The Vietoris-Rips Complex of the Circle
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Notation

Let \( S^1 \) be the circle of unit circumference with the geodesic metric.

Definition. For \( X \subset S^1 \) and \( r \geq 0 \), let \( \text{VR}(X, r) \) be the Vietoris-Rips complex with vertex set \( X \) and connectivity parameter \( r \). That is, simplex \( \sigma \subset X \) is in \( \text{VR}(X, r) \) when \( \text{diam}(\sigma) \leq r \).

Persistent Homology of \( \text{VR}(S^1, r) \)

Theorem 1. The odd-dimensional persistent homology of \( \text{VR}(S^1, r) \) is

\[
\text{dgm}(H_{2l+1}(\text{VR}(S^1))) = \left\{ \left\lfloor \frac{l}{2l+1} \right\rfloor \right\},
\]

and within this interval \( \text{VR}(S^1, r) \simeq S^{2l+1} \). The even-dimensional persistent homology has no intervals of positive length.

Homotopy Types of \( \text{VR}(X, r) \) for \( X \subset S^1 \)

Theorem 2. For \( X \subset S^1 \) finite, \( \text{VR}(X, r) \) is homotopy equivalent to either a point, an odd sphere, or a wedge sum of spheres of the same even dimension.

For example, let \( X_n \subset S^1 \) consist of \( n \) evenly-spaced points. This case is given in [1].

Example. We have \( \text{VR}(X_9, 1/3) \simeq \vee S^2 \).

Corollary 6.7 from [1]. Let \( k < n/2 \). Then

\[
\text{VR}(X_n, k/n) \simeq \begin{cases} 
\vee_{n-2k-1} S^{2l} & \text{if } \frac{k}{n} = \frac{l}{2l+1} < \frac{l+1}{2l+3} \\
S^{2l+1} & \text{for some } l \geq 0.
\end{cases}
\]

References