1;\frac{1}{3}) \simeq S^3$.

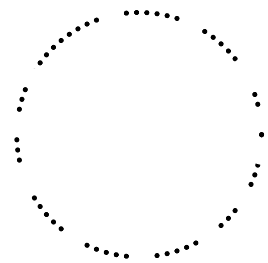




A, Mémoli, Moy, Wang, 2021, "The persistent homology of optimal transport based metric thickenings"

<u>Thm</u> For X totally bounded, $VR^m(X;r)$ and $VR(X;r)$ have the same (undecorated) persistence diagrams.

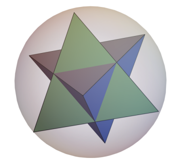


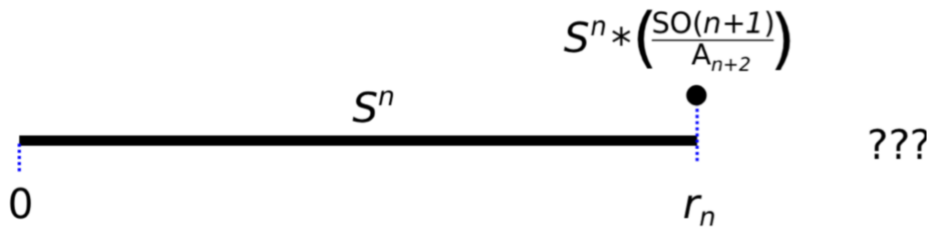


<u>Question</u> Is $VR^m_<(X;r) \simeq VR_<(X;r)$?

More generally,

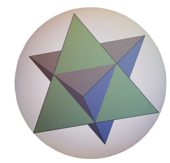
Thm $$\mathrm{VR}^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$$

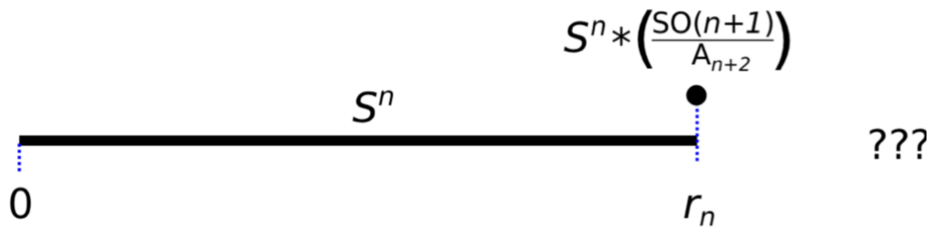




More generally,

<u>Thm</u> $VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$





<u>Sketch</u> $VR^m(S^n;r_n)$

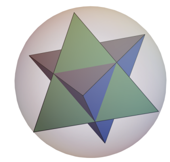
$= VR^m(S^n;r_n) \setminus \left(\text{interiors of regular } \Delta^{n+1}\right) \cup \Delta^{n+1} \times \left(\frac{SO(n+1)}{A_{n+2}}\right)$

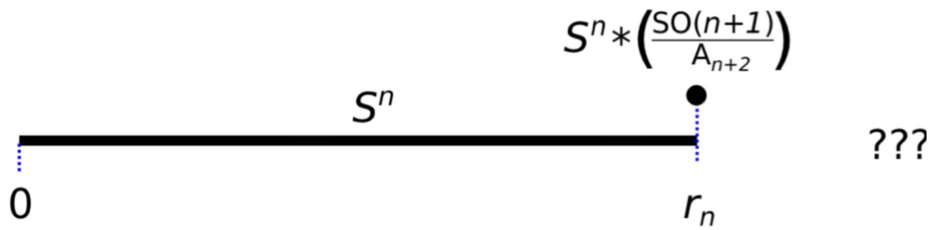
$\simeq S^n \times C\left(\frac{SO(n+1)}{A_{n+2}}\right) \cup C(S^n) \times \left(\frac{SO(n+1)}{A_{n+2}}\right)$

$= S^n * \frac{SO(n+1)}{A_{n+2}}$

More generally,

Thm $VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$

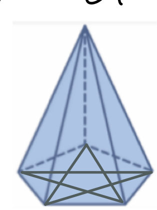




Katz, 1991, "On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional"

Conjecture The next change in homotopy type for $VR^m(S^2;r)$ occurs at the diameter of a pentagonal pyramid, with homotopy type an 8-dimensional CW complex

