Vietoris-Rips complexes: Open questions

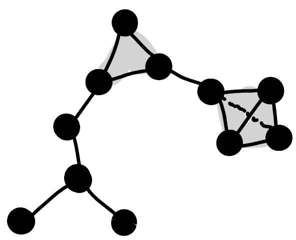
Henry Adams, Colorado State University

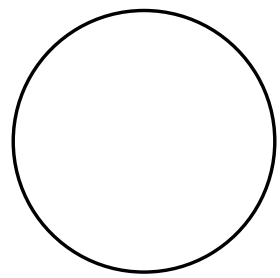
Joint with Michał Adamaszek, Florian Frick, Baris Coskunuzer, Johnathan Bush, Joshua Mirth, Michael Moy, Facundo Mémoli, Qingsong Wang

X metric space, $r \geq 0$.

Def The *Vietoris-Rips simplicial complex* has

- vertex set X
- finite simplex $\sigma \subseteq X$ when $\mathrm{diameter}(\sigma) \leq r$.

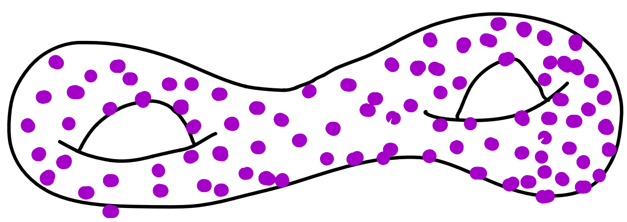




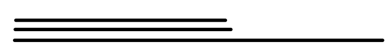
History

- Cohomology theory for metric spaces
- Geometric group theory
- Applied topology

Stability



$PH_1(VR(M;r))$



$PH_1(VR(X;r))$

Chazal, de Silva, Oudot, 2014

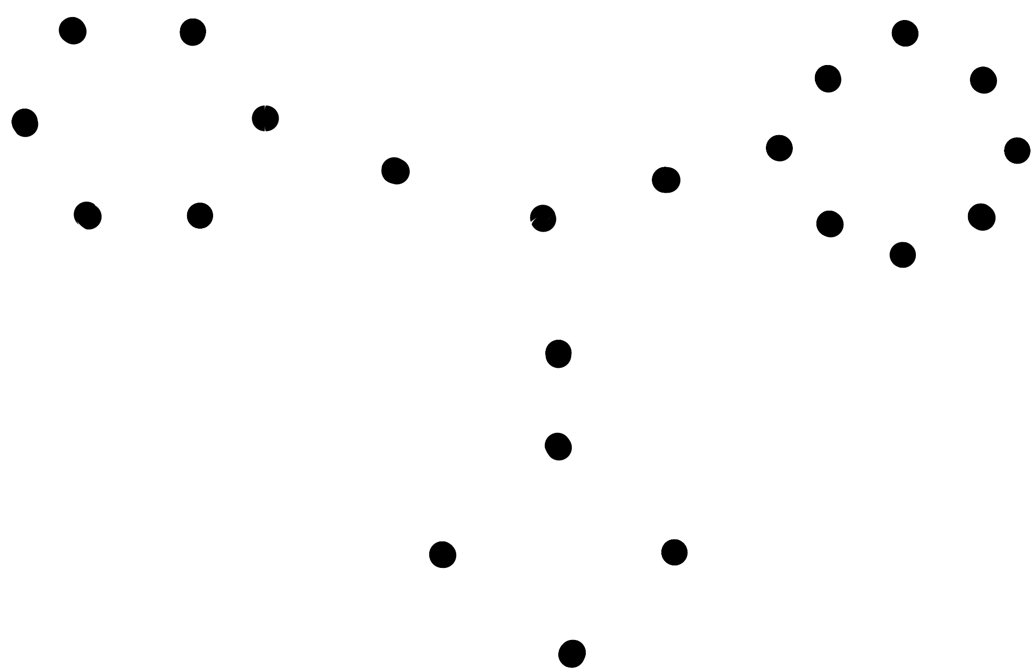
Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, 2009

Open question #1

For $X \subseteq \mathbb{R}^2$, is $VR(X;r)$ always a wedge of spheres?

