# Vietoris–Rips thickenings of spheres

Henry Adams (Colorado State University), Michał Adamaszek (MOSEK ApS), Florian Frick (Carnegie Mellon)

Thanks to my graduate students: Johnathan Bush, Brittany Carr, Mark Heim, Lara Kassab, Joshua Mirth, Alex Williams

Idea 1: Vietoris–Rips complexes of the circle

Idea 2: Metric thickenings

Idea 3: Vietoris–Rips thickenings of spheres

Idea 4: Borsuk–Ulam theorems

## Idea 1: Vietoris–Rips complexes of the circle

With Michał Adamaszek

*Pacific Journal of Mathematics* 290, 1–40, 2017

## Idea 2: Metric thickenings

With Michał Adamaszek and Florian Frick

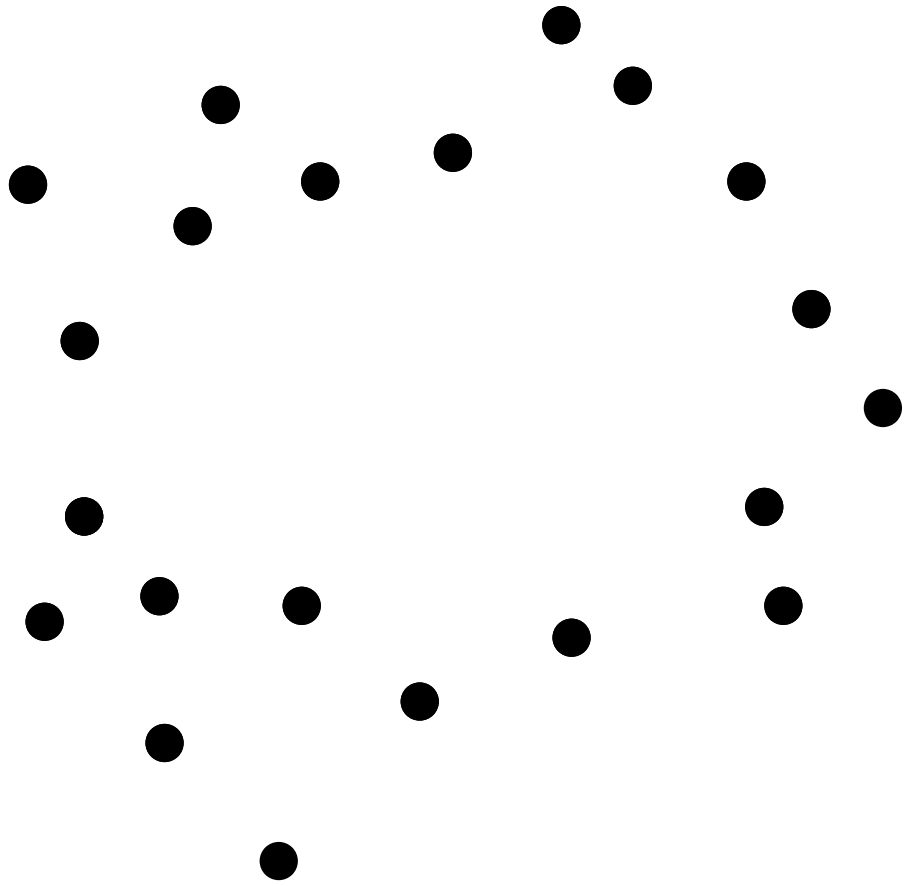
*SIAM Journal on Applied Algebra and Geometry* 2, 597–619, 2018

## Idea 3: Vietoris–Rips thickenings of spheres

## Idea 4: Borsuk–Ulam theorems

With Johnathan Bush and Florian Frick

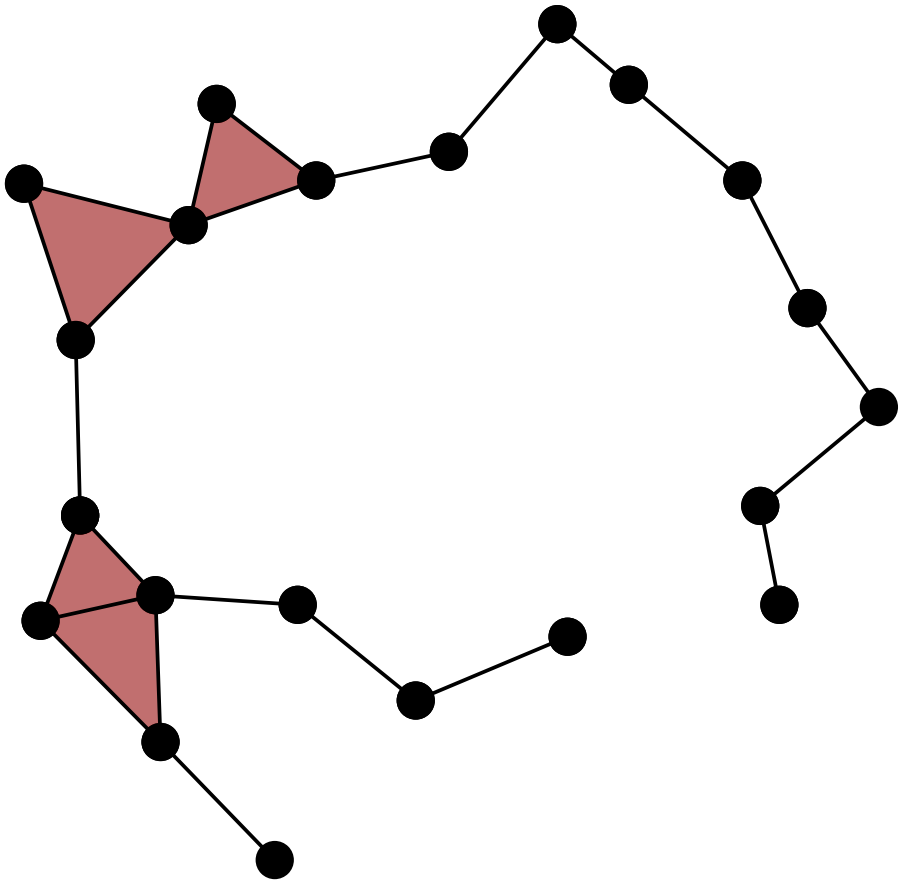
*Mathematika* 66, 79–102, 2020



**Definition**

For $X$ a metric space and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

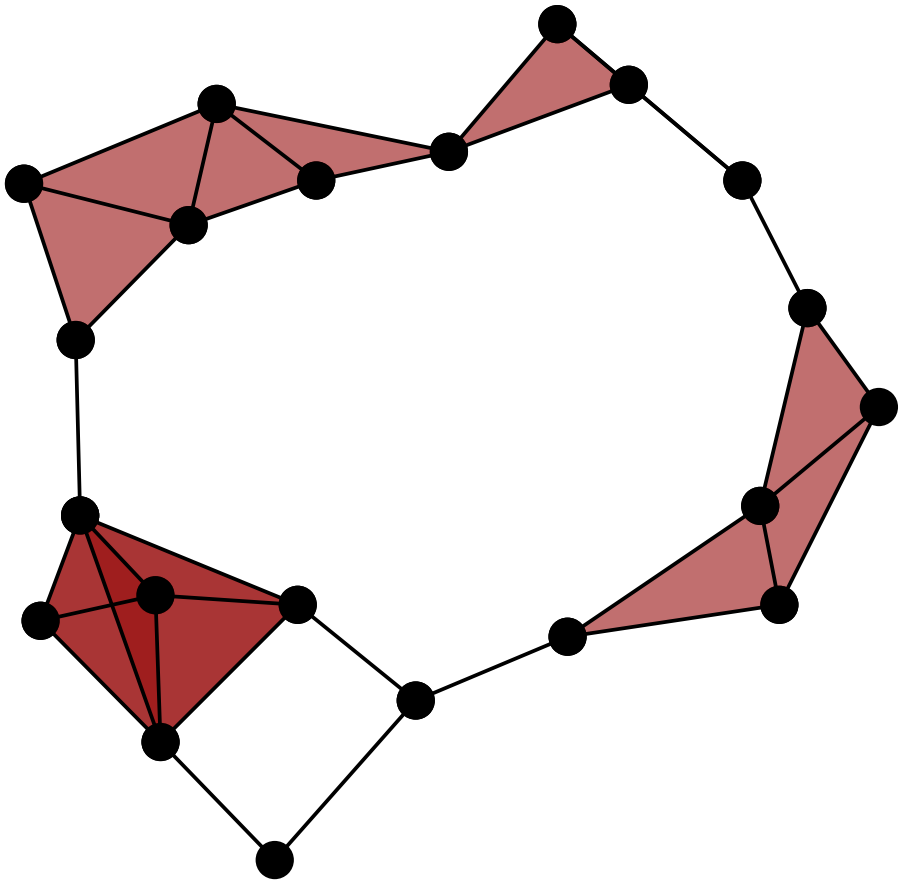
- vertex set $X$
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



**Definition**

For $X$ a metric space and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

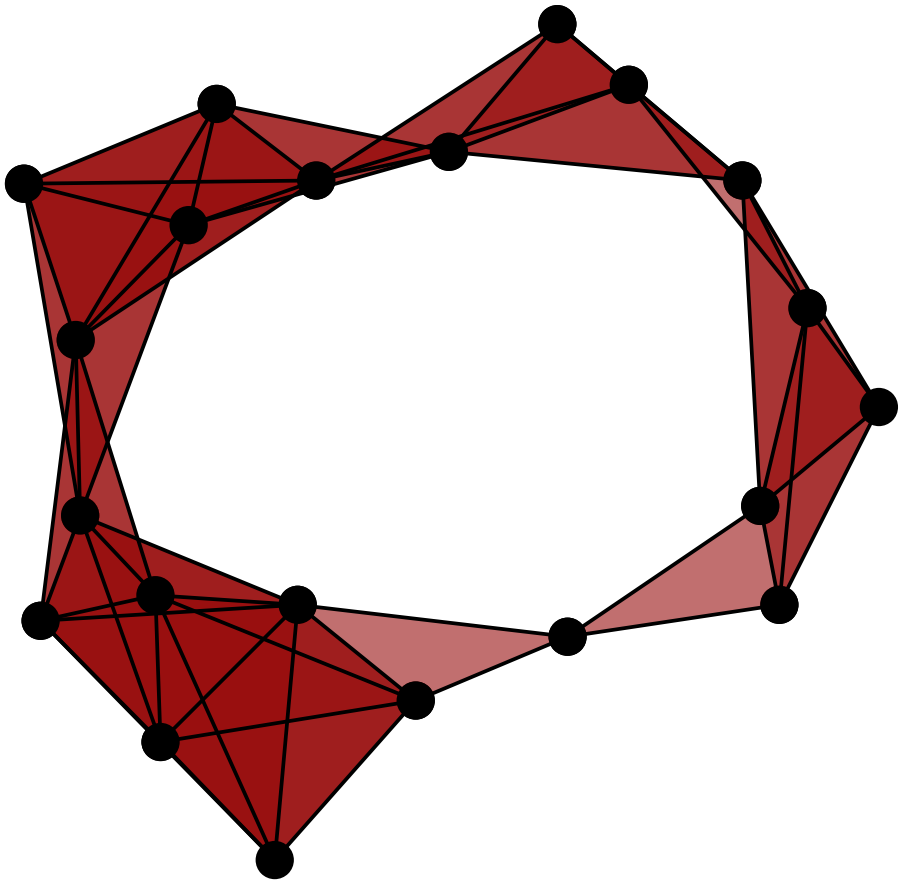
- vertex set $X$
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



**Definition**

For $X$ a metric space and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

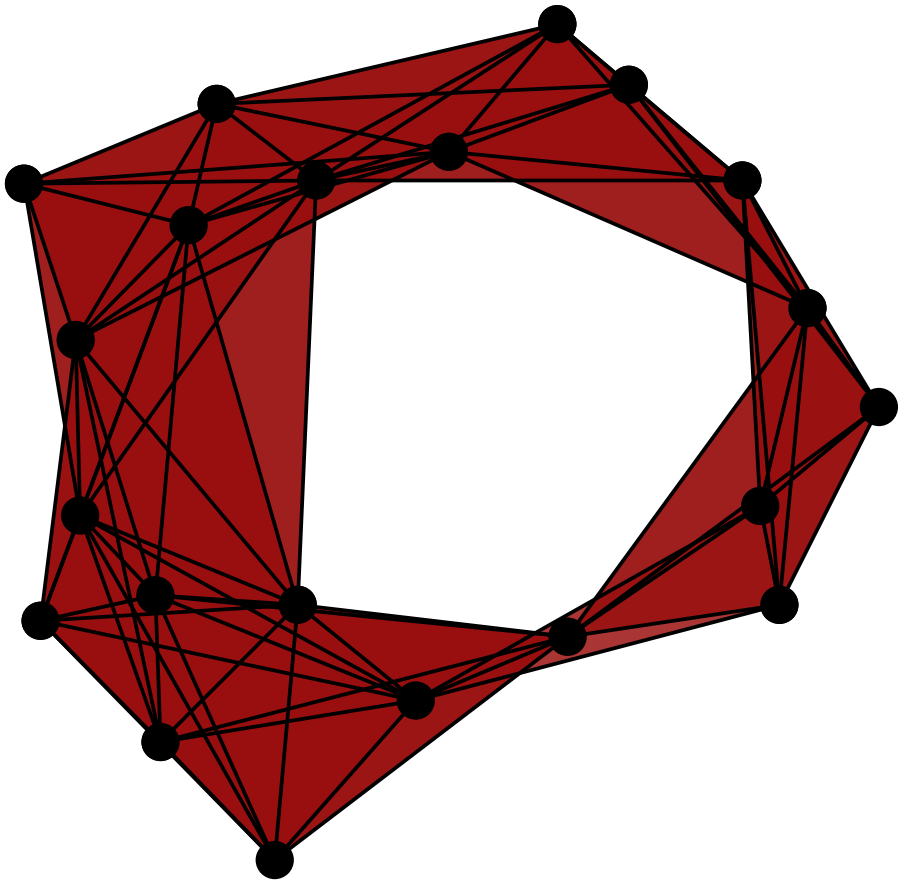
- vertex set $X$
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