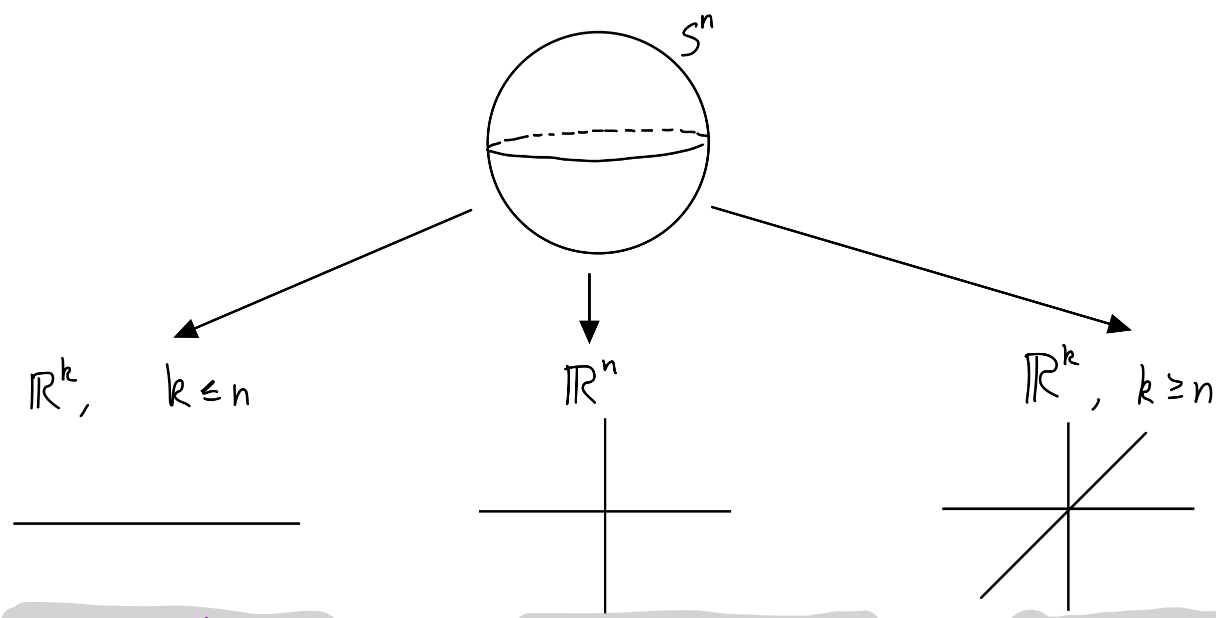
$$\left(S^2 * \frac{SO(3)}{A_4}\right) \cup_f \left(\Delta^5 \times \frac{SO(3)}{\mathbb{Z}/5\mathbb{Z}}\right).$$



Here $\partial\Delta^5 \times \frac{SO(3)}{\mathbb{Z}/5\mathbb{Z}} \xrightarrow{f} S^2 * \frac{SO(3)}{A_4}$

with $\pi_4\left(S^2 * \frac{SO(3)}{A_4}\right) \cong \mathbb{Z}/3\mathbb{Z}$.

Application: Borsuk-Ulam theorems



Almgren (1965):
Geometric measure theory
Gromov (2003):
Filling radius, characteristic classes

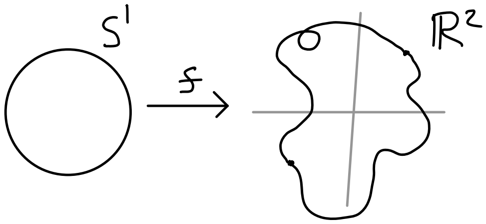
Borsuk-Ulam (1933)

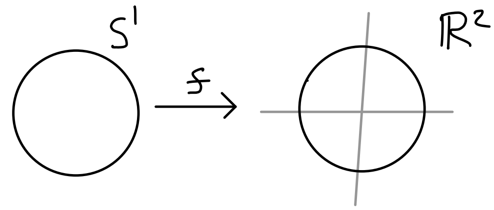
A, Bush, Frick (2020)
"Metric thickenings, Borsuk-Ulam theorems, and orbitopes"

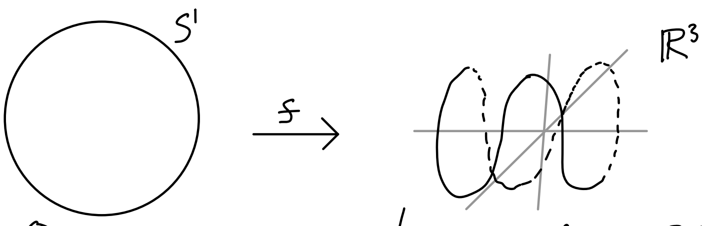
"Waist of sphere" theorem For $f : S^n \to \mathbb{R}^k$ with $k \le n$, $\exists\, y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \ge \mathrm{Vol}_{n-k}(S^{n-k})$.

