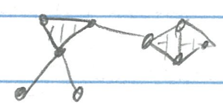
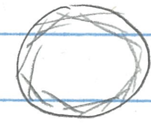
5/1/18

Using homotopy colimits to understand infinite simplicial complexes

X metric space, $r > 0$

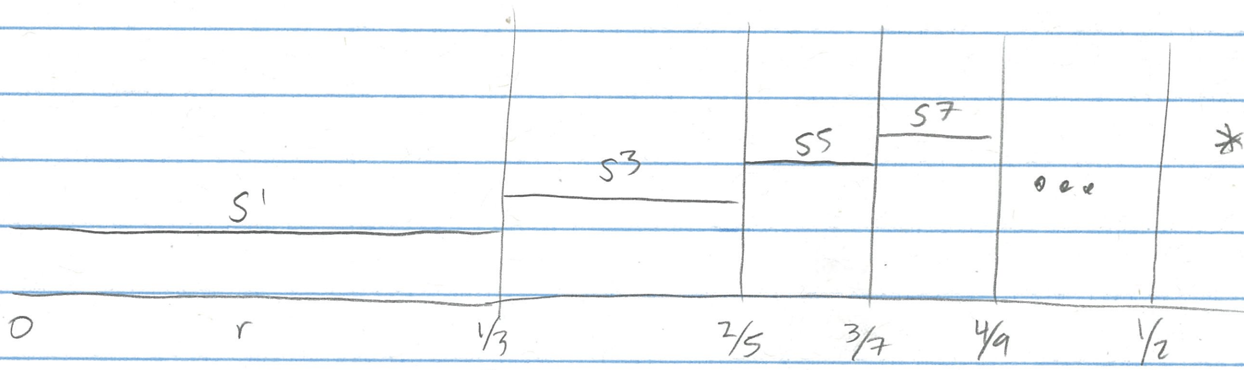
Def The Vietoris–Rips simplicial complex $VR(X;r)$ has
- vertex set X
- finite simplex $\sigma \subseteq X$ when $\mathrm{diam}(\sigma) \le r$

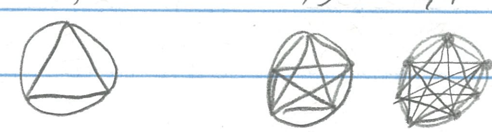


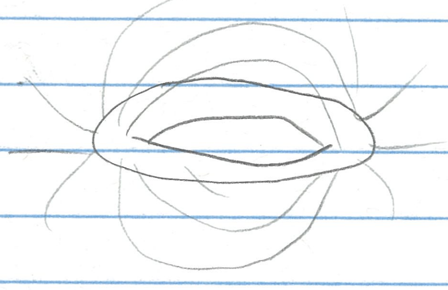


Let S^1 be the circle w/ the geodesic metric & unit circumference.

Thm $VR(S^1;r) \simeq S^{2\ell+1}$ if $\frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3}$







Our proof will use homotopy colimits and the following fact:

Fact For all $\frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3}$, $\exists\ \varepsilon > 0$ s.t. if finite $Y_0 \subseteq S^1$ is ε-dense, then
- $VR(Y_0;r) \simeq S^{2\ell+1}$, and
- $VR(Y;r) \xrightarrow{\simeq} VR(\tilde{Y};r)$ $\forall$ $Y_0 \subseteq Y \subseteq \tilde{Y}$ with $Y, \tilde{Y}$ finite.

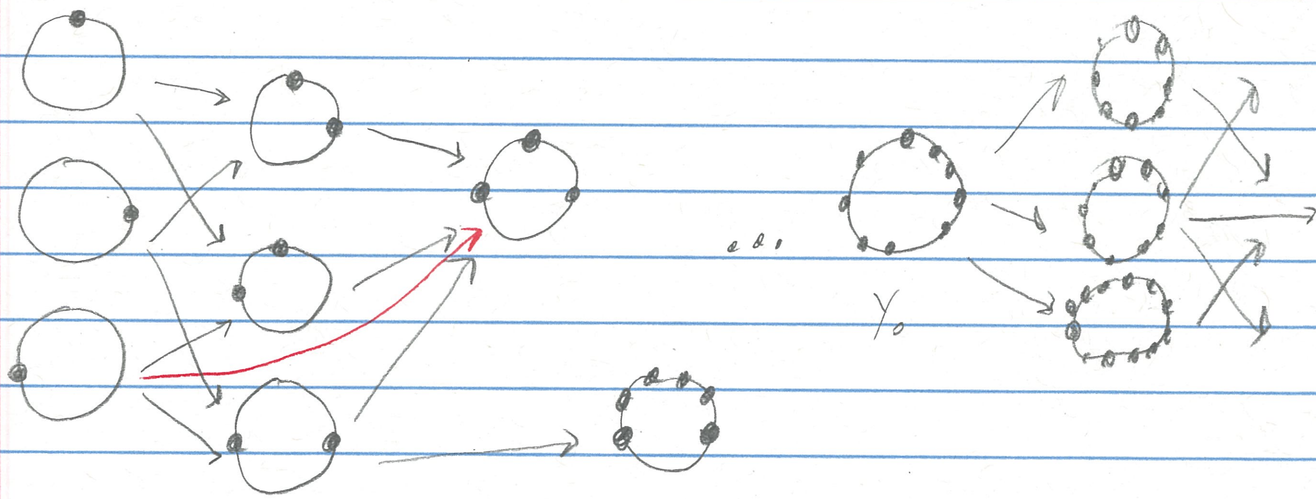
<u>Pf of Thm</u> Let D be the poset of a finite subsets of S^1. Pick some finite $Y_0 \subseteq S^1$ that's ε-dense; let $D(Y_0)$ be the subset of all finite subsets of S^1 containing Y_0.