## Definition

For $X$ a metric space and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

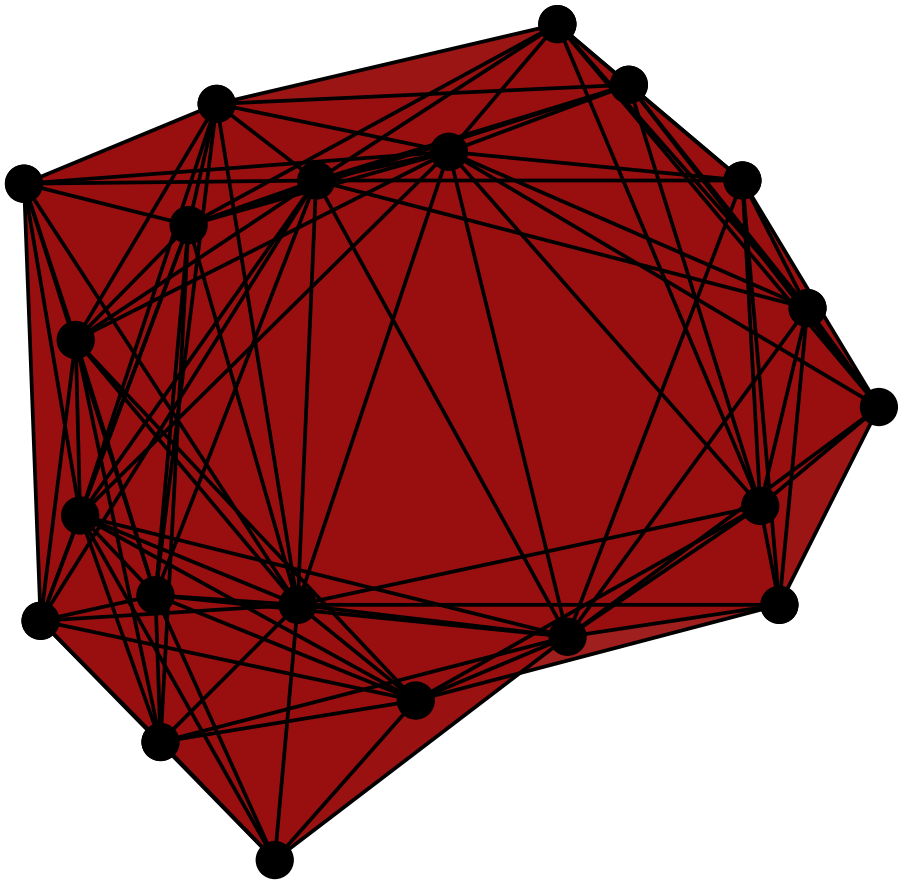
- vertex set $X$
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



**Definition**

For $X$ a metric space and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

- vertex set $X$
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



## Definition

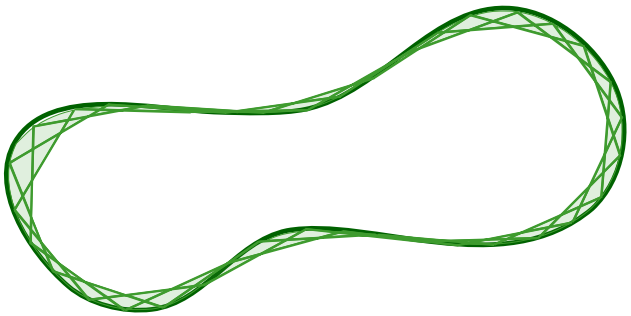
For $X$ a metric space and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

- vertex set $X$
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.

**Theorem (Hausmann, 1995)**

*For $M$ a compact Riemannian manifold and $r$ small,*

$$\mathrm{VR}(M; r) \simeq M.$$



**Theorem (Hausmann, 1995)**

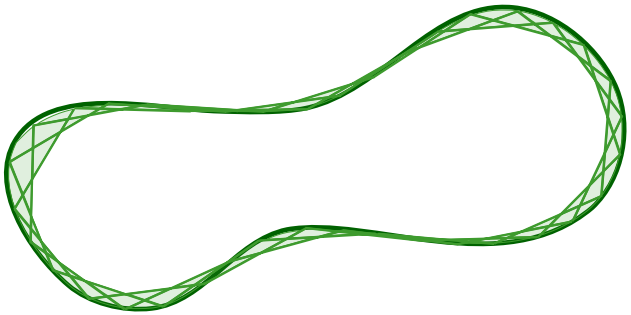
*For $M$ a compact Riemannian manifold and $r$ small,*

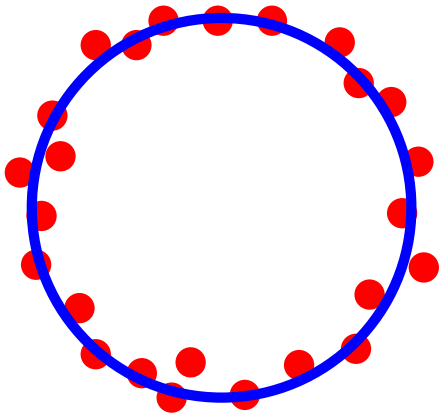
$$\mathrm{VR}(M; r) \simeq M.$$

**Theorem (Latschev, 2001)**

*For $r$ small and $d_{\mathrm{GH}}(X, M)$ small (depending on $r$),*

$$\mathrm{VR}(X; r) \simeq M.$$





**Theorem (Hausmann, 1995)**

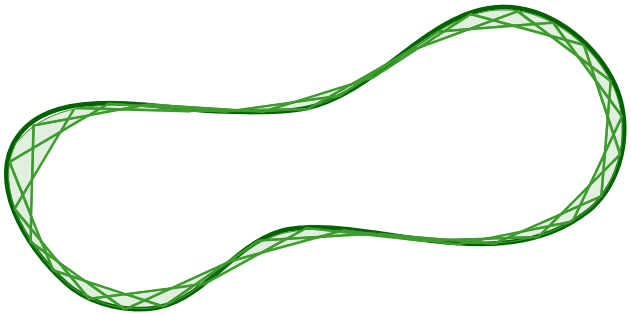
*For $M$ a compact Riemannian manifold and $r$ small,*

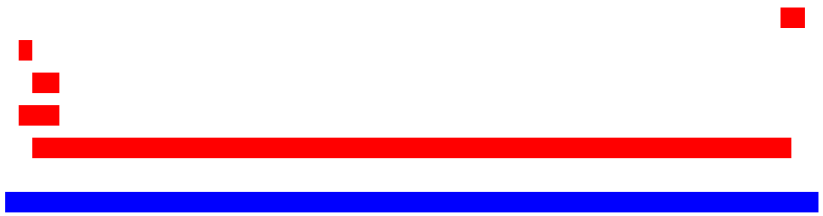
$$\mathrm{VR}(M; r) \simeq M.$$

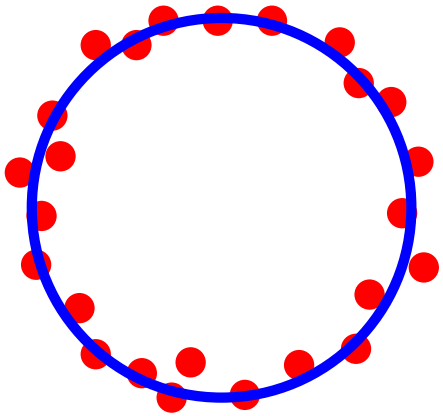
**Theorem (Latschev, 2001)**

*For $r$ small and $d_{\mathrm{GH}}(X, M)$ small (depending on $r$),*

$$\mathrm{VR}(X; r) \simeq M.$$



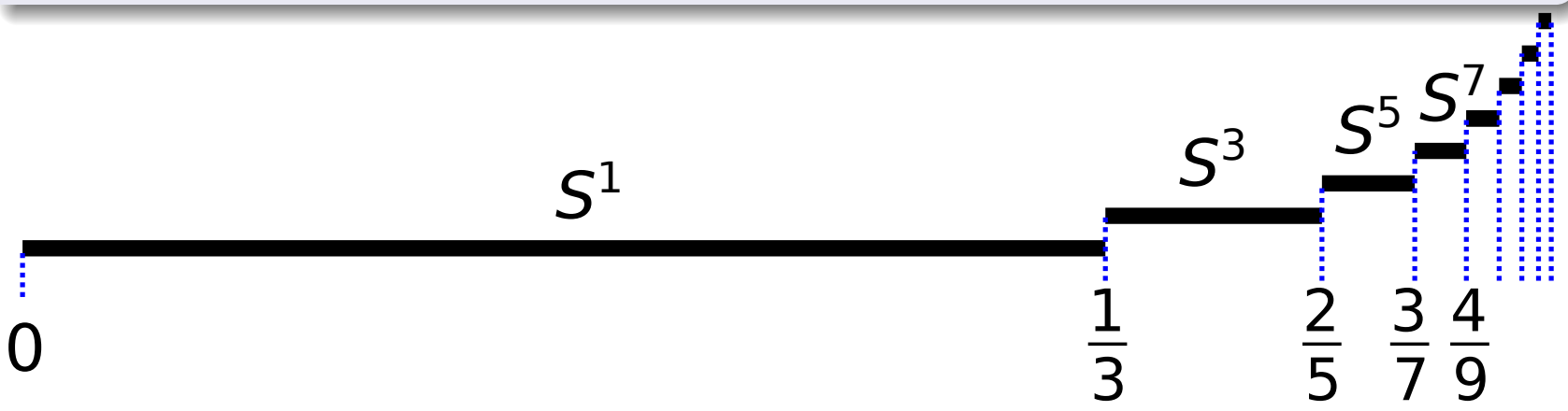




Idea 1: Let $S^1$ be the circle of unit circumference.

**Theorem (Adamaszek, A)**

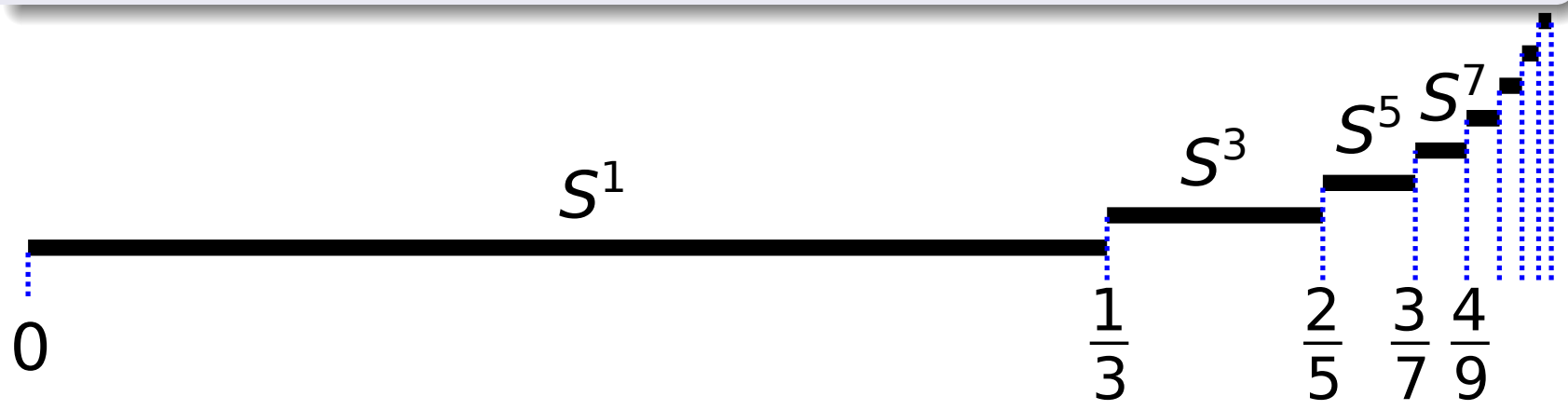
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$

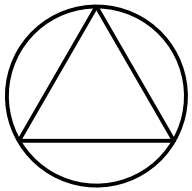


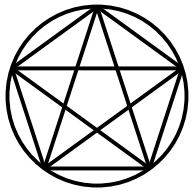
Idea 1: Let $S^1$ be the circle of unit circumference.

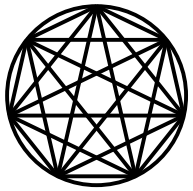
**Theorem (Adamaszek, A)**

$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$




$r = \frac{1}{3}$ 

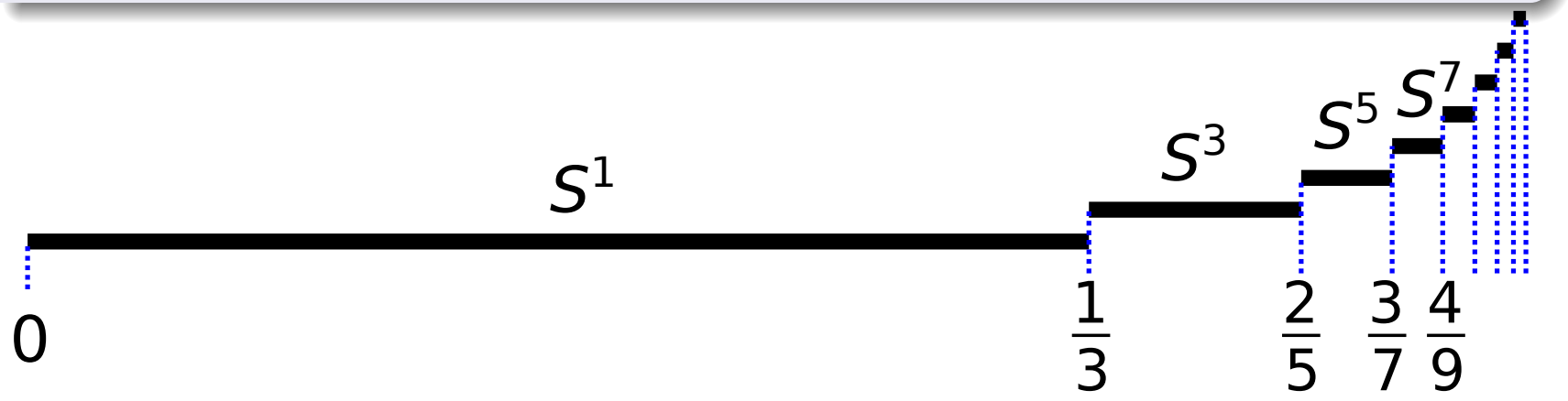
$r = \frac{2}{5}$ 

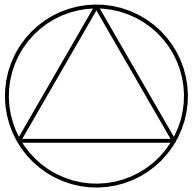
$r = \frac{3}{7}$ 

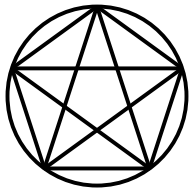
Idea 1: Let $S^1$ be the circle of unit circumference.

