

The persistent topology of optimal transport based thickenings



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arXiv:2109.15061

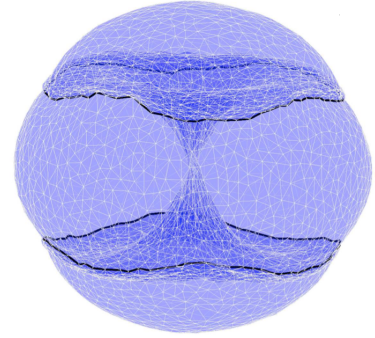
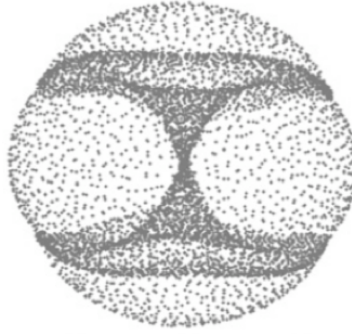
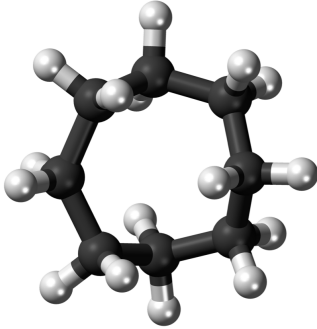
www.math.colostate.edu/~adams/talks/PTOTBT.pdf



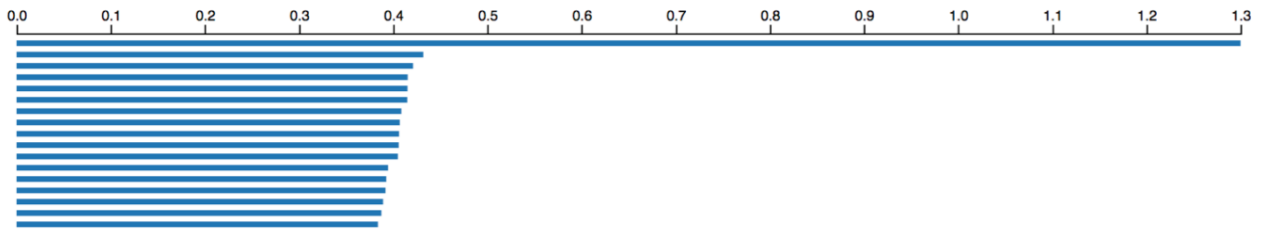
AATR.NE.T, www.aatr.net, 1-2 live talks per week
YouTube: 4,150+ subscribers, 24 hours watched per day
With Hana Dal Poz Kourimská, Teressa Heiss, Sara Kalisnik, Bastian Rieck, Elchanan Solomon

Cyclo-octane molecule C_8H_{16}

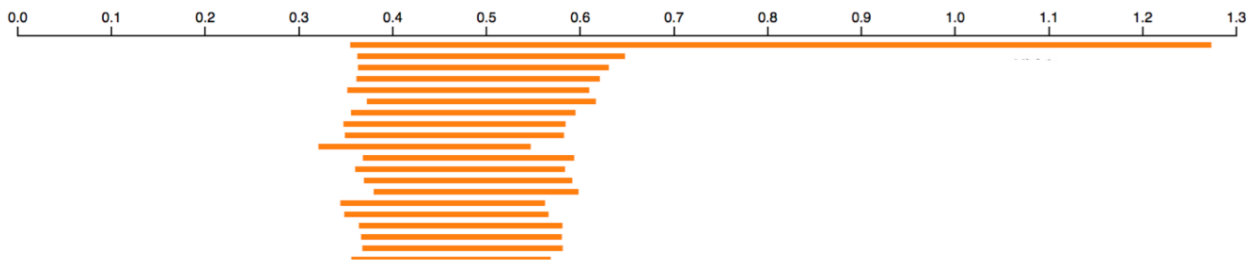
Martin, Thompson, Coutsias, Watson 2010



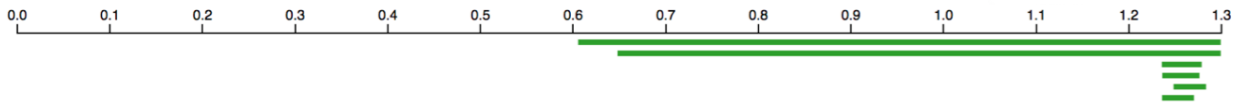
Persistence intervals in dimension 0:



Persistence intervals in dimension 1:



