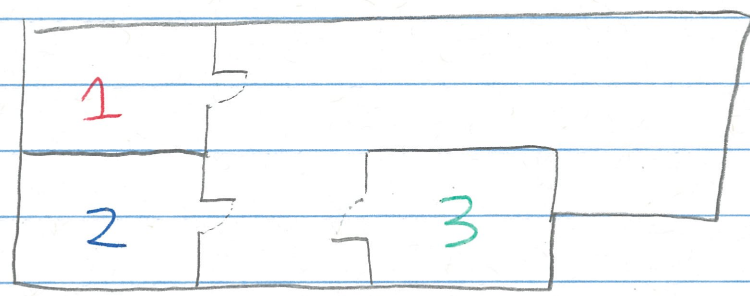
Sperner's lemma and fair division

Problem Suppose 3 roommates need to pay \$3,000/month rent for their 3-bedroom apartment. The bedrooms are not equivalent, and each roommate has different preferences!



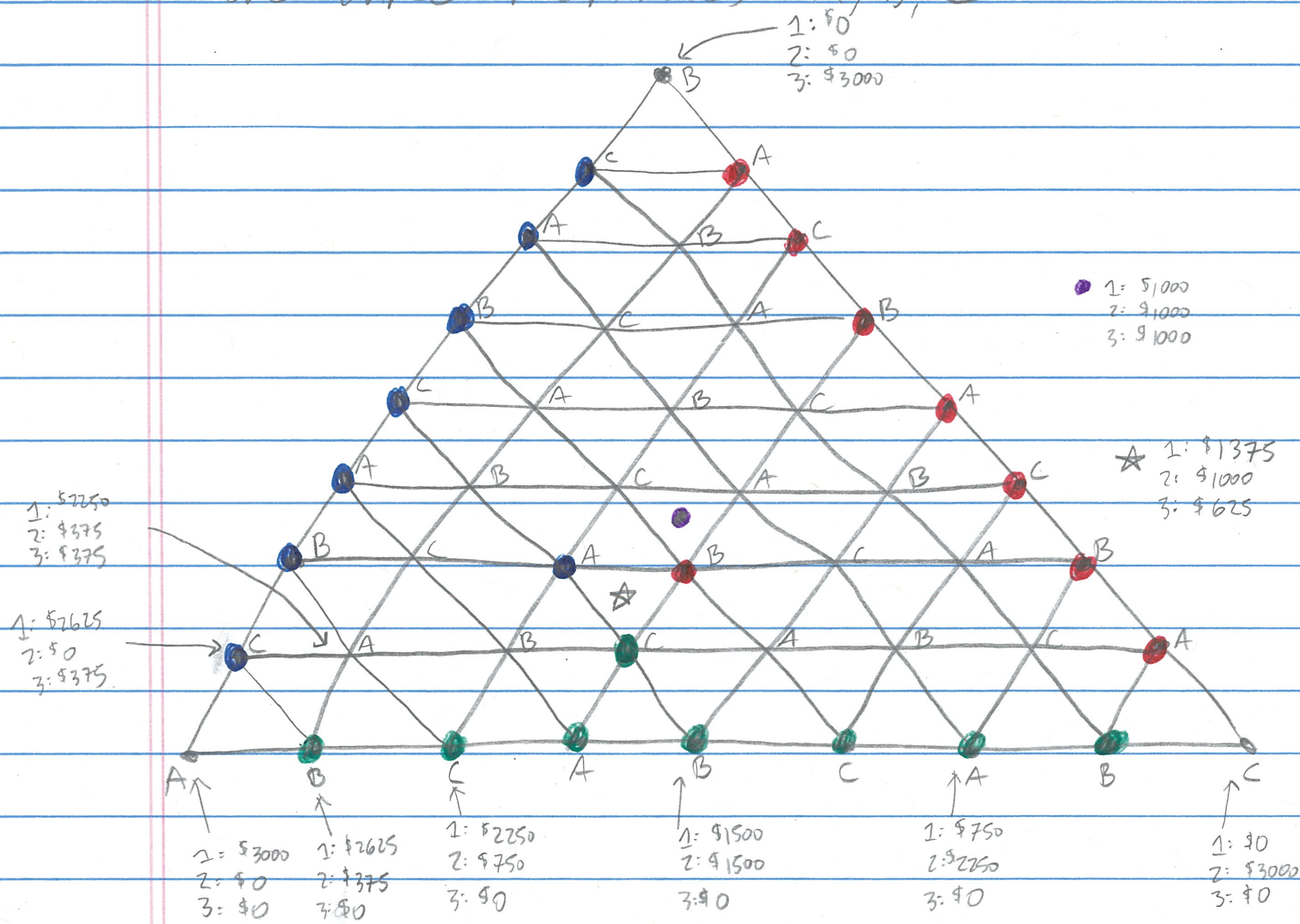
How do you fairly divide the rent, and decide who gets which room, in an envy-free fashion?