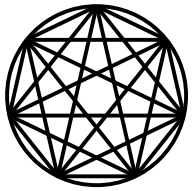
**Theorem (Adamaszek, A)**

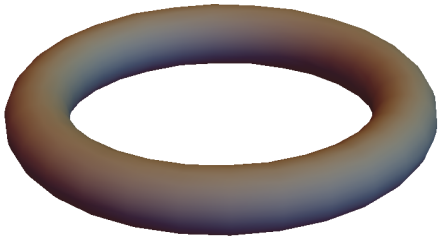
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$




$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

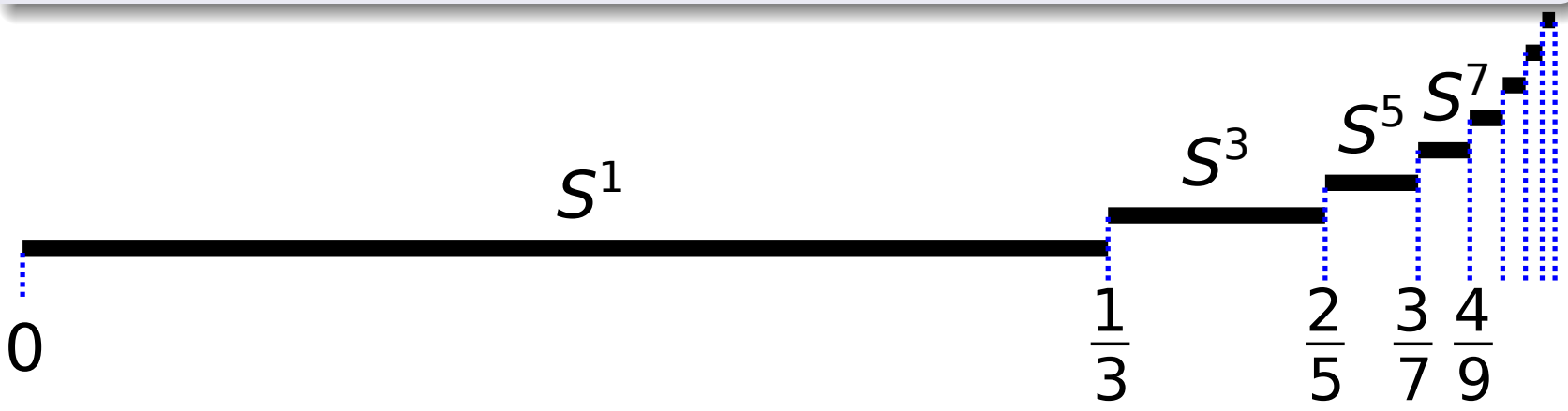
$r = \frac{3}{7}$ 

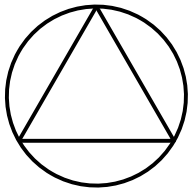


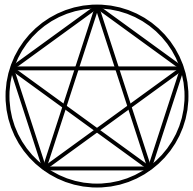
Idea 1: Let $S^1$ be the circle of unit circumference.

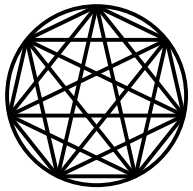
**Theorem (Adamaszek, A)**

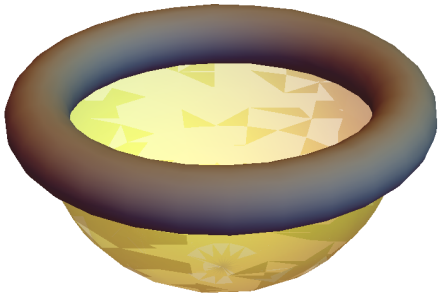
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$




$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

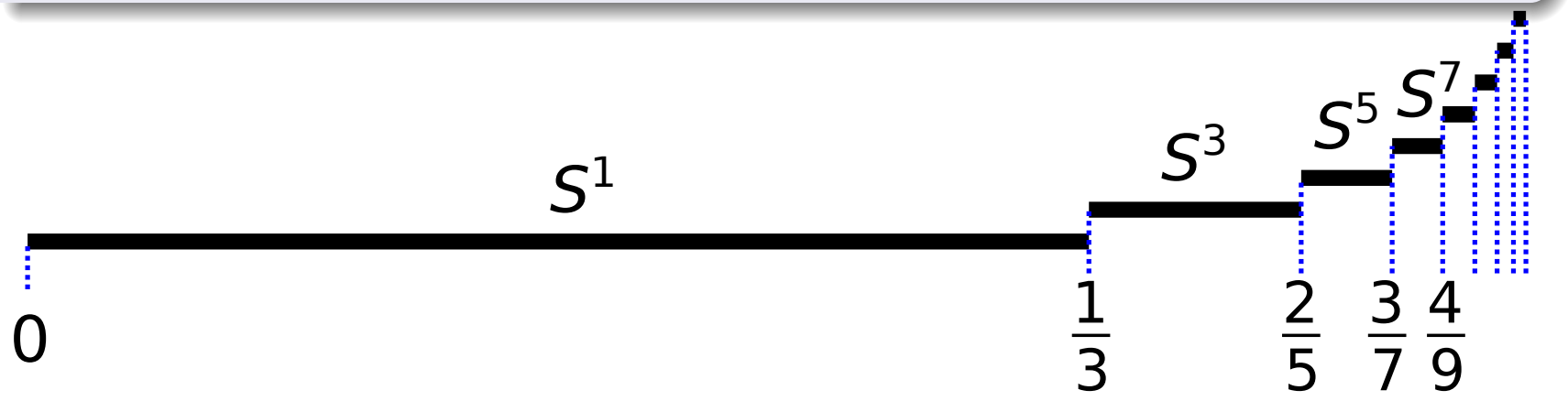
$r = \frac{3}{7}$ 

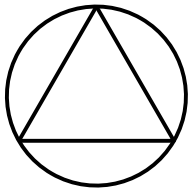


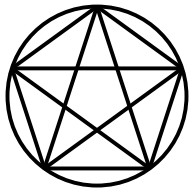
Idea 1: Let $S^1$ be the circle of unit circumference.

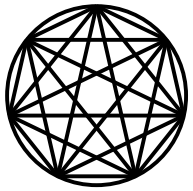
**Theorem (Adamaszek, A)**

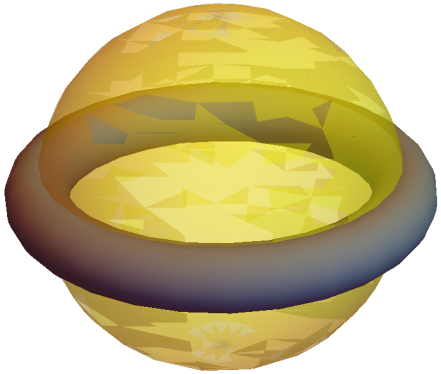
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$




$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

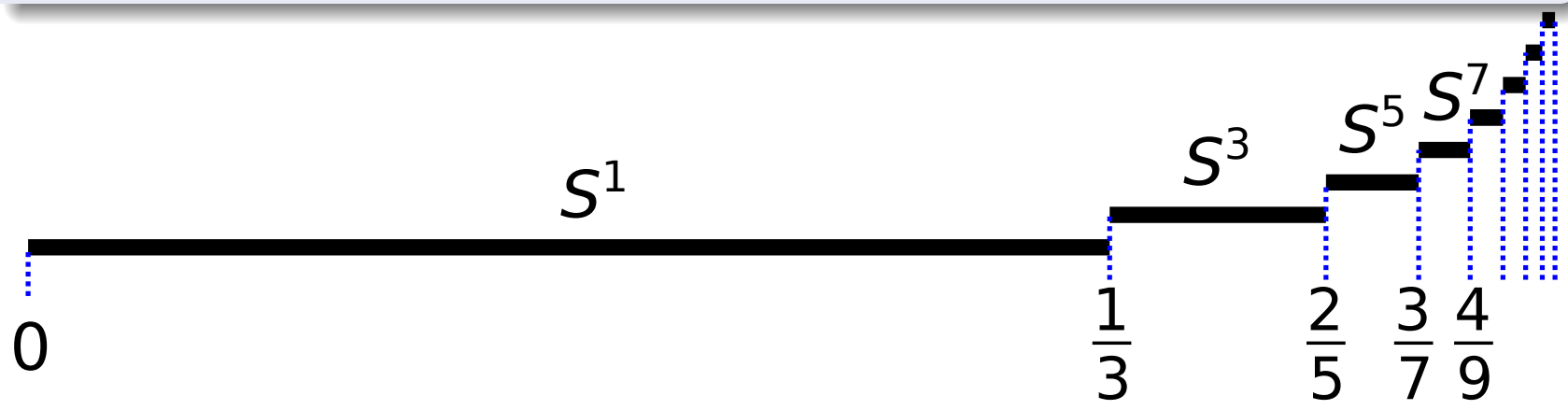
$r = \frac{3}{7}$ 

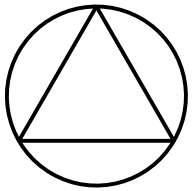


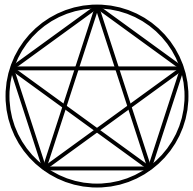
Idea 1: Let $S^1$ be the circle of unit circumference.

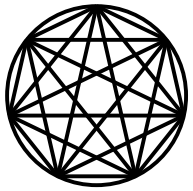
**Theorem (Adamaszek, A)**

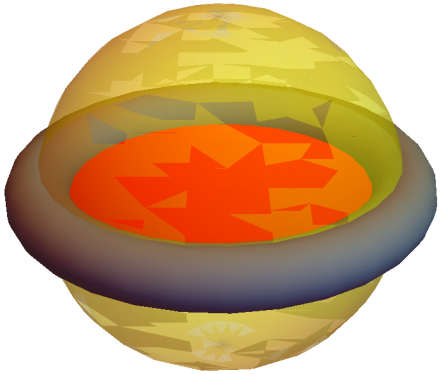
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$




$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

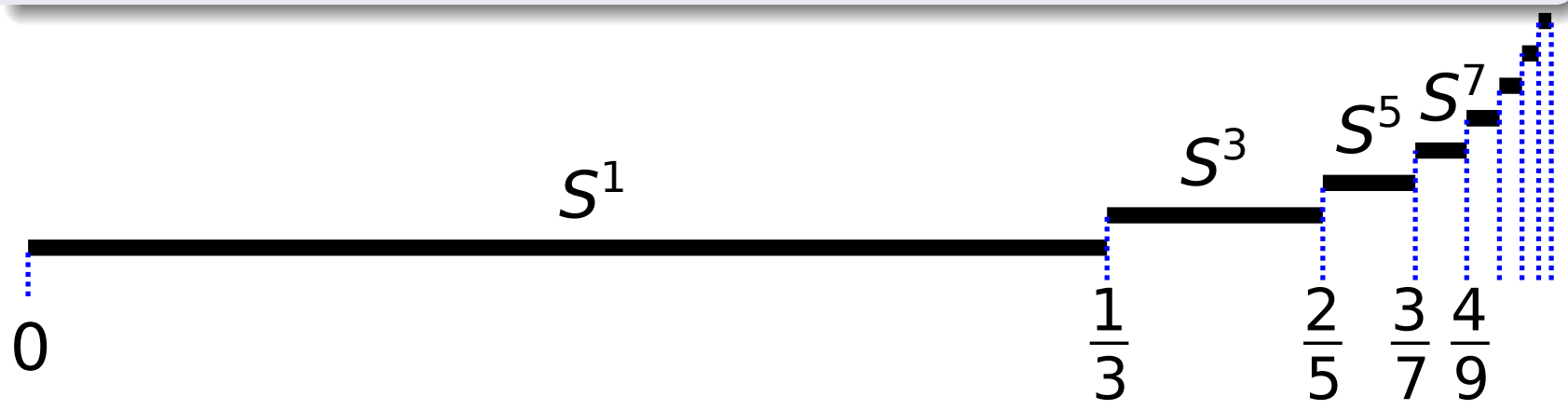
$r = \frac{3}{7}$ 

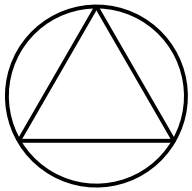


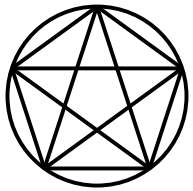
Idea 1: Let $S^1$ be the circle of unit circumference.