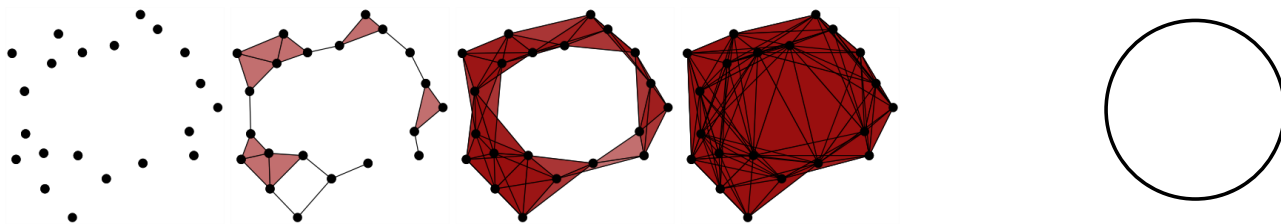
Persistence intervals in dimension 2:



X metric space, $r \geq 0$.

Def The Vietoris-Rips simplicial complex $VR(X, r)$ has

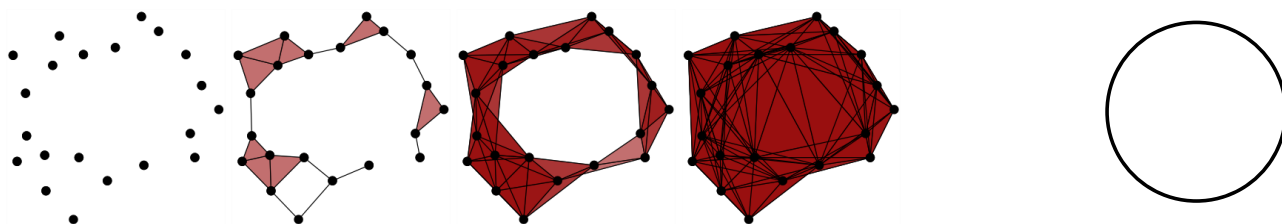
- vertex set X
- finite simplex $\sigma \subseteq X$ when $\text{diameter}(\sigma) < r$.



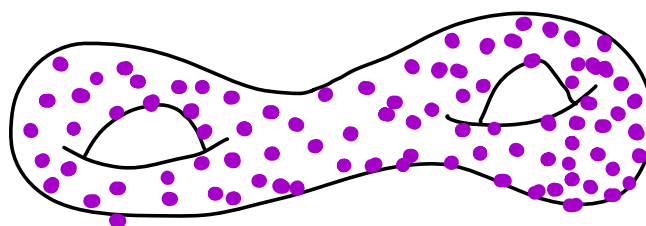
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Stability



$$PH_1(VR(M; r)) \equiv \equiv \equiv \equiv \equiv$$

$$PH_1(VR(X; r)) \equiv \equiv \equiv \equiv \equiv$$

Chazal, de Silva, Oudot, 2014

Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, 2009

Metric Reconstruction

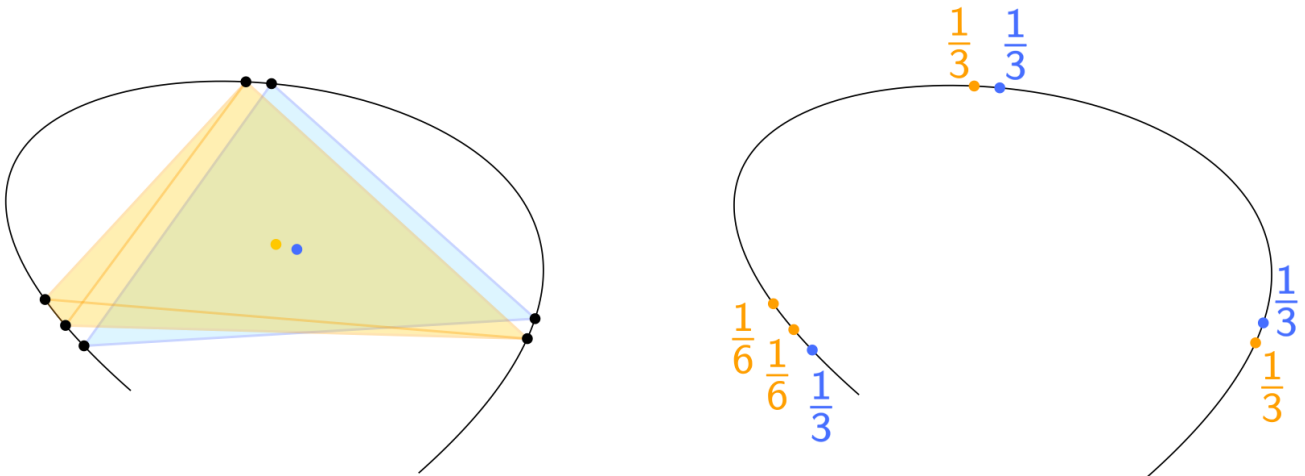
A simplicial complex whose vertex set is a metric space should often be equipped with an optimal transport metric (instead of the simplicial complex topology).