Then $VR(S^1;r) = \operatorname{colim}_{Y \in D} VR(Y;r)$ $\left[VR(S^1;r) = \bigcup_{\substack{\text{finite} \\ Y \subseteq S^1}} VR(Y;r) \right]$

$\simeq \operatorname{hocolim}_{Y \in D} VR(Y;r)$ Projection Lemma (inclusions of simplicial complexes are cofibrations)

$\simeq \operatorname{hocolim}_{Y \in D(Y_0)} VR(Y;r)$ Cofinality Theorem ($D(Y_0)$ is cofinal in D)

$\simeq S^{2\ell+1}$ Quasifibration Lemma + Fact



From "Homotopy colimits – comparison lemmas for combinatorial applications" by Volkmar Welker, Günter Ziegler, and Rade Živaljević.

<u>Projection Lemma (Prop 3.1)</u> If $F: D \to \mathrm{Top}$ is a diagram of spaces such that $F(\alpha): F(d_0) \to F(d_1)$ is a <u>closed cofibration</u> $\forall \alpha: d_0 \to d_1$, then $\mathrm{hocolim}(F) \to \mathrm{colim}(F)$ is a homotopy equivalence.

Think "nice inclusions"

$F = VR(\_\_; r)$

<u>Ex</u> Inclusions of simplicial complexes are closed cofibrations.

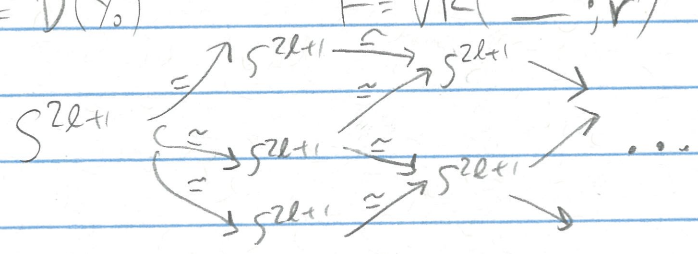
<u>Quasifibration Lemma (Prop 3.6)</u>

If $F: D \to \mathrm{Top}$ is a diagram of spaces such that

- $F(\alpha)$ is a (weak) homotopy equivalence $\forall$ arrows α in D, and
- the "nerve" or "classifying space" $B(D)$ is contractible,

then $\forall d \in D$ the map $F(d) \to \mathrm{hocolim}(F)$ is a (weak) homotopy equivalence

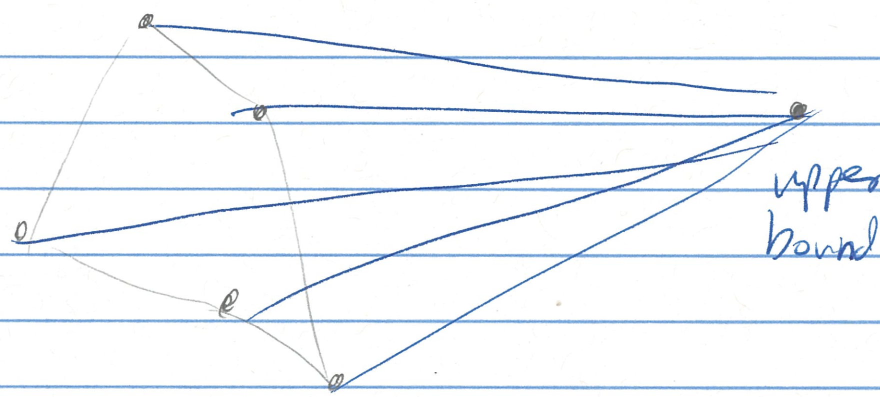
<u>Ex</u> $D = D(Y_0)$ $F = VR(\_\_; r)$



$\mathrm{hocolim}_{Y \in D(Y_0)} VR(Y; r)$

The nerve of D is a simplicial set with one vertex for each vertex of D, and an n-simplex for each sequence of n composable arrows.

For us, $B(D(Y_0))$ is contractible since any collection of objects has an "upper bound" (i.e. a cone vertex)



Cofinality Theorem (Prop 3.10)

If $C \subseteq D$ is right cofinal, i.e. if $\forall d \in D$ $\exists c \in C$ and an arrow $\alpha : d \to c$ in D, then $\mathrm{hocolim}_C(F) \longrightarrow \mathrm{hocolim}_D(F)$ is a homotopy equivalence.

Ex $C = D(Y_0)$ $D = D$ $F = VR(- ; r)$

Note $D(Y_0)$ is cofinal in D since for any finite subset $Y \subseteq S^1$, we have
$$Y \hookrightarrow Y \cup Y_0 .$$