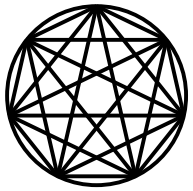
**Theorem (Adamaszek, A)**

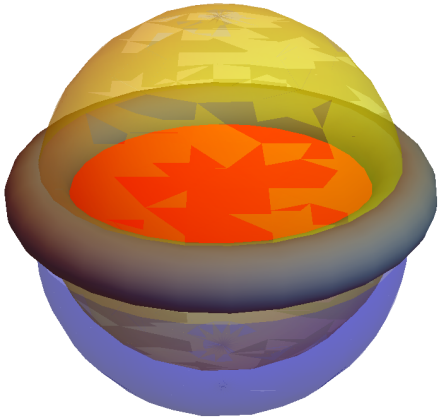
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$




$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

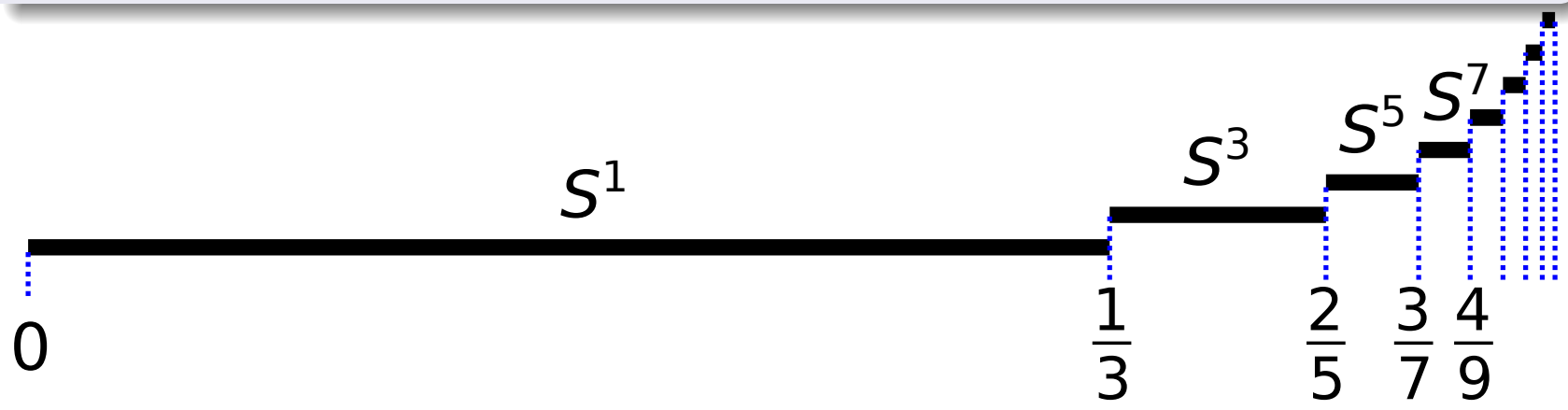
$r = \frac{3}{7}$ 

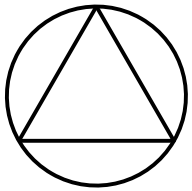


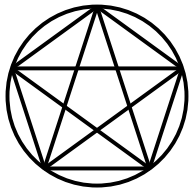
Idea 1: Let $S^1$ be the circle of unit circumference.

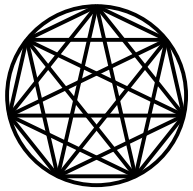
**Theorem (Adamaszek, A)**

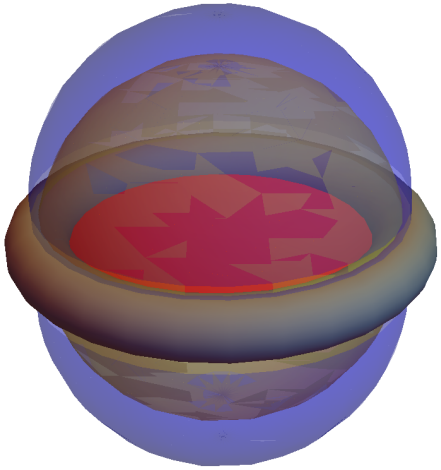
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$




$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

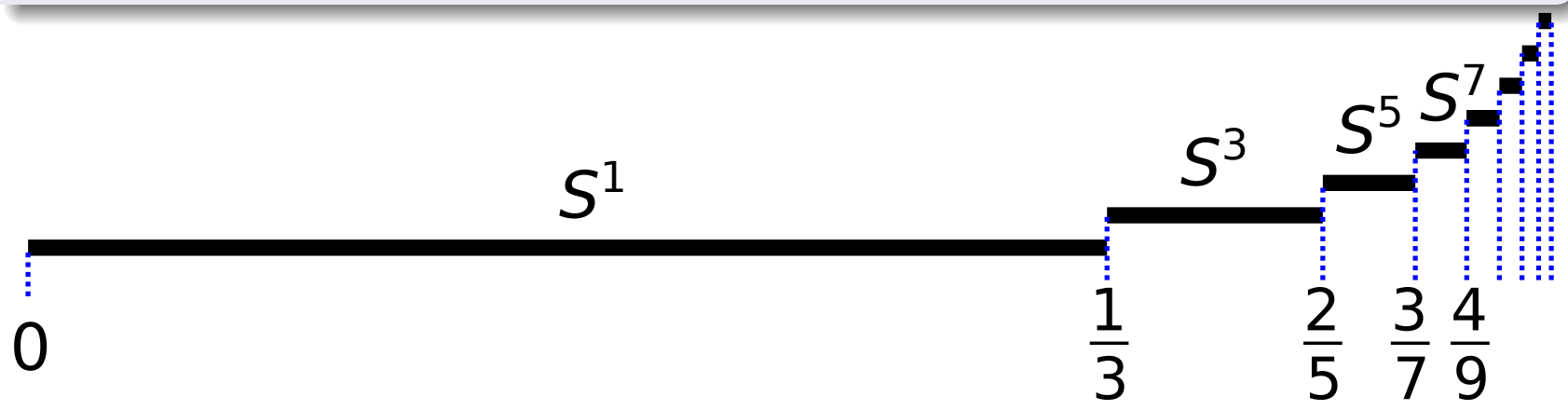
$r = \frac{3}{7}$ 

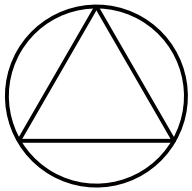


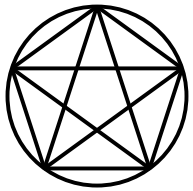
Idea 1: Let $S^1$ be the circle of unit circumference.

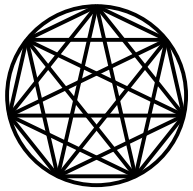
**Theorem (Adamaszek, A)**

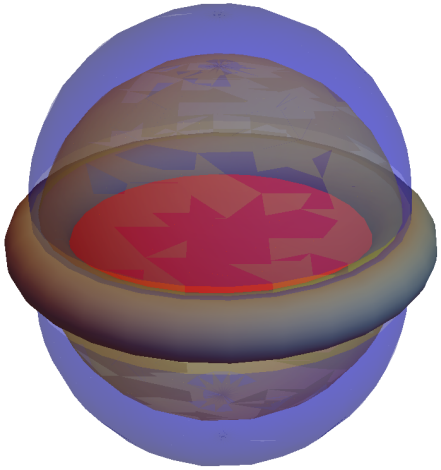
$$\mathrm{VR}(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ \bigvee^{\infty} S^{2k} & \text{if } r = \frac{k}{2k+1} \end{cases} \quad \text{for some } k \in \mathbb{N}.$$




$r = \frac{1}{3}$ 

$r = \frac{2}{5}$ 

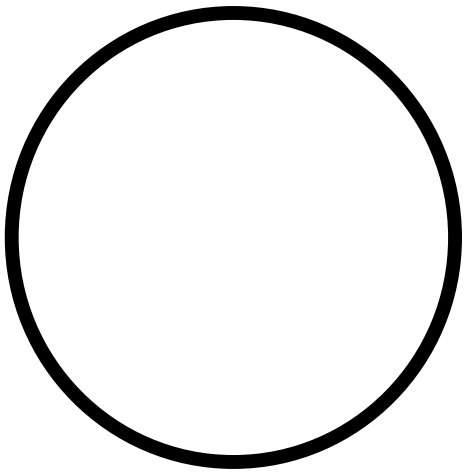
$r = \frac{3}{7}$ 

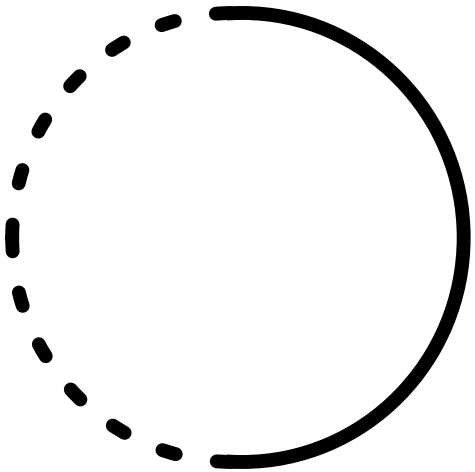


# Idea 2: Metric thickenings

The Vietoris–Rips simplicial complex may not be metrizable!

Metric space $M$ $\rightsquigarrow$ $X \subseteq M$ $\rightsquigarrow$ $\mathrm{VR}(X; r)$





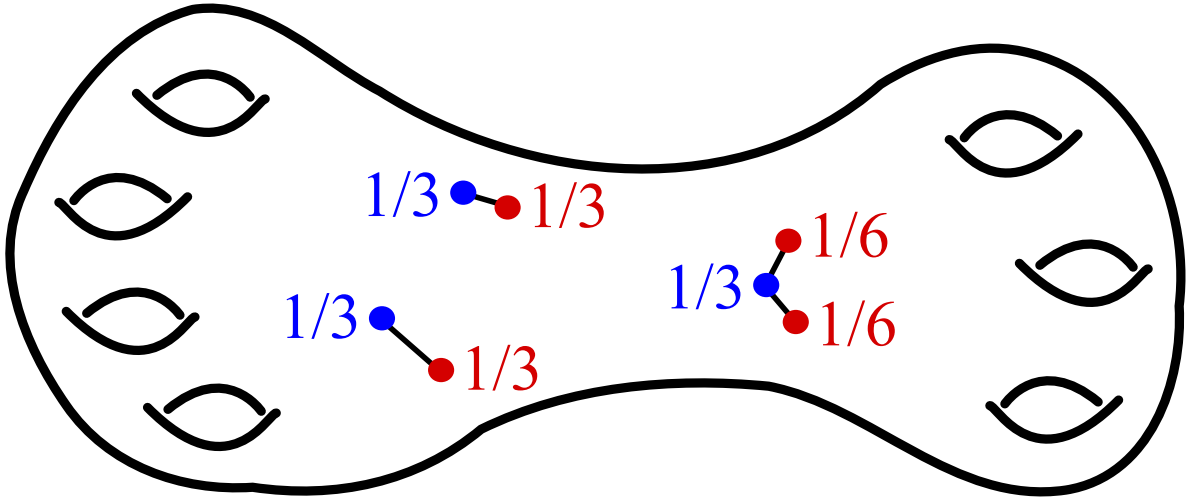
Let $X$ be a metric space and $r > 0$.

## Definition

The *Vietoris–Rips metric thickening* is $\mathrm{VR}^m(X; r)$

$$= \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; \lambda_i \geq 0,\ \sum_i \lambda_i = 1,\ x_i \in X,\ \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\},$$

equipped with the 1-Wasserstein metric.



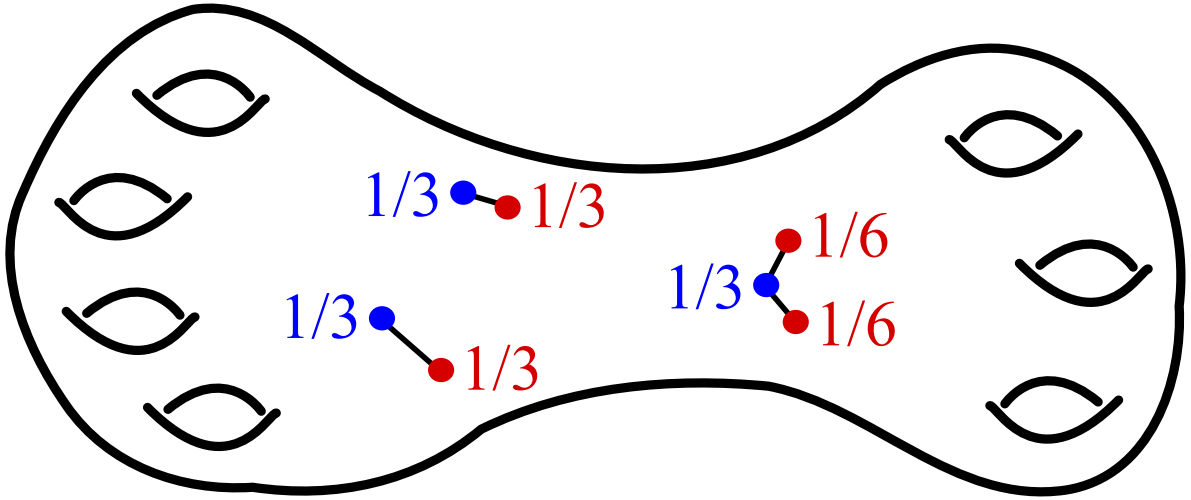
Let $X$ be a metric space and $r > 0$.

## Definition

The *Vietoris–Rips metric thickening* is $\mathrm{VR}^m(X; r)$

$$= \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; \lambda_i \geq 0,\ \sum_i \lambda_i = 1,\ x_i \in X,\ \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \right\},$$

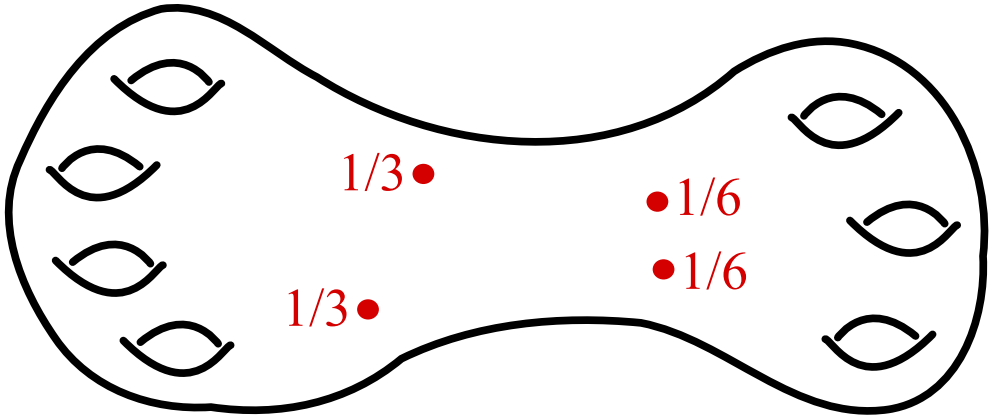
equipped with the 1-Wasserstein metric.