Adamaszek, A, Frick, 2018

Def The Vietoris-Rips metric thickening is

$$\begin{aligned} VR_\infty(X; r) &= \left\{ \sum_{i=0}^k \lambda_i \delta_{x_i} \mid x_i \in X, \lambda_i \geq 0, \sum_i \lambda_i = 1, \text{diam}(\{x_0, \dots, x_k\}) < r \right\} \\ &= \left\{ \text{probability measures } \mu \mid \text{diam}(\text{supp}(\mu)) < r \right\}, \end{aligned}$$

equipped with an optimal transport metric.



Metric Reconstruction

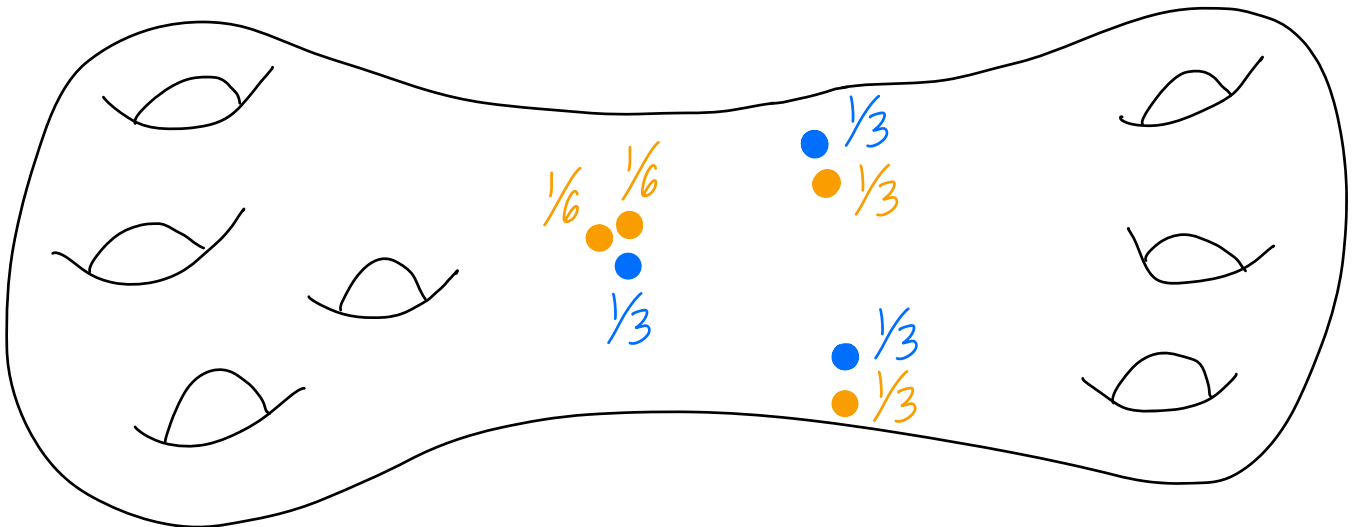
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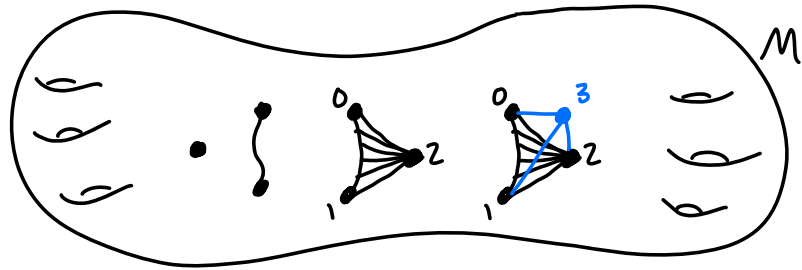
equipped with an optimal transport metric.



Thm Hausmann 1995

M compact Riemannian manifold.
 Then $\exists r_0 > 0$ such that $VR(M; r) \approx M \forall r < r_0$.