- Known: Any wedge of spheres is possible.
- Known: $\pi_1(VR(X;r))$ is free.

(Chambers, de Silva, Erickson, Ghrist 2010)
(Adamaszek, Frick, Vakili 2017)



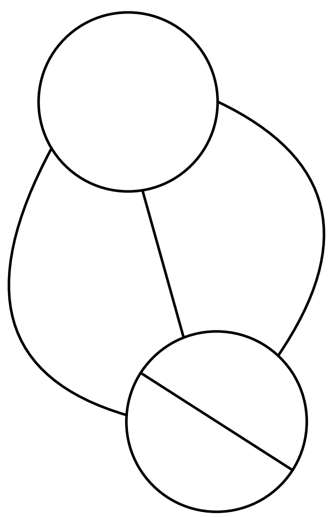
Open question #2

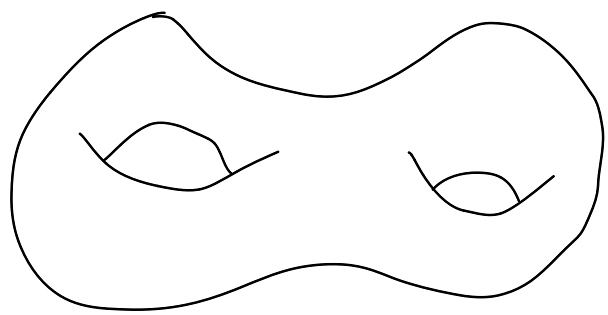
For X an arbitrary metric space, how do we bound the length of a k-dimensional persistent homology class?

- Known: k=1, X geodesic

  Gasparovic, Gommel, Purvine, Sazdanovic, Wang, Wang, Ziegelmeier 2018

  Virk 2020





Open question #2

For X an arbitrary metric space, how do we bound the length of a k-dimensional persistent homology class?

- Unknown: $k \geq 2$?

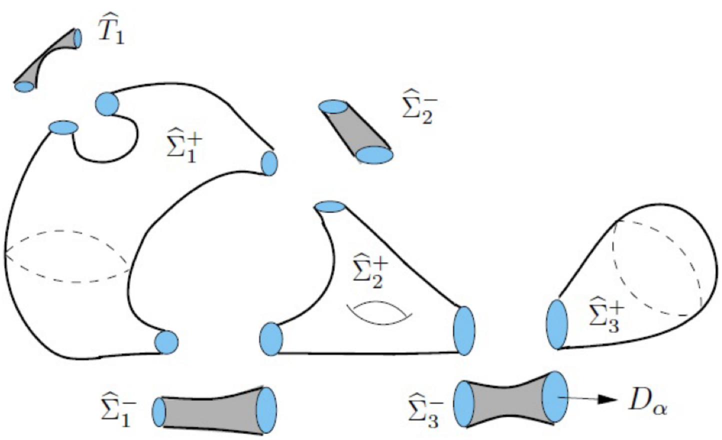
Lim, Mémoli, Okutan, 2020, "Vietoris-Rips persistent homology, injective metric spaces, and the filling radius"

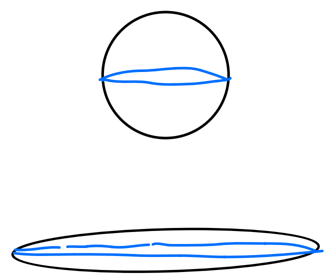
Open question #2

For X an arbitrary metric space, how do we bound the length of a k-dimensional persistent homology class?

- Unknown: $k \geq 2$?

A, Coskunuzer, 2021, "Geometric approaches on persistent homology





X vertex set of unweighted graph.