Invariance of dimension.

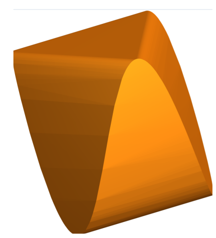
Borsuk-Ulam theorems for $f: S^n \to \mathbb{R}^k$ with $k \geq n$?







$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta)$$



<u>Thm</u> For $f: S^1 \to \mathbb{R}^{2k+1}$, $\exists$ $X \subset S^1$ of diameter at most $\frac{k}{2k+1}$ such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.

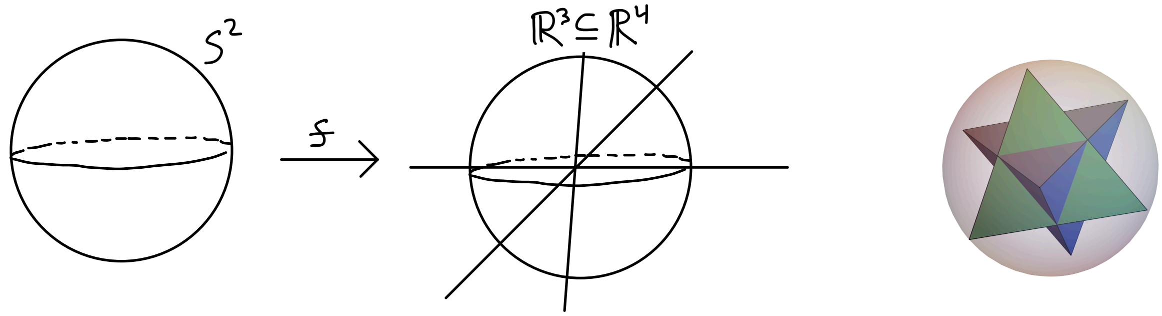
<u>Proof</u>

$$S^1 \xrightarrow{\quad f \quad} \mathbb{R}^{2k+1} \quad \text{induces}$$
$$VR(S^1; r) \xrightarrow{\quad f \quad} \mathbb{R}^{2k+1}$$

Sharpness of diameter bound

$$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$$
$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$$

Thm For $f: S^n \to \mathbb{R}^{n+2}$, $\exists\ X \subset S^n$ of diameter at most r_n such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.



Proof

$$S^n \xrightarrow{\ f\ } \mathbb{R}^{n+2} \text{ induces}$$

$$S^n * \frac{SO(n+1)}{A_{n+2}} \simeq VR^m(S^n; r) \xrightarrow{\ f\ } \mathbb{R}^{n+2}$$

Application?: Gromov-Hausdorff distances between spheres

Lim, Mémoli, Smith, 2021, "The Gromov-Hausdorff distance between spheres"