Proof

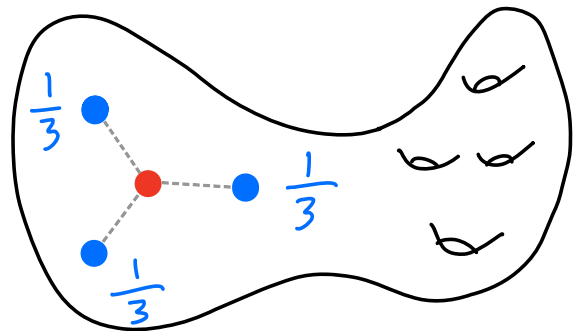


- Not canonical
- $M \hookrightarrow VR(M; r)$ not continuous.

Our Proof for VR_∞



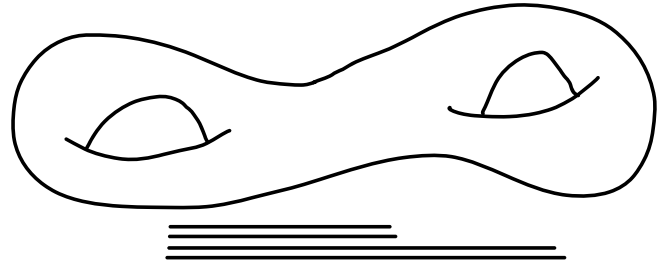
$\sum \lambda_i \delta_{x_i}$
 \downarrow
 Fréchet mean



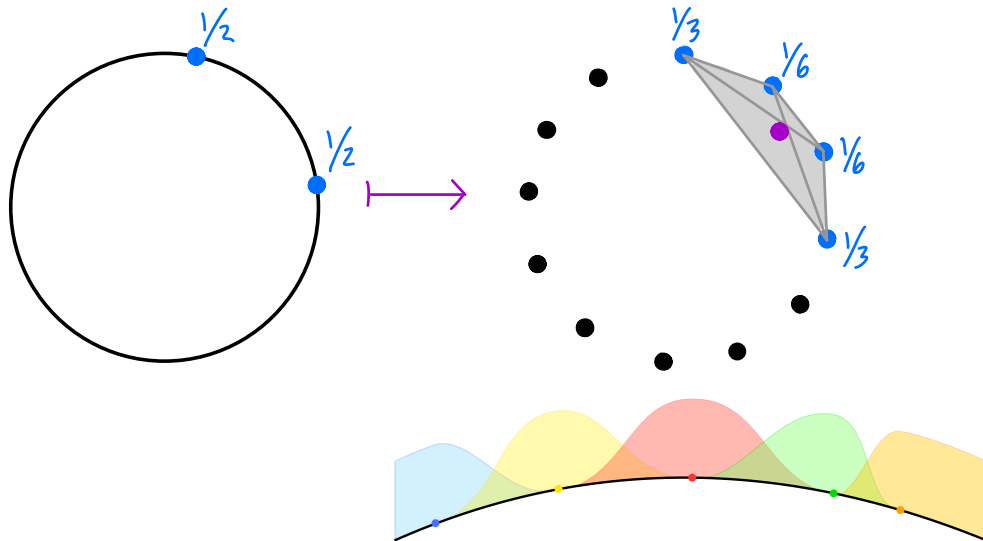
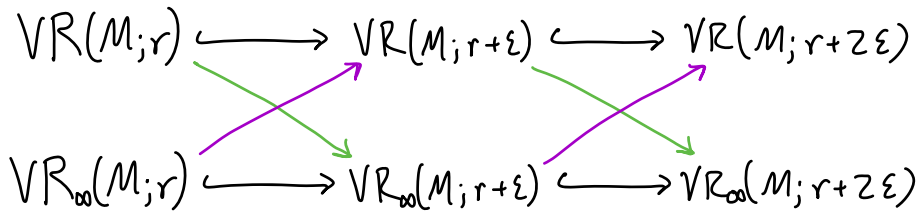
Thm For X totally bounded, $VR_\infty(X;r)$ and $VR(X;r)$ have the same (undecorated) persistence diagrams.

A., Mémoli, Moy, Wang, 2021

Corollary The persistence of $VR_\infty(X;r)$ is stable.