## Theorem (Adamaszek, A, Frick)

*If $M$ is a Riemannian manifold and $r$ is sufficiently small, then*

$$\mathrm{VR}^m(M;r) \simeq M.$$




## Proof.

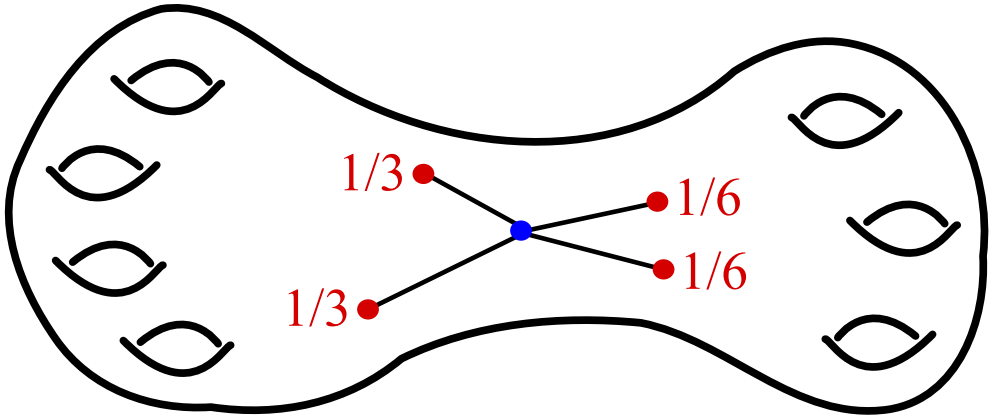
$$\begin{tikzcd} \mathrm{VR}^m(M;r) \arrow[r, bend left, "\sum_i \lambda_i \delta_{x_i} \ \mapsto \ \text{Fréchet mean}"] & M \arrow[l, bend left, "\delta_x \ \leftarrow\!\shortmid \ x"] \end{tikzcd}$$

□

## Theorem (Adamaszek, A, Frick)

*If $M$ is a Riemannian manifold and $r$ is sufficiently small, then*

$$\mathrm{VR}^m(M;r) \simeq M.$$




## Proof.

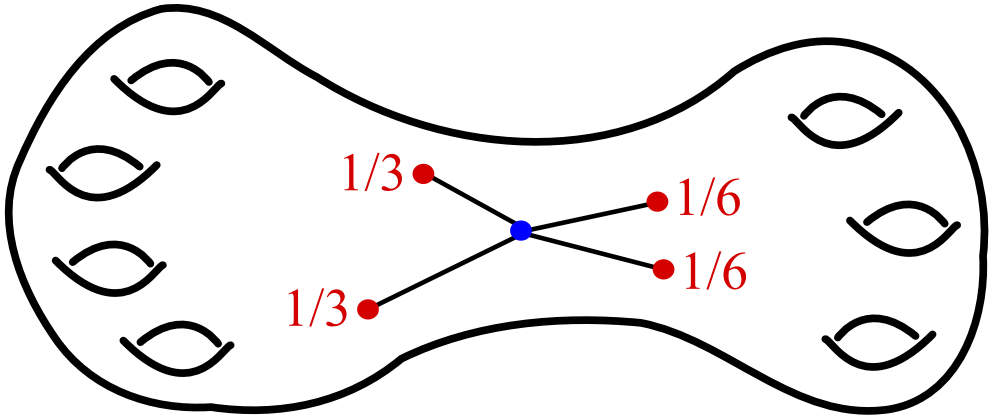
$$\begin{tikzcd} \mathrm{VR}^m(M;r) \arrow[r, bend left, "\sum_i \lambda_i \delta_{x_i} \ \mapsto \ \text{Fréchet mean}"] & M \arrow[l, bend left, "\delta_x \ \leftarrow\!\shortmid \ x"] \end{tikzcd}$$

□

## Theorem (Adamaszek, A, Frick)

*If $M$ is a Riemannian manifold and $r$ is sufficiently small, then*

$$\mathrm{VR}^m(M;r) \simeq M.$$




## Proof.

$$\begin{tikzcd} \mathrm{VR}^m(M;r) \arrow[r, bend left, "\sum_i \lambda_i \delta_{x_i} \ \mapsto \ \text{Fréchet mean}"] & M \arrow[l, bend left, "\delta_x \ \leftarrow\!\shortmid \ x"] \end{tikzcd}$$

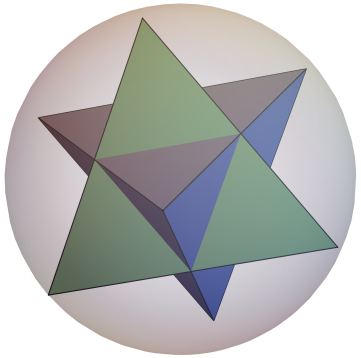
□

With Joshua Mirth: version for Euclidean sets of positive reach.

Idea 3: We can now say something about the $n$-sphere $S^n$.

**Theorem (Adamaszek, A, Frick)**

$$\mathrm{VR}^m(S^n; r) \simeq \begin{cases} S^n & r < r_c \\ S^n * \frac{\mathrm{SO}(n+1)}{A_{n+2}} & r = r_c. \end{cases}$$

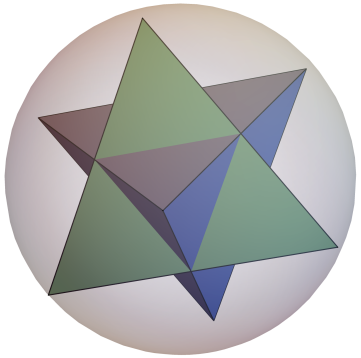


$r_c$ = diameter of inscribed regular $\Delta^{n+1}$.
$\mathrm{SO}(n+1)$ = group of rotations in $\mathbb{R}^{n+1}$.
$A_{n+2}$ = group of rotations of $\Delta^{n+1}$.

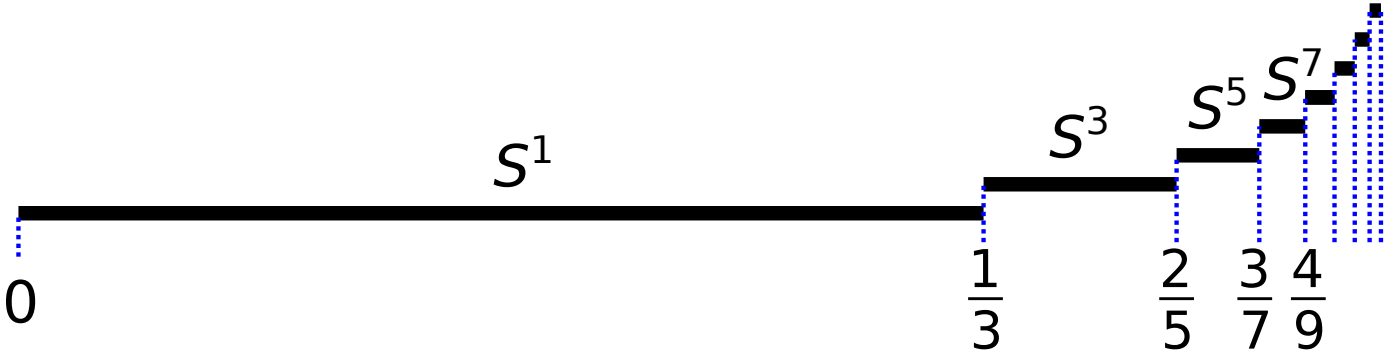
Idea 3: We can now say something about the $n$-sphere $S^n$.

**Theorem (Adamaszek, A, Frick)**

$$\mathrm{VR}^m(S^n; r) \simeq \begin{cases} S^n & r < r_c \\ S^n * \frac{\mathrm{SO}(n+1)}{A_{n+2}} & r = r_c. \end{cases}$$



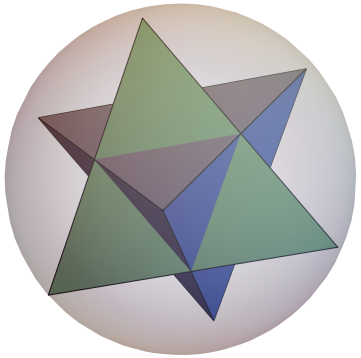
$r_c$ = diameter of inscribed regular $\Delta^{n+1}$.
$\mathrm{SO}(n+1)$ = group of rotations in $\mathbb{R}^{n+1}$.
$A_{n+2}$ = group of rotations of $\Delta^{n+1}$.



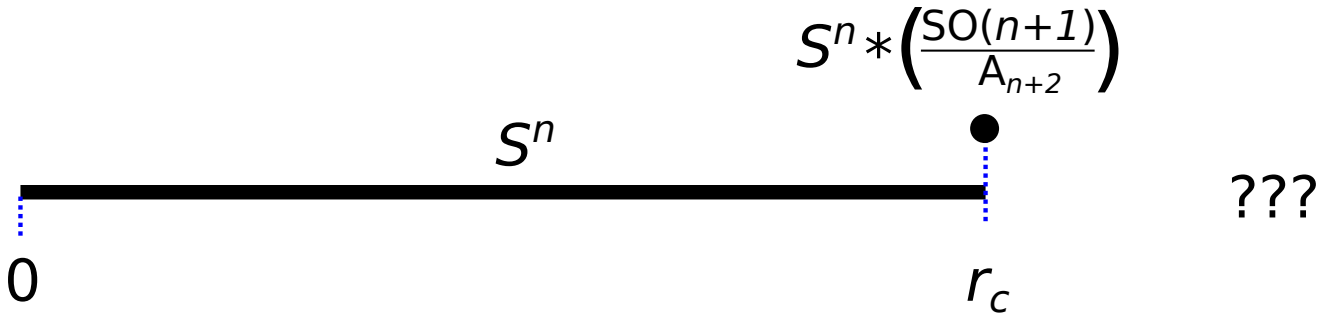
Idea 3: We can now say something about the $n$-sphere $S^n$.

**Theorem (Adamaszek, A, Frick)**

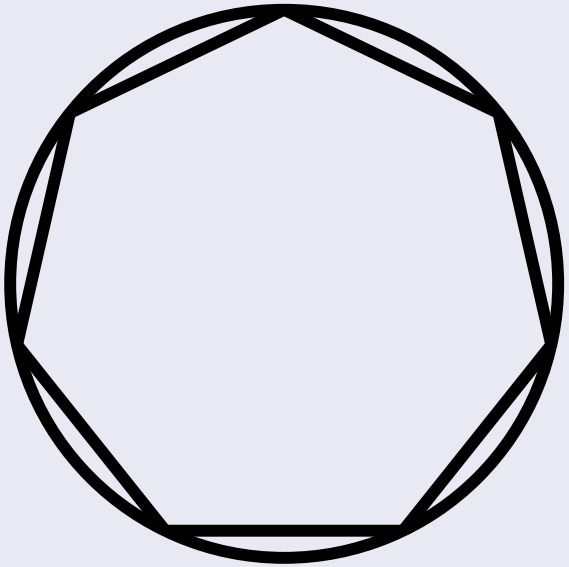
$$\mathrm{VR}^m(S^n; r) \simeq \begin{cases} S^n & r < r_c \\ S^n * \frac{\mathrm{SO}(n+1)}{A_{n+2}} & r = r_c. \end{cases}$$

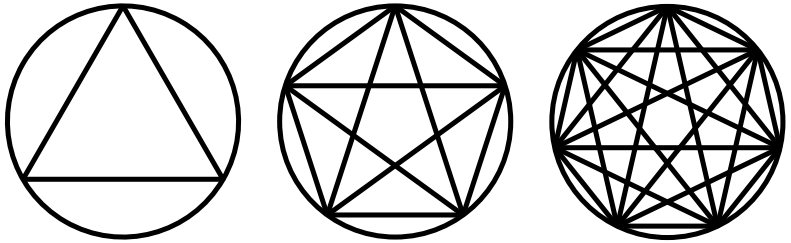


$r_c$ = diameter of inscribed regular $\Delta^{n+1}$.
$\mathrm{SO}(n+1)$ = group of rotations in $\mathbb{R}^{n+1}$.
$A_{n+2}$ = group of rotations of $\Delta^{n+1}$.