Thm A 2-dimensional homology class σ in $VR(X;r)$ has persistence $\leq \sqrt{\text{area}(\sigma)} + 1$.

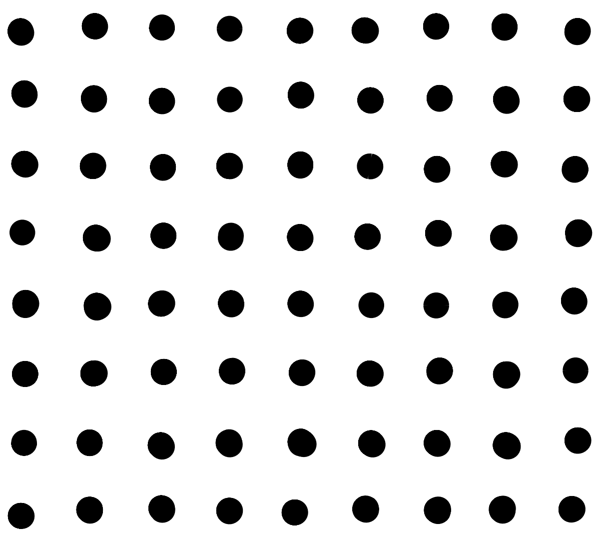
Thm A k-dimensional homology class σ in $VR(X;r)$ has persistence $\leq \text{width}(\sigma) + 1$.

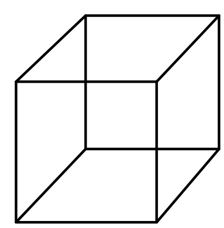
Open question #3

Is $VR((\mathbb{Z}^n; L^1); r)$ contractible for $r \geq n$?

(Case $n=2,3$ by Mallery, Zaremsky)

More generally: Cayley graphs of groups, finiteness properties





Open question #4

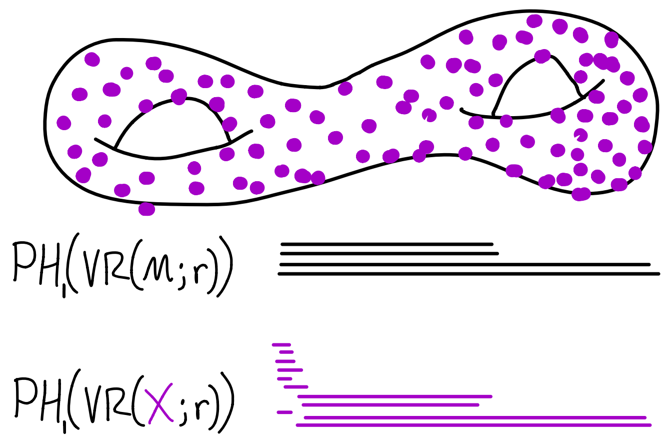
Homotopy types of $VR((\{0,1\}^n; L^1); r)$ for $r < n$?

Carlsson, Fillippenko 2020

Adamaszek, Adams 2021

Open question #5

What are the homotopy types of Vietoris-Rips complexes of manifolds, specifically spheres, ellipsoids, tori, $\mathbb{R}P^n$, $\mathbb{C}P^n$?