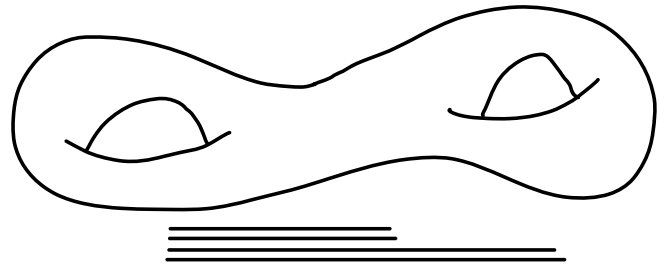


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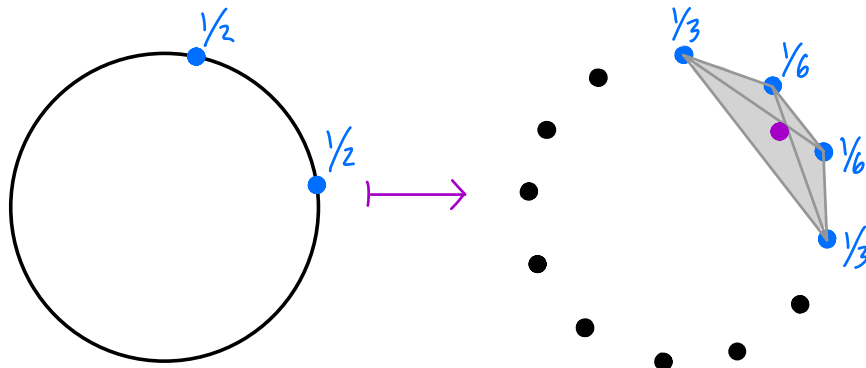
A., Mémoli, Moy, Wang, 2021



Corollary The persistence of $VR_\infty(X;r)$ is stable.

Proof

$$\begin{array}{ccccc}
 VR(M;r) & \hookrightarrow & VR(M;r+\epsilon) & \hookrightarrow & VR(M;r+2\epsilon) \\
 & \searrow^{\text{green}} & \nearrow^{\text{magenta}} & & \nearrow^{\text{green}} \\
 VR_\infty(M;r) & \hookrightarrow & VR_\infty(M;r+\epsilon) & \hookrightarrow & VR_\infty(M;r+2\epsilon)
 \end{array}$$



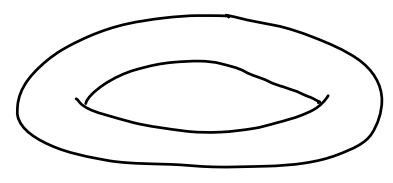
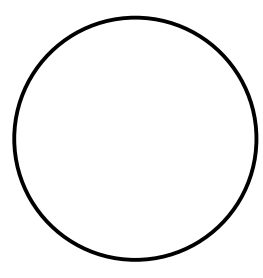
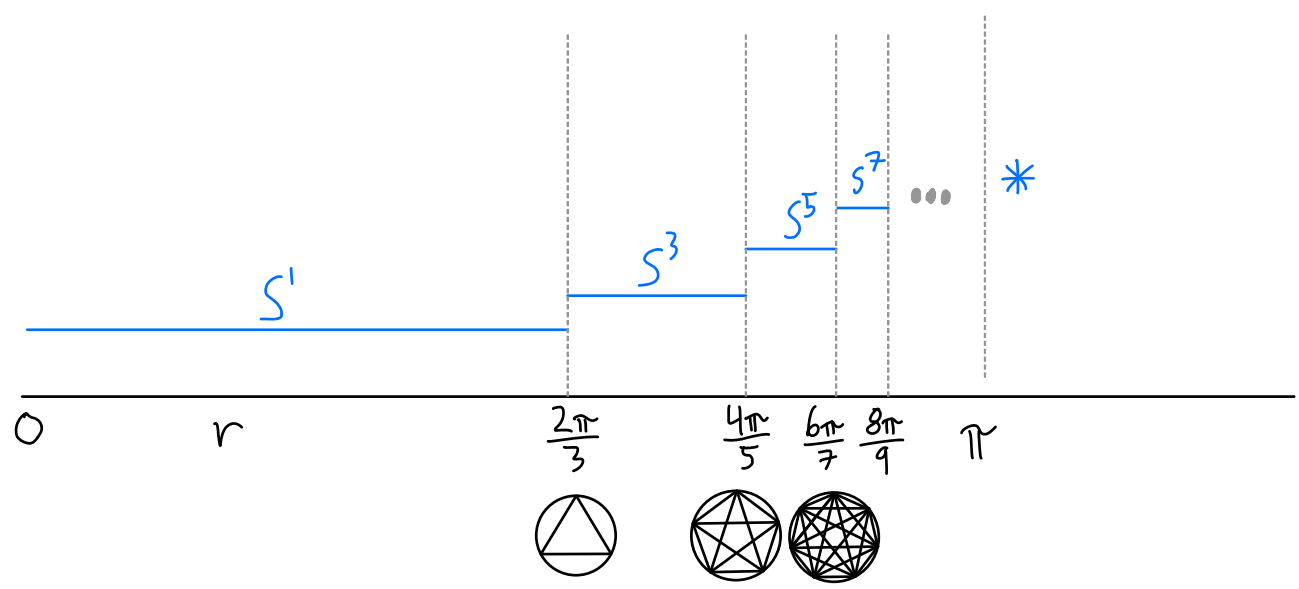
Question Is $VR_\infty(X;r) \simeq VR(X;r)$?