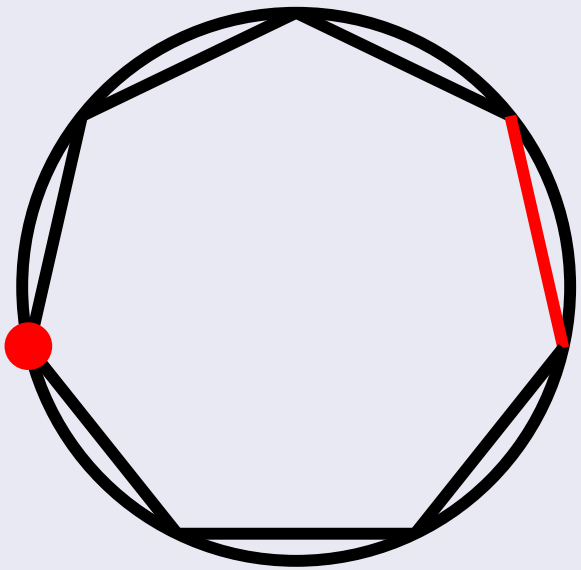
## Lovász' strongly self-dual polytopes

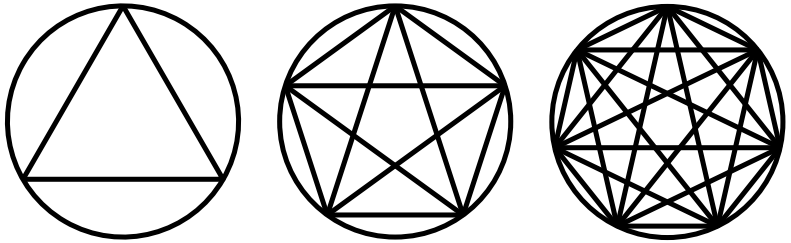




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

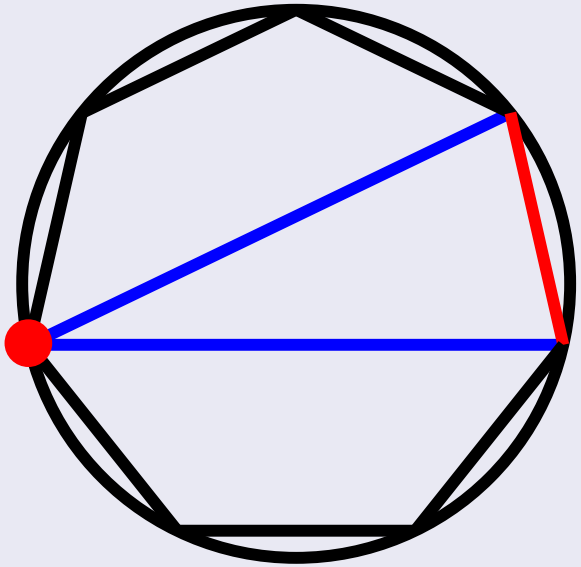
## Lovász' strongly self-dual polytopes

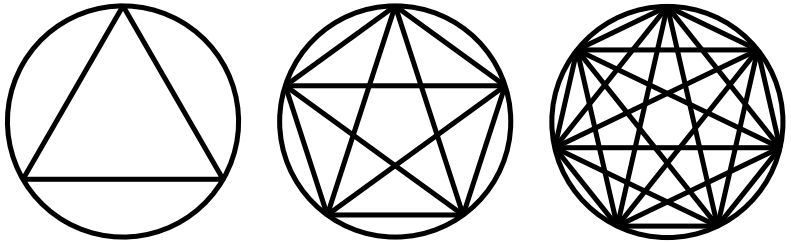




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

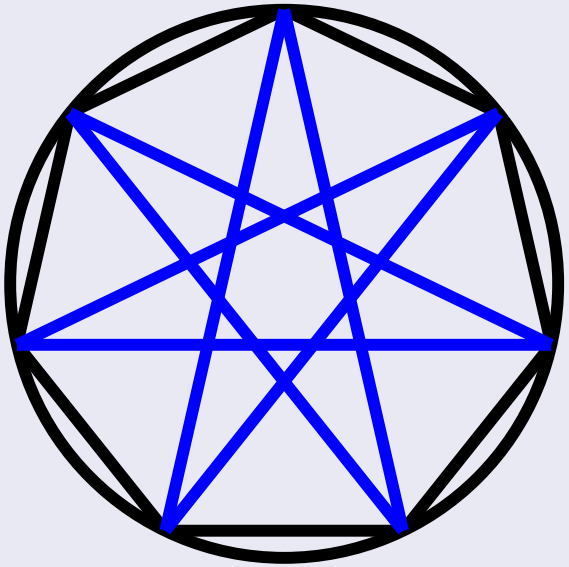
## Lovász' strongly self-dual polytopes

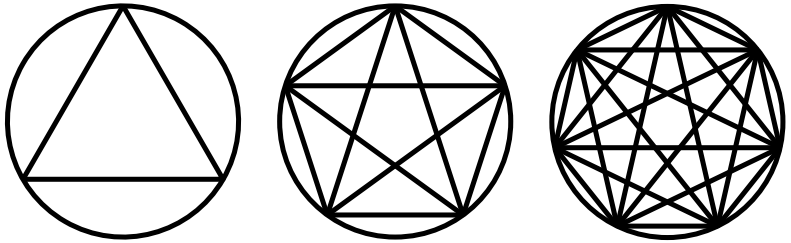




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

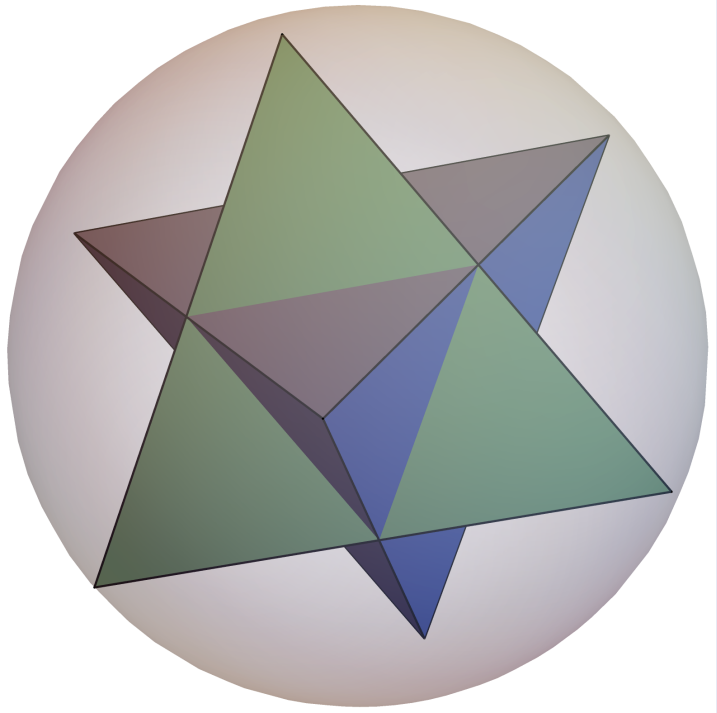
## Lovász' strongly self-dual polytopes

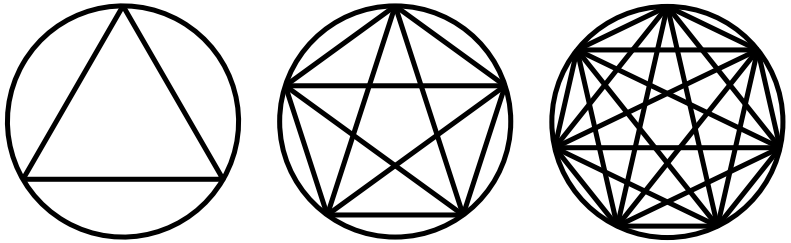




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

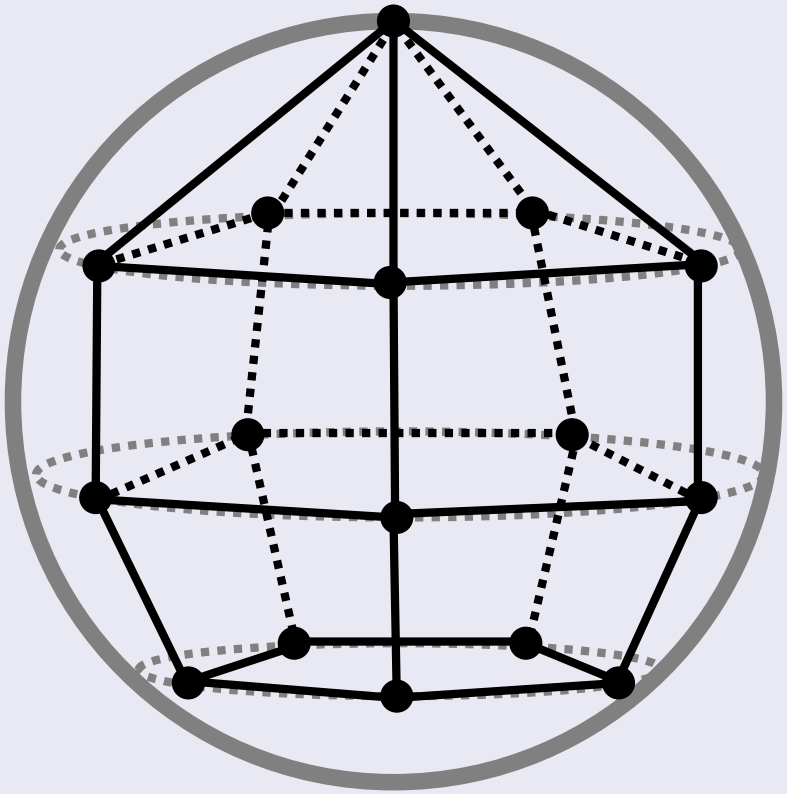
## Lovász' strongly self-dual polytopes

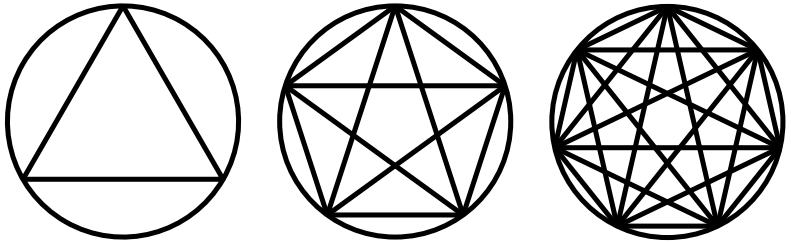




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

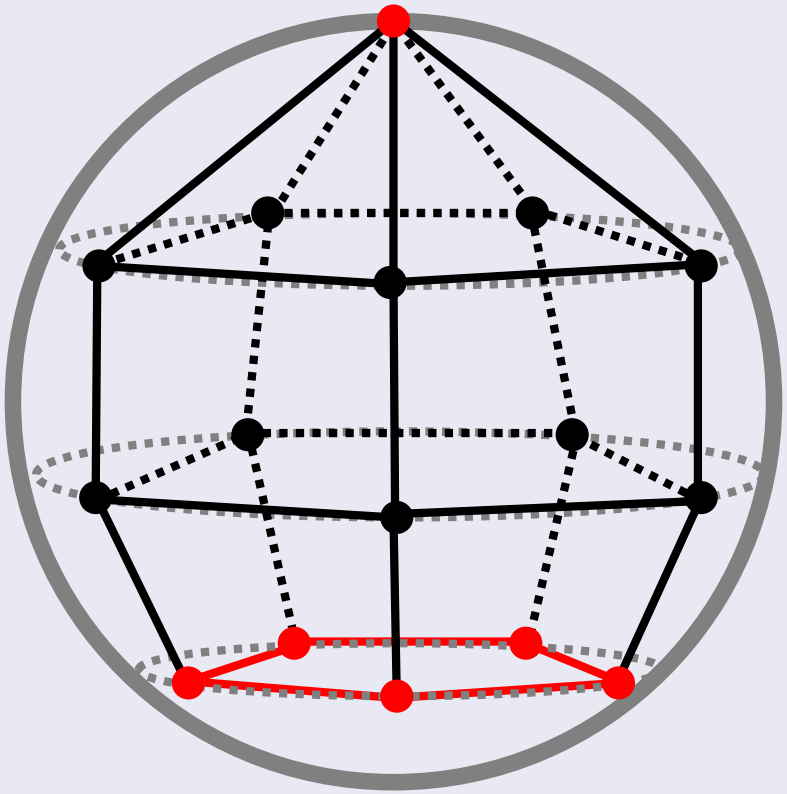
## Lovász' strongly self-dual polytopes

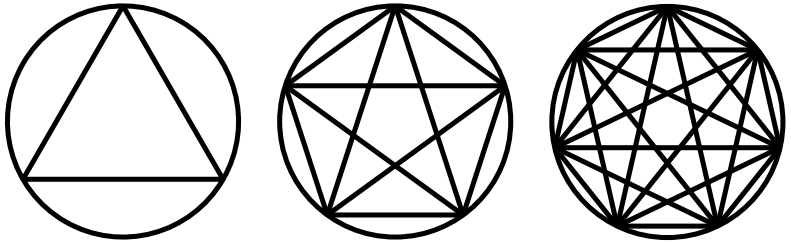




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

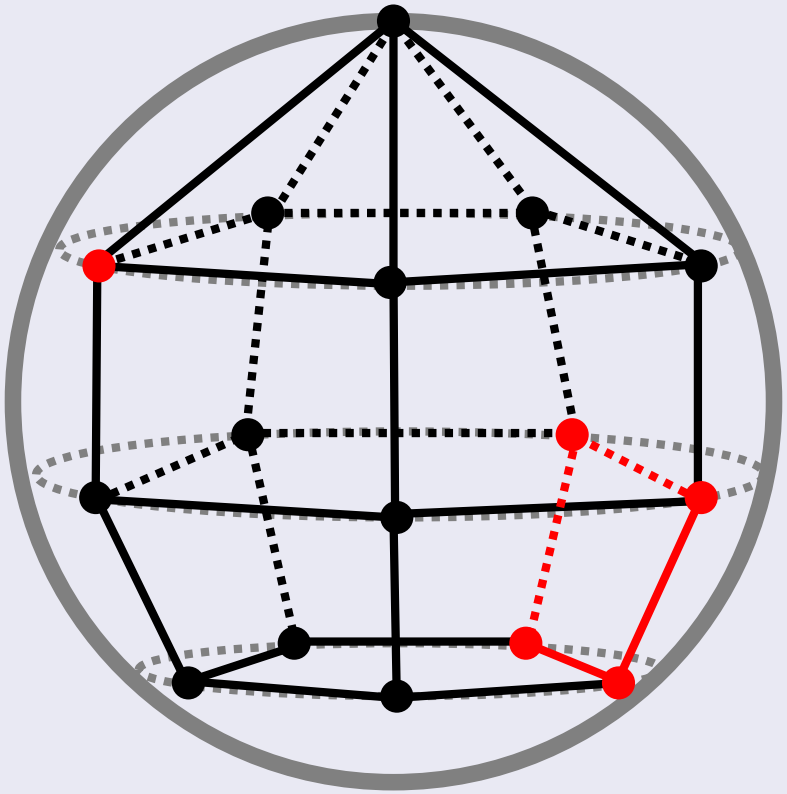
## Lovász' strongly self-dual polytopes

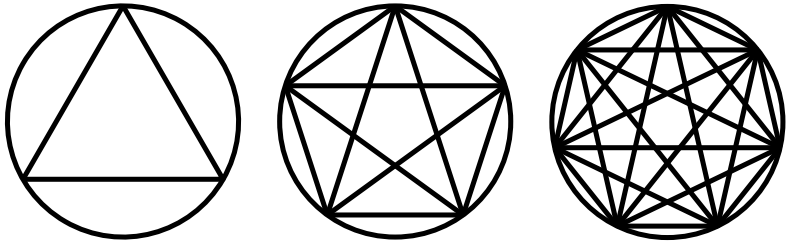




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