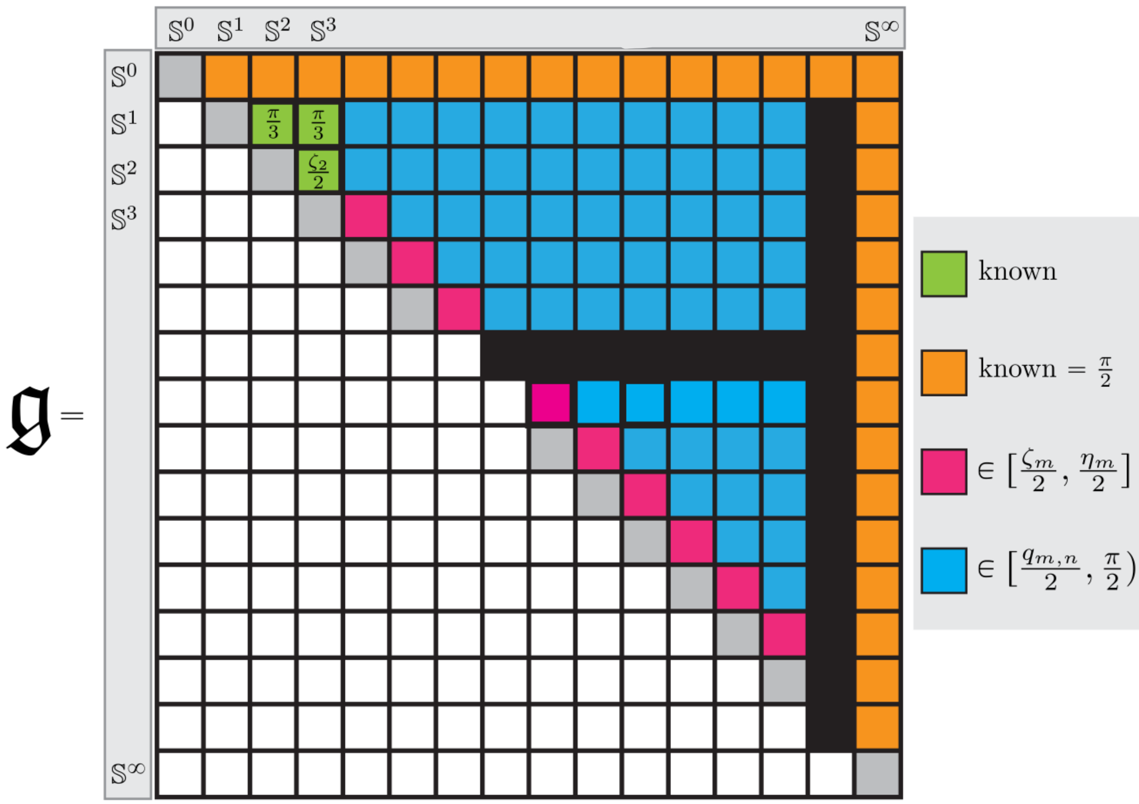
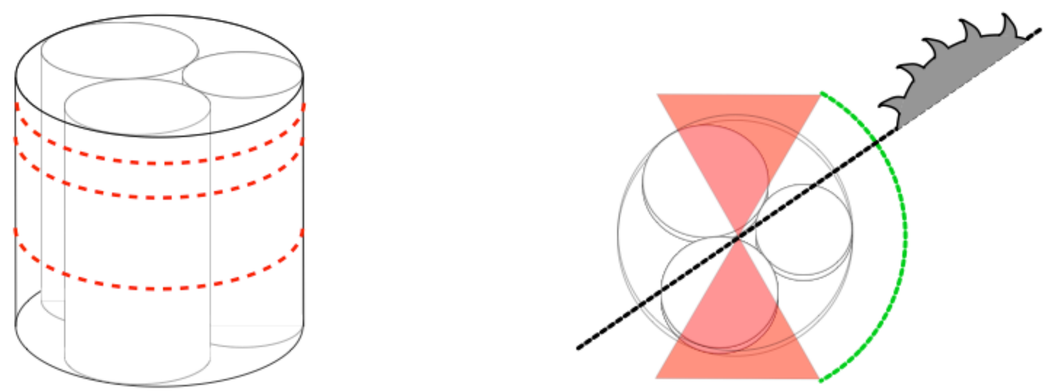


FIGURE 2. **The matrix $\mathfrak{g}$ such that** $\mathfrak{g}_{m,n} := d_{\mathrm{GH}}(\mathbb{S}^m, \mathbb{S}^n)$. According to Remark 1.5 and Corollary 1.14, all non-zero entries of the matrix $\mathfrak{g}$ are in the range $[\frac{\pi}{4}, \frac{\pi}{2}]$. In the figure, $\zeta_m = \arccos\left(\frac{-1}{m+1}\right)$ is the edge length of the regular geodesic simplex inscribed in $\mathbb{S}^m$, η_m is the diameter of a face of the regular geodesic simplex in $\mathbb{S}^m$ (see equation (5)), and $q_{m,n} = \max\left\{\frac{\zeta_m}{2}, \frac{\pi}{2} - \mathrm{cov}_{\mathbb{S}^m}(n+1)\right\}$.

For $n \leq k$, is $2 \cdot d_{GH}(S^n, S^k) = \inf\{ r \mid VR(S^n; r) \text{ is } (k-1)\text{-connected}\}$?

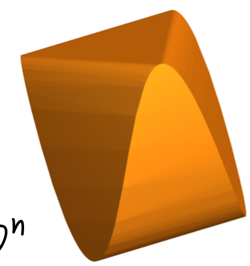
Application

PROJECTIVE CODES AND ODD MAPS



A, Bush, Frick, 2021, "The topology of projective codes and the distribution of zeros of odd maps"

<u>Questions</u>



(1) $VR^m(S^n;r)$ for larger r?

(2) $\check{C}ech^m(S^n;r)$?

(3) Other manifolds? Tori, ellipsoids, $\mathbb{RP}^n$, $\mathbb{CP}^n$

(4) $VR^m_<(X;r) \simeq VR_<(X;r)$?

(5) Morse and Morse-Bott theories

(6) Measures with infinite support

(8) Tighter connections between $VR^m(X;r)$ and $B_{L^\infty(X)}(X;r)$.

(7) In $VR^m(X;r)$ replace ∞-diam with p-diam.
In $\check{C}ech^m(X;r)$ replace ∞-variance with p-variance.