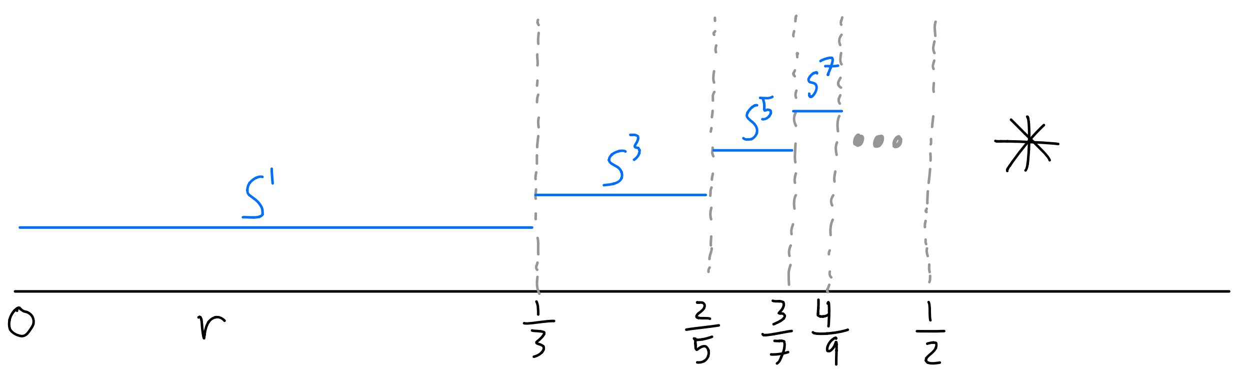


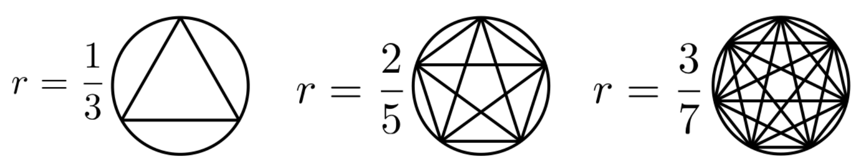
Open question #5

What are the homotopy types of Vietoris-Rips complexes of manifolds, specifically spheres, ellipsoids, tori, $\mathbb{R}P^n$, $\mathbb{C}P^n$?

A, Adamaszek, "The Vietoris-Rips complexes of a circle", 2017

Thm $VR(S^1;r) \simeq S^{2k+1}$ if $\frac{k}{2k+1} < r < \frac{k+1}{2k+3}$ $k \in \mathbb{N}$

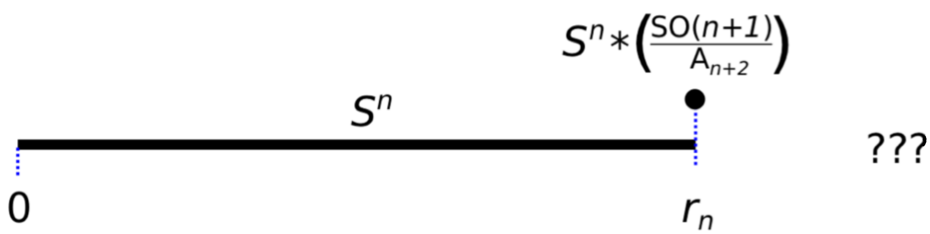


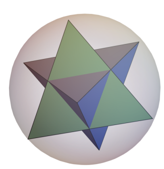


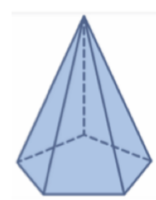
Open question #5

What are the homotopy types of Vietoris-Rips complexes of manifolds, specifically spheres S^n, ellipsoids, tori, $\mathbb{RP}^n$, $\mathbb{CP}^n$?

Thm $VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$







Adamaszek, A, Frick, 2018, "Metric reconstruction via optimal transport"

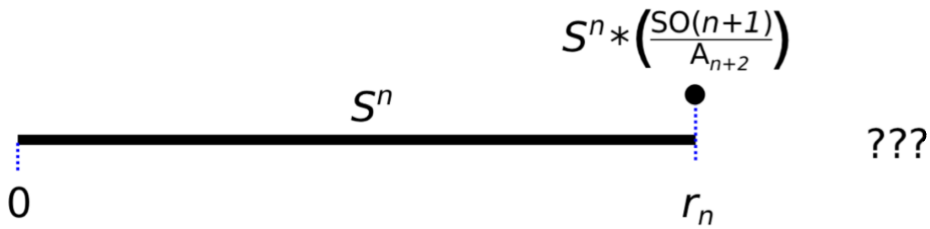
Lim, Mémoli, Okutan, 2020, "Vietoris-Rips persistent homology, injective metric spaces, and the filling radius"

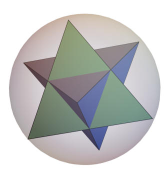
Katz, 1991, "On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional"

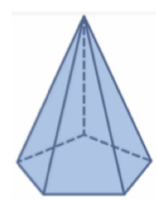
Open question #5

What are the homotopy types of Vietoris-Rips complexes of manifolds, specifically spheres S^n, ellipsoids, tori, $\mathbb{R}P^n$, $\mathbb{C}P^n$?