Remark $VR_\infty(X;r)$ and $VR(X;r)$ have the same homotopy groups.

A., Frick, Virk 2022

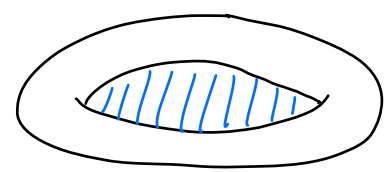
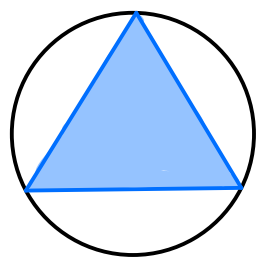
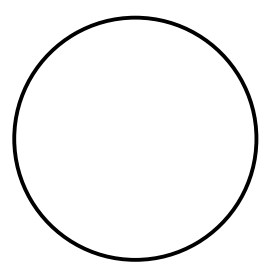
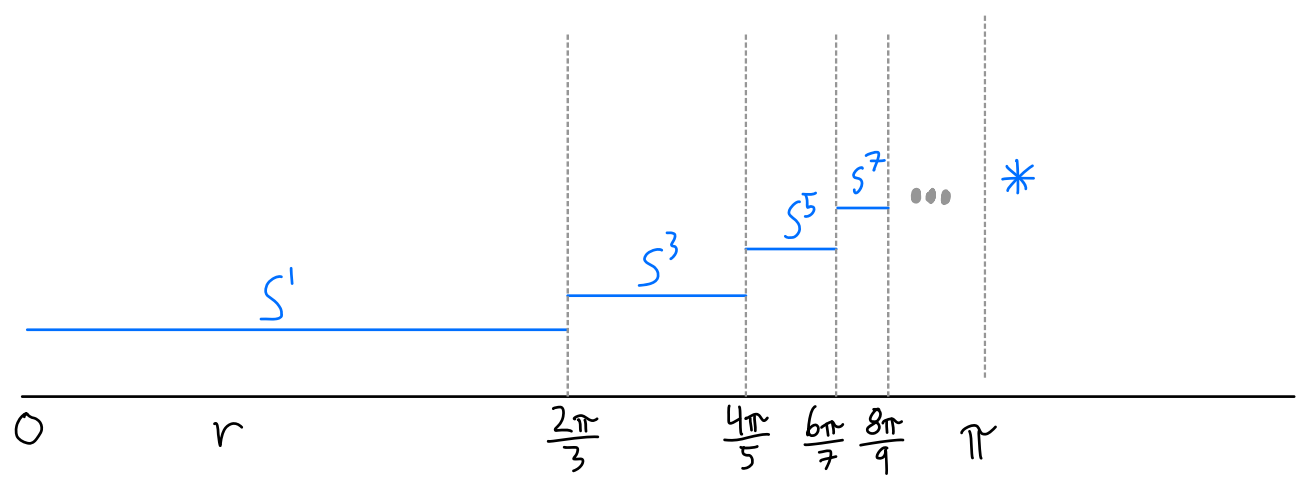
Adamaszek, A. 2017 Moy 2022

Ihm $VR(S'; r) \approx S^{2k+1} \approx VR_\infty(S'; r)$ if $\frac{2\pi k}{2k+1} < r < \frac{2\pi(k+1)}{2k+3}$