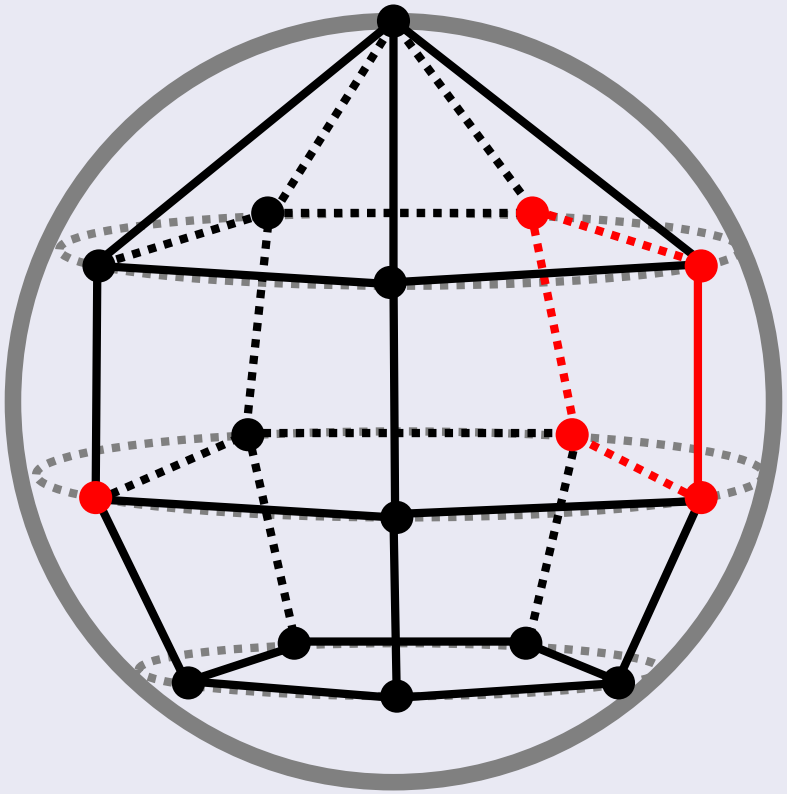
## Lovász' strongly self-dual polytopes

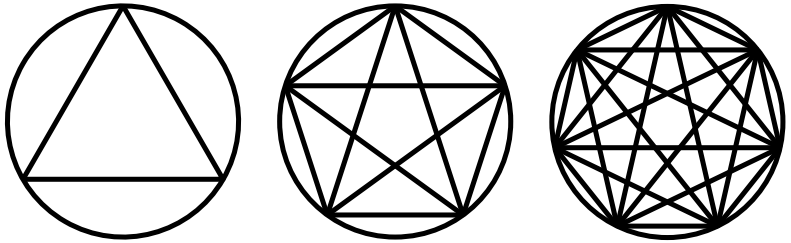




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

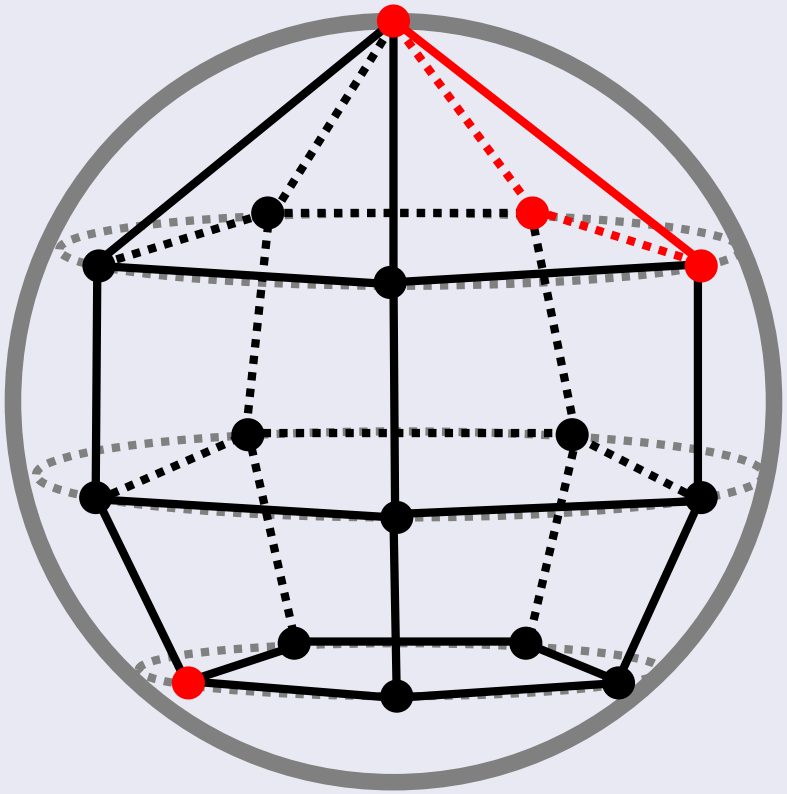
## Lovász' strongly self-dual polytopes

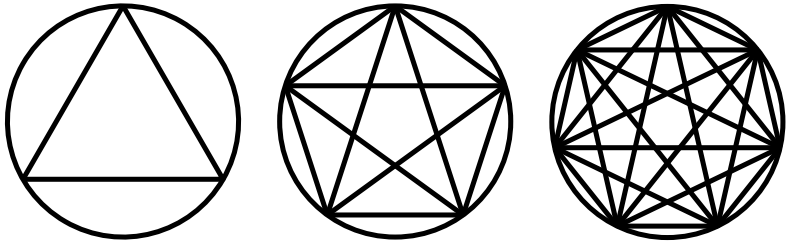




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

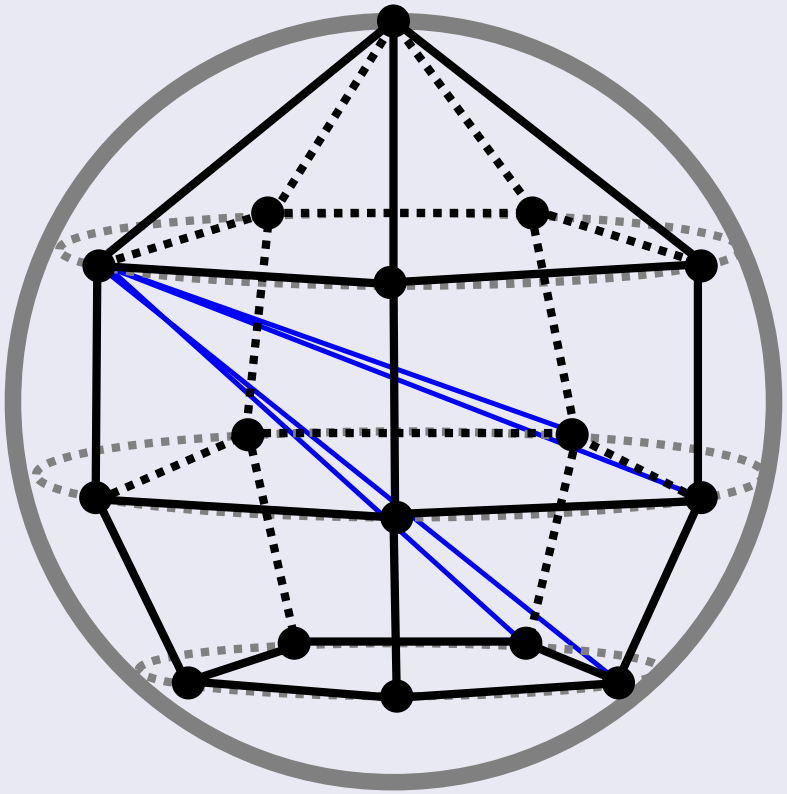
## Lovász' strongly self-dual polytopes

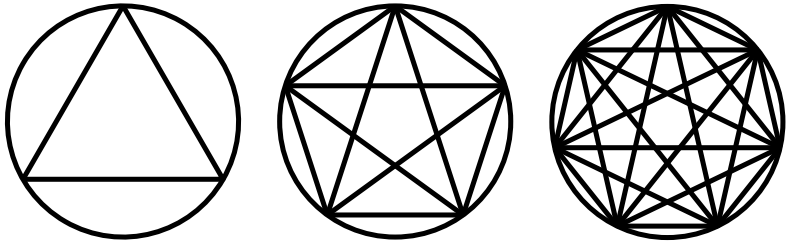




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

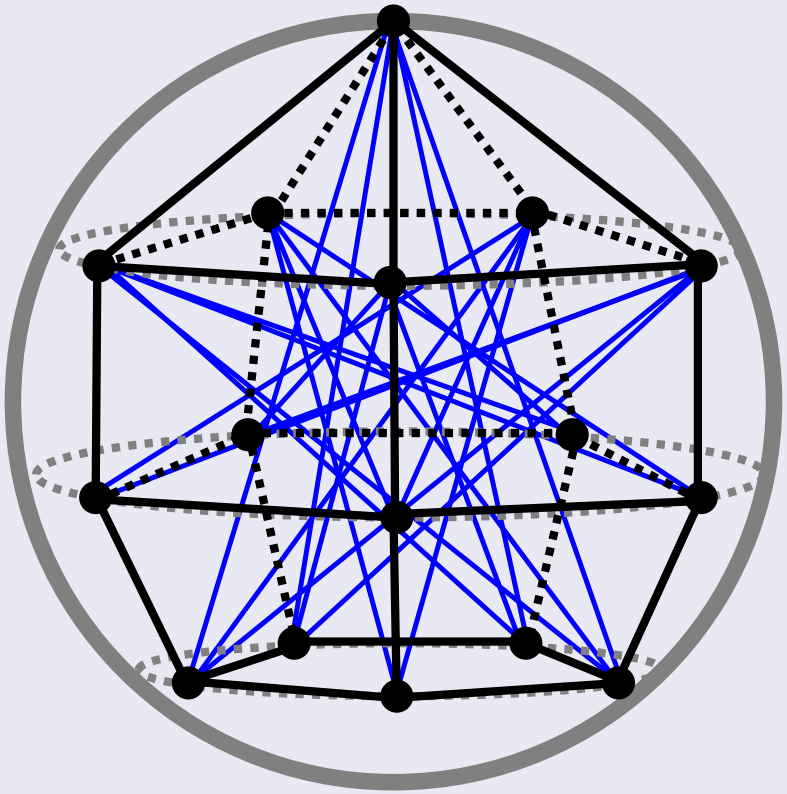
## Lovász' strongly self-dual polytopes

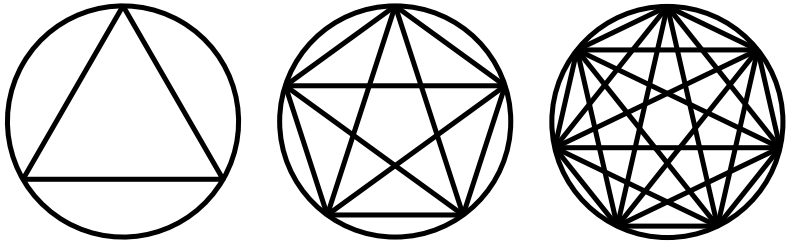




László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

## Lovász' strongly self-dual polytopes





László Lovász, *Self-dual polytopes and the chromatic number of distance graphs on the sphere*, Acta Sci. Math. 45, 317–323, 1983.

**Theorem (Borsuk–Ulam)**

*For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

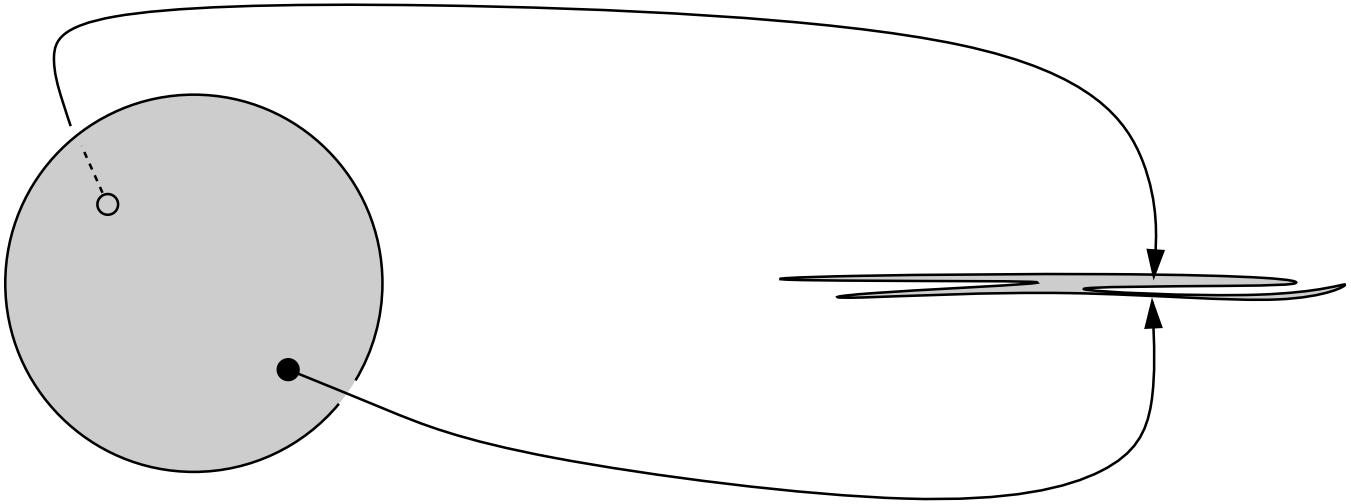


Figure credit: Jiří Matoušek, *Using the Borsuk–Ulam theorem*

**Theorem (Borsuk–Ulam)**

*For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

Gromov's "waist of sphere" theorem is for $f\colon S^n \to \mathbb{R}^k$ with $k \leq n$.