Assumption Assume the house is good enough that each person always prefers a free room over a non-free room.

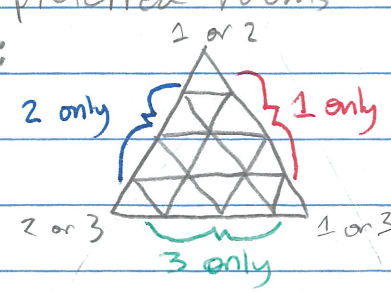
References See the New York Times article "To divide the rent, start with a triangle" and the paper "Rental Harmony: Sperner's lemma in fair division" by Francis Su.

Algorithm

- Draw a triangle representing rent divisions b/w rooms
- Subdivide into smaller triangles
  (up to some acceptable approximation level)
- Label the vertices in an alternating fashion by the three roommates A, B, C



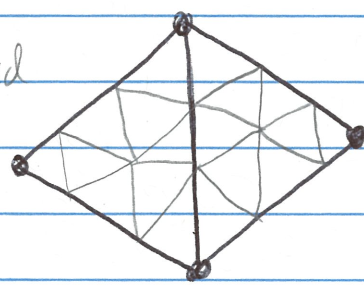
- Ask A, B, C which room they prefer at the prices given by each of their labeled vertices.
- By the "free room" assumption, the preferred rooms are of the following form:



- By a variant of Sperner's lemma, there exists a small triangle which is rainbow-colored (all 3 colors present).
- This gives an assignment of rooms and an (approximate) envy-free pricing scheme!

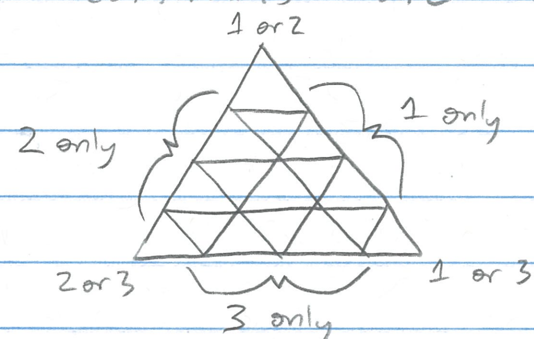
Remark This can also be done with n roommates and n rooms.

$\underline{n=4}$ Tetrahedron instead of a triangle



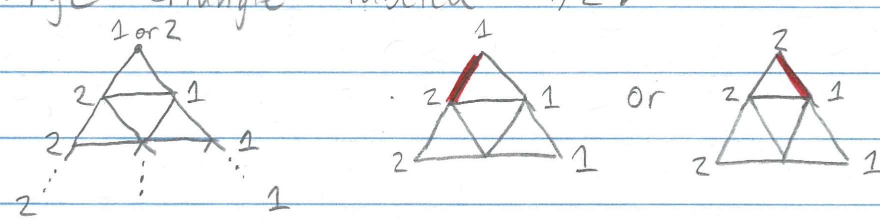
A variant of Sperner's lemma

Consider a subdivision of a triangle into smaller triangles, such that each vertex is labeled 1, 2, or 3, and such that the following boundary constraints are satisfied:



Then there exists a small triangle with vertices labeled 1, 2, and 3.

Proof Note there is exactly one edge on the boundary of the large triangle labeled 1,2.

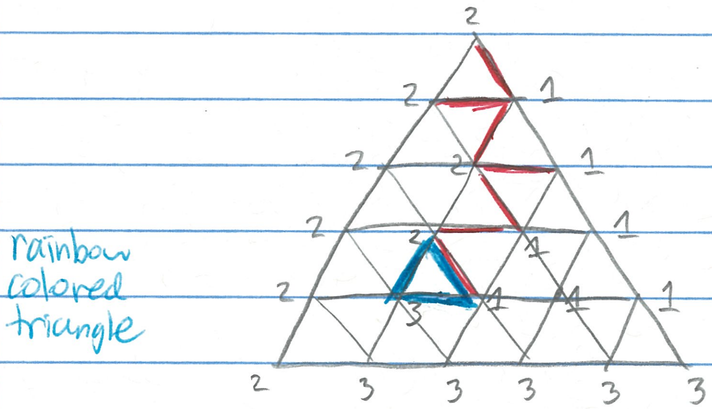


The small triangle containing the first 1,2 edge either has a second 1,2 edge, or else a vertex labeled 3 (in which case we're done).

[Remark: The initial small triangle is necessarily in the former case.]

The adjacent triangle containing the second 1,2 edge either has a third 1,2 edge, or else a vertex labeled 3 (in which case we're done).

We continue to get a path of such 1,2 edges.



This path must end in a rainbow triangle containing label 3.

- The path can't end with a 1,2 edge on the boundary of the large triangle since there is only one such edge (our initial edge)
- The path can't loop back on itself since no small triangle contains three 1,2 edges.
- The path can't go on forever since our triangulation is finite.