Thm $$VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$$







For $n \le k$, do we have:

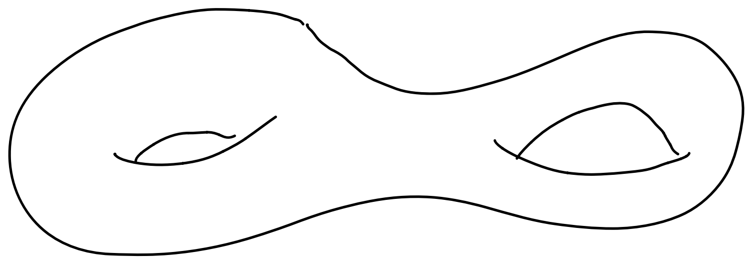
$$2 \cdot d_{GH}(S^n, S^k) = \text{smallest } r \text{ such that } VR(S^n; r) \text{ is } (k-1)\text{-connected ?}$$

Lim, Mémoli, Smith, 2021

Bridge #1: Filling Radius

Gromov, 1983, "Filling Riemannian manifolds"

Isosystolic inequality For M an essential n-dimensional Riemannian manifold, $\text{systole}(M) \leq C \operatorname{vol}(M)^{1/n}$.

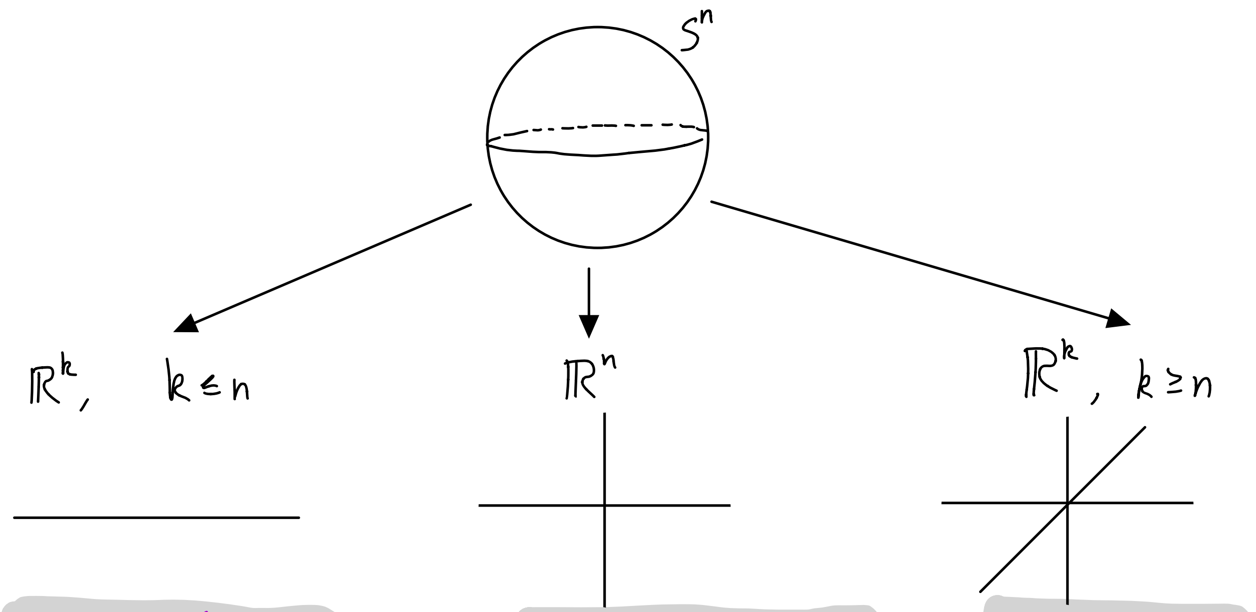


The systole of M is the length of the shortest non-contractible loop.