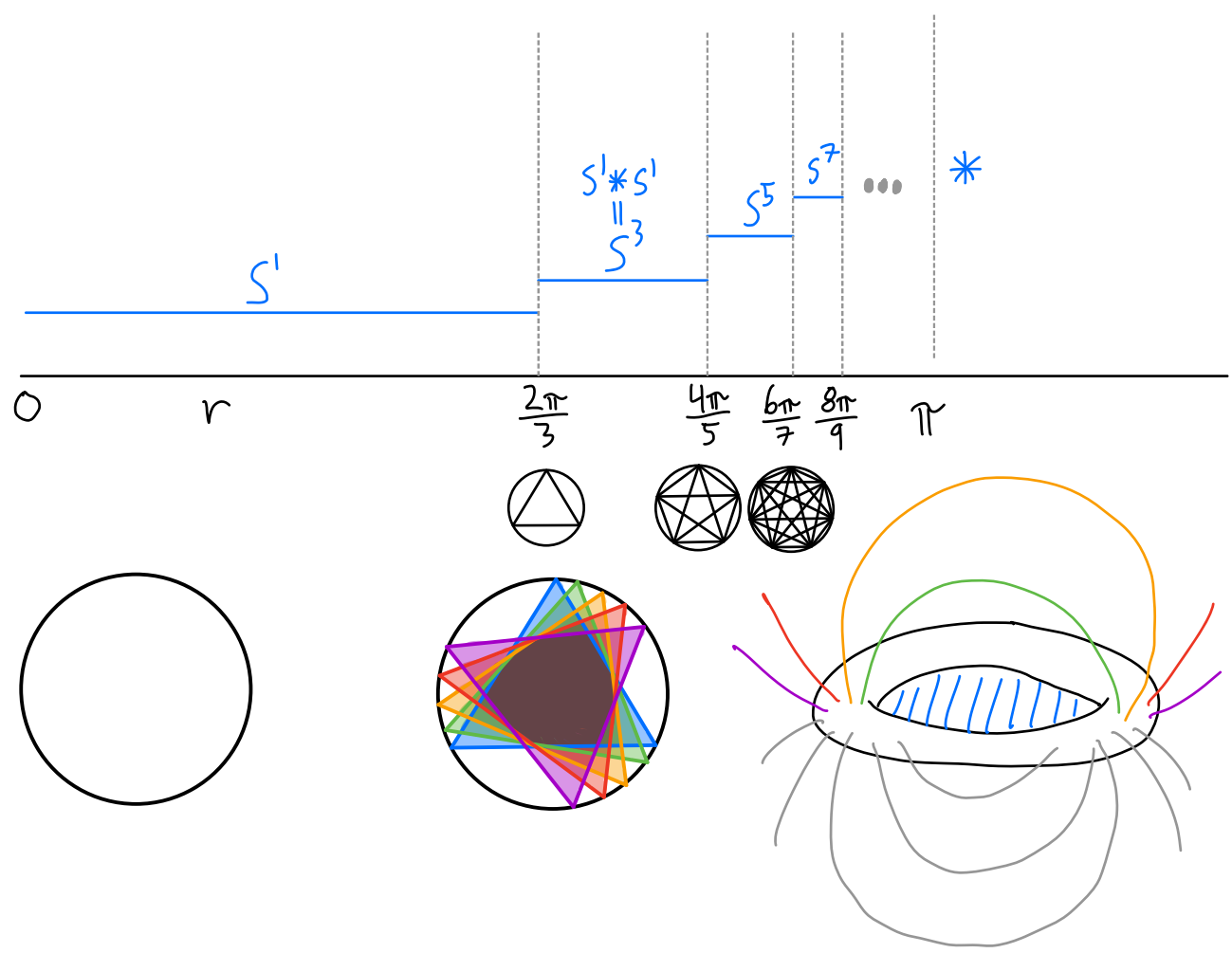
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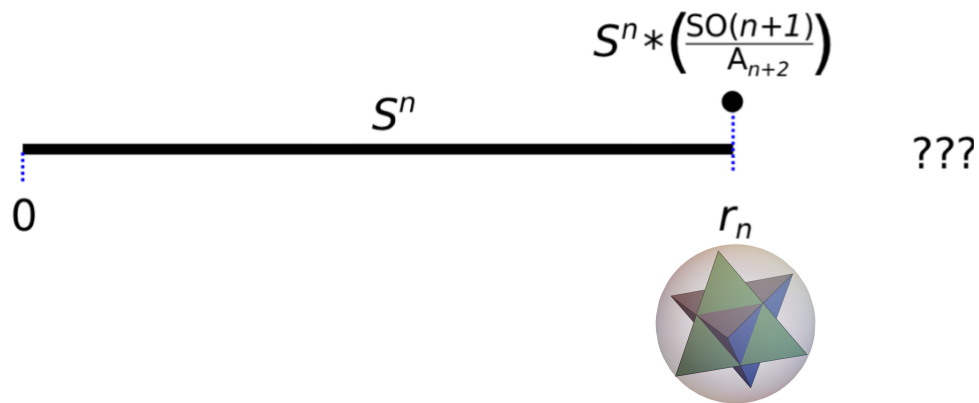
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More generally,