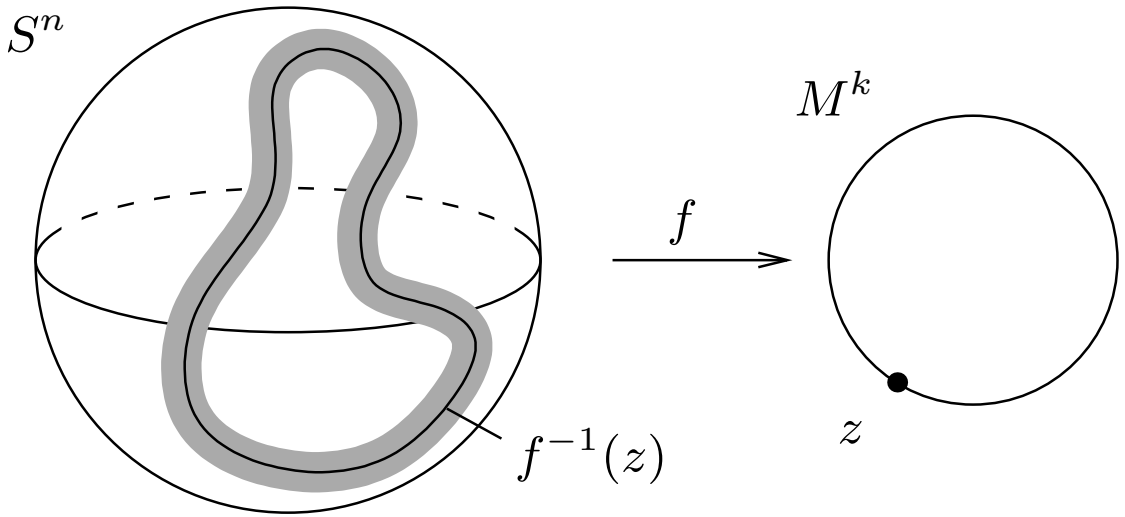


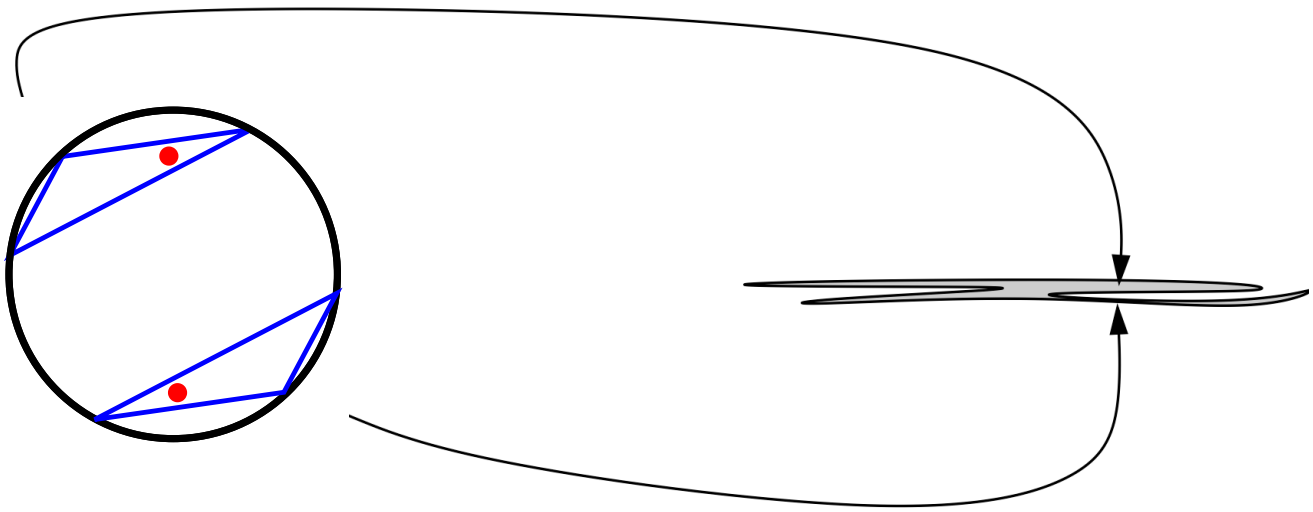

Figure credit: Benjamin Matschke, *Journal of Topology & Analysis*

**Theorem (Borsuk–Ulam)**

*For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

Gromov's "waist of sphere" theorem is for $f\colon S^n \to \mathbb{R}^k$ with $k \leq n$.

What about $f\colon S^n \to \mathbb{R}^k$ with $k \geq n$?

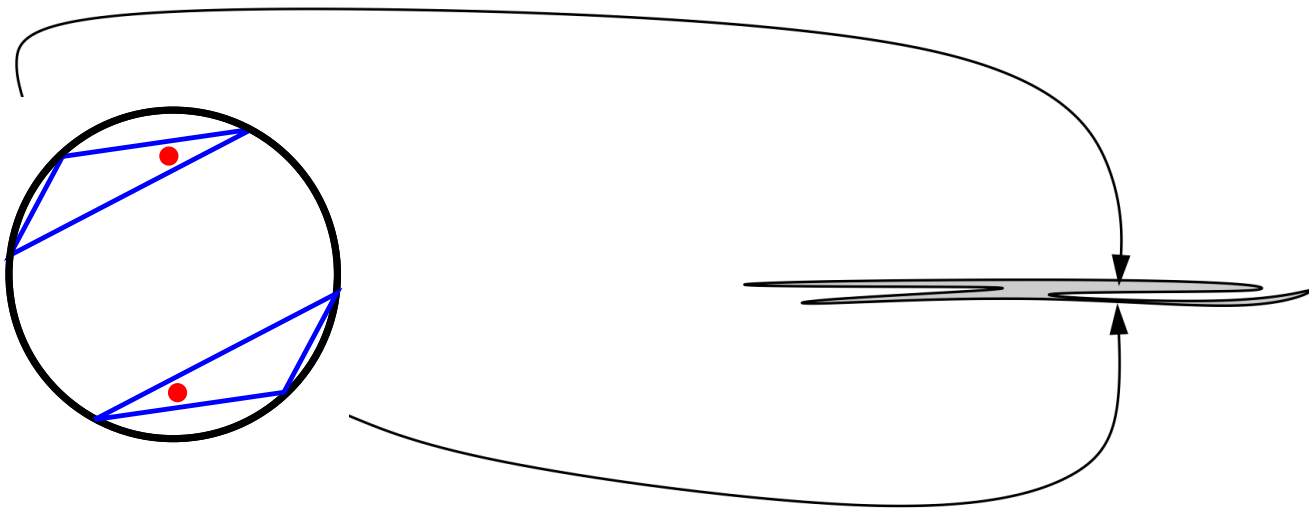


**Theorem (Borsuk–Ulam)**

*For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

**Theorem (A, Bush, Frick)**

*For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.*

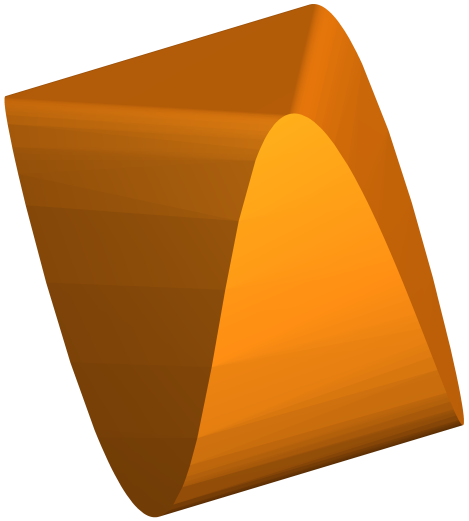


**Theorem (Borsuk–Ulam)**

*For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

**Theorem (A, Bush, Frick)**

*For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.*



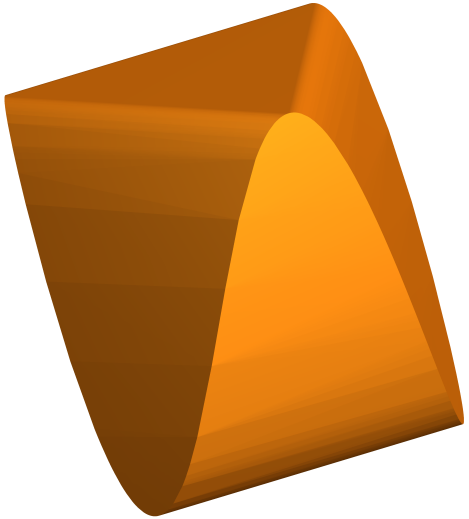
**Theorem (Borsuk–Ulam)**

*For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

**Theorem (A, Bush, Frick)**

*For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.*

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

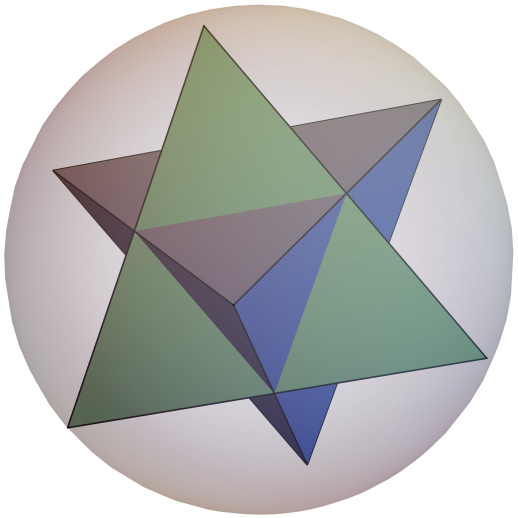
**Theorem (Borsuk–Ulam)**

*For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

**Theorem (A, Bush, Frick)**

*For $f\colon S^n \to \mathbb{R}^{n+2}$, there exists a set $\{x_0, \ldots, x_{n+2}\}$ of diameter at most $r_c$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.*

Proof: $S^{n+2}$ "$\subseteq$" $\mathrm{VR}^m(S^n; r_c) \xrightarrow{f} \mathbb{R}^{n+2}$

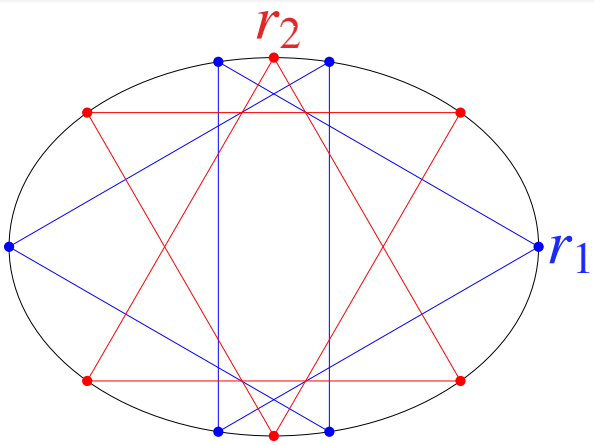


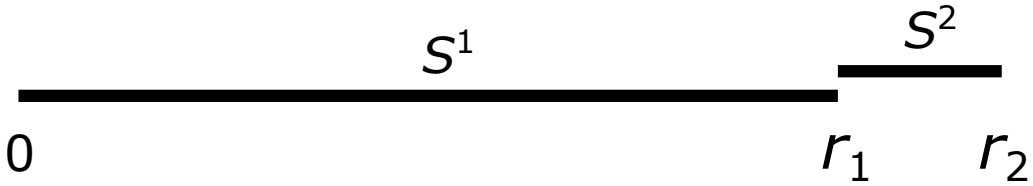
## Open questions & future work

1. Čech and Nerve complexes (Borsuk)
2. Other manifolds $M$?
3. $\mathrm{VR}_<(X;r) \simeq \mathrm{VR}^m_<(X;r)$?
4. Morse, Morse–Bott, and Bestvina–Brady Morse theories.
5. Michael Crabb's Borsuk–Ulam generalizations using characteristic classes.

## Open questions & future work

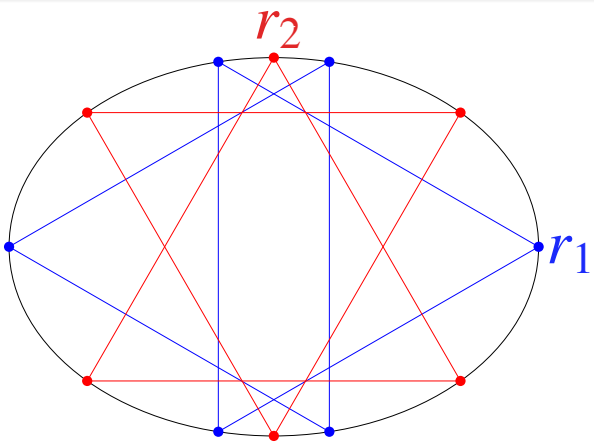
1. Čech and Nerve complexes (Borsuk)
2. Other manifolds $M$?
3. $\mathrm{VR}_<(X;r) \simeq \mathrm{VR}^m_<(X;r)$?
4. Morse, Morse–Bott, and Bestvina–Brady Morse theories.
5. Michael Crabb's Borsuk–Ulam generalizations using characteristic classes.

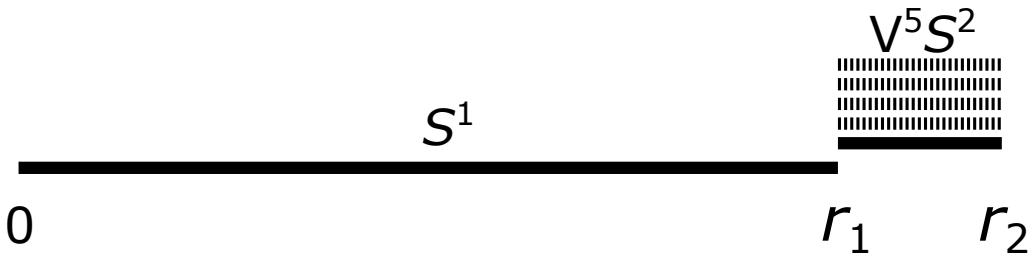


## Open questions & future work

1. Čech and Nerve complexes (Borsuk)
2. Other manifolds $M$?
3. $\mathrm{VR}_<(X;r) \simeq \mathrm{VR}^m_<(X;r)$?
4. Morse, Morse–Bott, and Bestvina–Brady Morse theories.
5. Michael Crabb's Borsuk–Ulam generalizations using characteristic classes.



## References


- Michał Adamaszek, Henry Adams, Florian Frick, *Metric reconstruction via optimal transport*, SIAM Journal on Applied Algebra and Geometry 2 (2018).
- Michał Adamaszek and Henry Adams, *The Vietoris–Rips complexes of a circle*, Pacific Journal of Mathematics 290 (2017), 1–40.
- Henry Adams, Johnathan Bush, and Florian Frick, *Metric thickenings, Borsuk–Ulam theorems, and orbitopes*, Mathematika 66 (2020), 79–102.
- Jean-Claude Hausmann, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, Annals of Mathematics Studies 138 (1995), 175–188.


Thank you!