"Essential" rules out counterexamples like $S^1 \times S^2$.

Proof $\text{sys}(M) \underset{\substack{\uparrow \\ M \text{ essential}}}{\leq} 6 \cdot \text{FillingRadius}(M) \underset{\substack{\uparrow \\ \text{all } M}}{\leq} C \operatorname{vol}(M)^{1/n}$

Bridge #2: Borsuk-Ulam theorems



Almgren (1965):
Geometric measure theory
Gromov (2003):
Filling radius, characteristic classes

Borsuk-Ulam (1933)

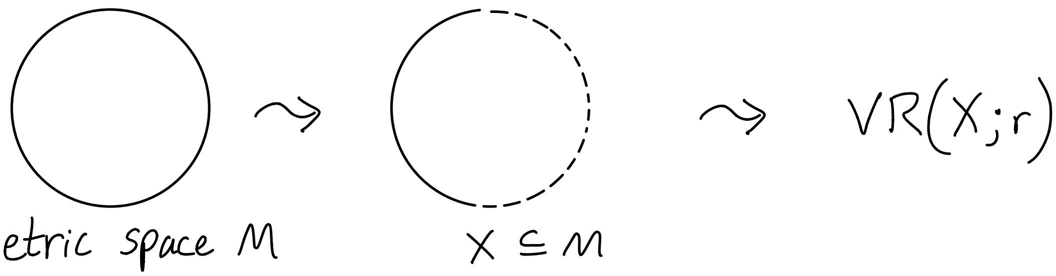
A, Bush, Frick (2020)

<u>"Waist of sphere" theorem</u> For $f: S^n \to \mathbb{R}^k$ with $k \leq n$, $\exists\, y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \geq \mathrm{Vol}_{n-k}(S^{n-k})$.

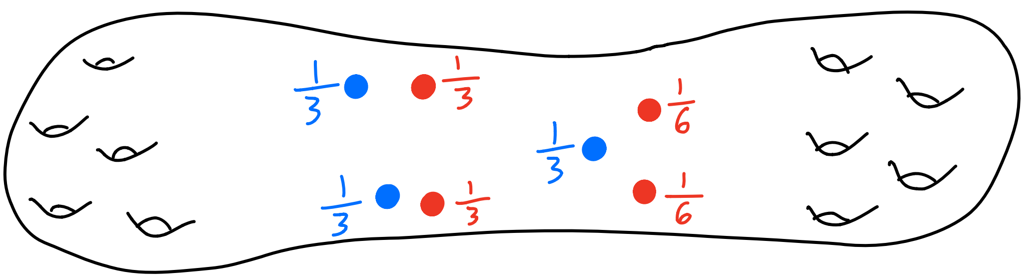
Invariance of dimension.

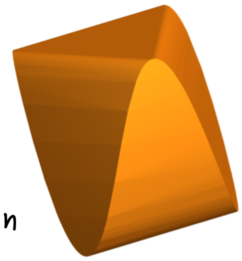
Bridge #3: Optimal transport

A simplicial complex whose vertex set is a metric space should often be equipped with an optimal transport metric (instead of the simplicial complex topology).



Adamaszek, A, Frick, 2018, "Metric reconstruction via optimal transport"





Questions

(1) $VR(S^n;r)$ for larger r?

(2) $\check{C}ech^m(S^n;r)$?

(3) Other manifolds ? Tori, ellipsoids, $\mathbb{RP}^n$, $\mathbb{CP}^n$

(4) Morse and Morse-Bott theories