**Definition.** For $X$ a metric space and $r > 0$, the **Vietoris–Rips simplicial complex** $\text{VR}(X; r)$ has $X$ as its vertex set, and a finite subset $\sigma \subset X$ as a simplex when $\text{diam}(\sigma) \leq r$.

**Remark.** If $X$ is not discrete then the inclusion $X \hookrightarrow \text{VR}(X; r)$ is not continuous, and if $\text{VR}(X; r)$ is not locally finite then $\text{VR}(X; r)$ is not metrizable.

In [3], Hausmann proves that for $M$ a Riemannian manifold and $r$ sufficiently small, there is a homotopy equivalence $\text{VR}(M; r) \cong M$. This proof is not as straightforward as one might hope: map $\text{VR}(M; r) \to M$ depends on the choice of a total ordering of all points in $M$, and the inclusion $M \hookrightarrow \text{VR}(M; r)$ is not a homotopy inverse since it’s not continuous.

![Vietoris–Rips metric thickenings](image)

**Definition.** For $X$ a metric space and $r > 0$, the **Vietoris–Rips metric thickening** $\text{VR}^m(X; r)$ is

$$\text{VR}^m(X; r) = \left\{ \sum_{i=0}^k \lambda_i \delta_{x_i} \mid \lambda_i \geq 0, \sum_i \lambda_i = 1, x_i \in X, \text{diam}\{x_0, \ldots, x_k\} \leq r \right\},$$

equipped with the 1-Wasserstein metric.

It is not hard to check that $\text{VR}^m(X; r)$ is a **metric $r$-thickening** of $X$.

**Remark.** As a set, $\text{VR}^m(X; r)$ is naturally identified with the geometric realization of $\text{VR}(X; r)$. However, these two topological spaces need not be homotopy equivalent.

**Main Theorem ([2]).** If $M$ is a Riemannian manifold and $r$ is sufficiently small, then $\text{VR}^m(M; r) \cong M$.

**Proof.** Consider the (now canonical) continuous map $f : \text{VR}^m(M; r) \to M$ defined by sending a point $\sum_i \lambda_i \delta_{x_i}$ to its Fréchet mean in $M$. This map has the (now continuous) inclusion $\iota : M \hookrightarrow \text{VR}^m(M; r)$ as a homotopy inverse. We have $f \circ \iota = \text{id}_M$, and via a linear homotopy we get $\iota \circ f \simeq \text{id}_{\text{VR}^m(M; r)}$. \qed

**Example differences between $\text{VR}(X; r)$ and $\text{VR}^m(X; r)$**

- **The circle.** Let $S^1$ be the circle of unit circumference with the geodesic metric. We have $\text{VR}(S^1; r) \cong S^2$ if $r \leq \frac{\pi}{6}$, and $\text{VR}(S^1; r) \cong S^1$ if $r > \frac{\pi}{6}$.

- **The $n$-sphere.** Let $S^n$ be the $n$-sphere, and let $r_n$ be the scale parameter when the first inscribed regular $(n+1)$-simplex $\Delta^{n+1}$ appears. We have $\text{VR}(S^n; r_n) \cong S^n$ if $r \leq \frac{\pi}{6}$, whereas $\text{VR}^m(S^n; r_n) \cong S^n$ if $r > \frac{\pi}{6}$.

**Questions**

- Are the homotopy types of $\text{VR}^m(S^n; r)$ related to strongly self-dual polytopes [4]?
- For $M$ a Riemannian manifold, are $\text{conn}(\text{VR}(M; r))$ and $\text{conn}(\text{VR}^m(M; r))$ non-decreasing functions of $r$ [3]?
- Are $\text{VR}_<(X; r)$ and $\text{VR}^m_<(X; r)$ homotopy equivalent?

**References**