Adamaszek, A., Frick, 2018

$$\underline{\text{Thm}} \quad \text{VR}_\infty(S^n; r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{\text{SO}(n+1)}{A_{n+2}} & r = r_n. \end{cases}$$



Katz 1991

Zaremsky 2018

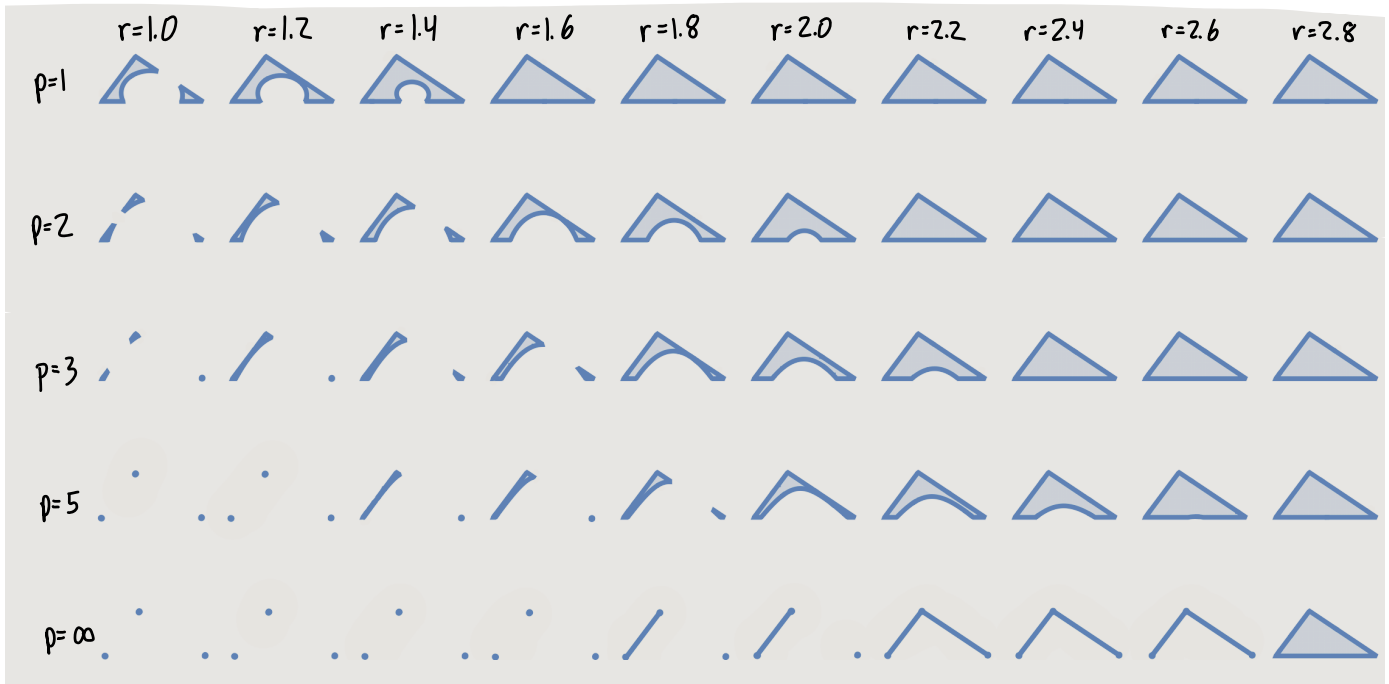
Lim, Mémoli, Okutan 2021

A., Bush, Frick 2021

Relaxation $VR_p(X; r) := \{ \text{probability measures } \mu \mid \text{diam}_p(\mu) < r \}$.

A., Mémoli, Moy, Wang, 2021

$$\text{diam}_p(\mu) = \left(\int_{X \times X} d(x, x')^p d(\mu \times \mu) \right)^{1/p}$$



Bifiltration $VR_p(X; r) \hookrightarrow VR_{p'}(X; r')$ for $r \leq r', p \geq p'$.

Theorem $VR_p(X; r)$ is stable for all $p \in [1, \infty]$.

Theorem $VR_2(S^n; r) \simeq S^n$ or $*$ for all r .

Connections to Morse theory

Katz 1991 Adamaszek, A., Frick, 2018 Mirth 2020 Moy 2022 Katz, Mémoli, Wang 2022

Questions

- (1) $VR_p(S^n; r)$ for larger r ?
- (2) Čech $(S^n; r)$?
- (3) Other manifolds ? Tori, ellipsoids, $\mathbb{R}P^n$, $\mathbb{C}P^n$
- (4) Morse and Morse-Bott theories
- (5) Deeper connections to Gromov-Hausdorff distances

