Open problems on clique complexes of graphs

Henry Adams, Colorado State University

The clique complex of a graph is a simplicial complex with a simplex for each clique. Clique complexes are frequently being computed in applications of topology to data, but we do not understand their algorithmic theory or their mathematical theory. I will introduce clique complexes of graphs, explain why applied topologists care about them, and survey open problems about the topology of clique complexes of unit disk graphs, powers of lattice graphs, and powers of hypercube graphs.

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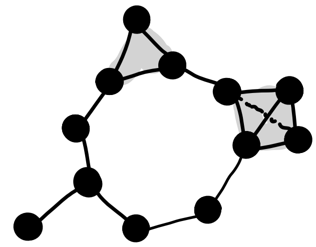
Let G be a simple undirected graph.



<u>Def</u> The <u>clique simplicial complex $cl(G)$</u> has vertex set $V(G)$ and a simplex for each clique in G.

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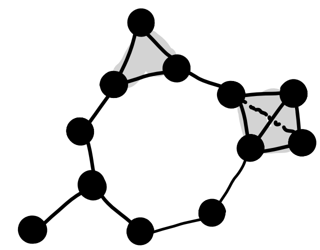

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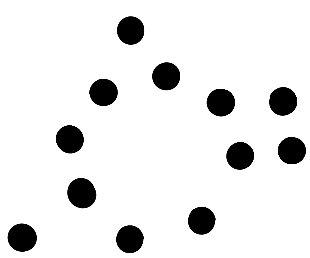
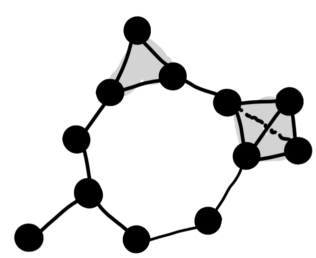
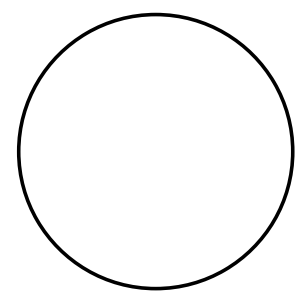
X metric space, $r \geq 0$.

<u>Def</u> The <u>Vietoris-Rips simplicial complex $VR(X;r)$</u> has

- vertex set X
- finite simplex $\sigma \subseteq X$ when $\text{diameter}(\sigma) \leq r$.







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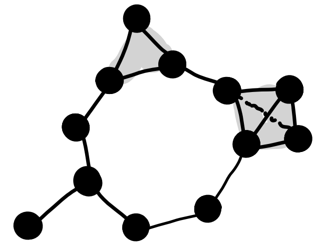

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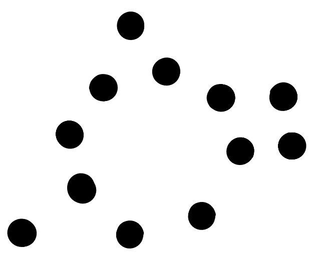
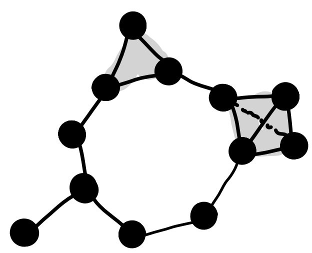
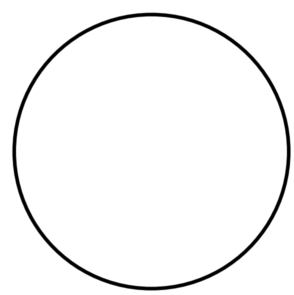
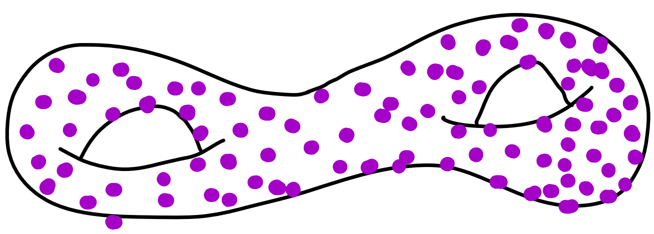


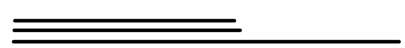




<u>Stability</u>



$PH_1(VR(M;r))$



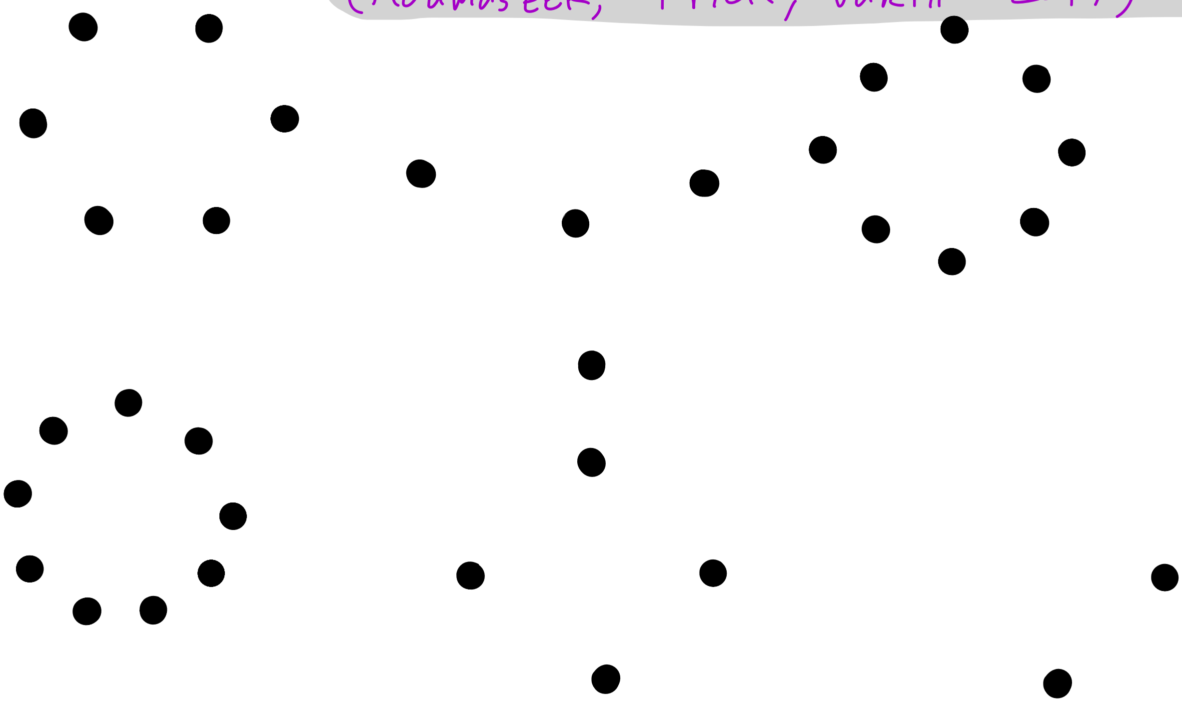
$PH_1(VR(X;r))$

Open question #1

For $X \subseteq \mathbb{R}^2$, is each component of $VR(X;r)$ a wedge of spheres?

- Known: Any wedge of spheres is possible.
- Known: $\pi_1(VR(X;r))$ is free.

(Chambers, de Silva, Erickson, Ghrist 2010)
(Adamaszek, Frick, Vakili 2017)



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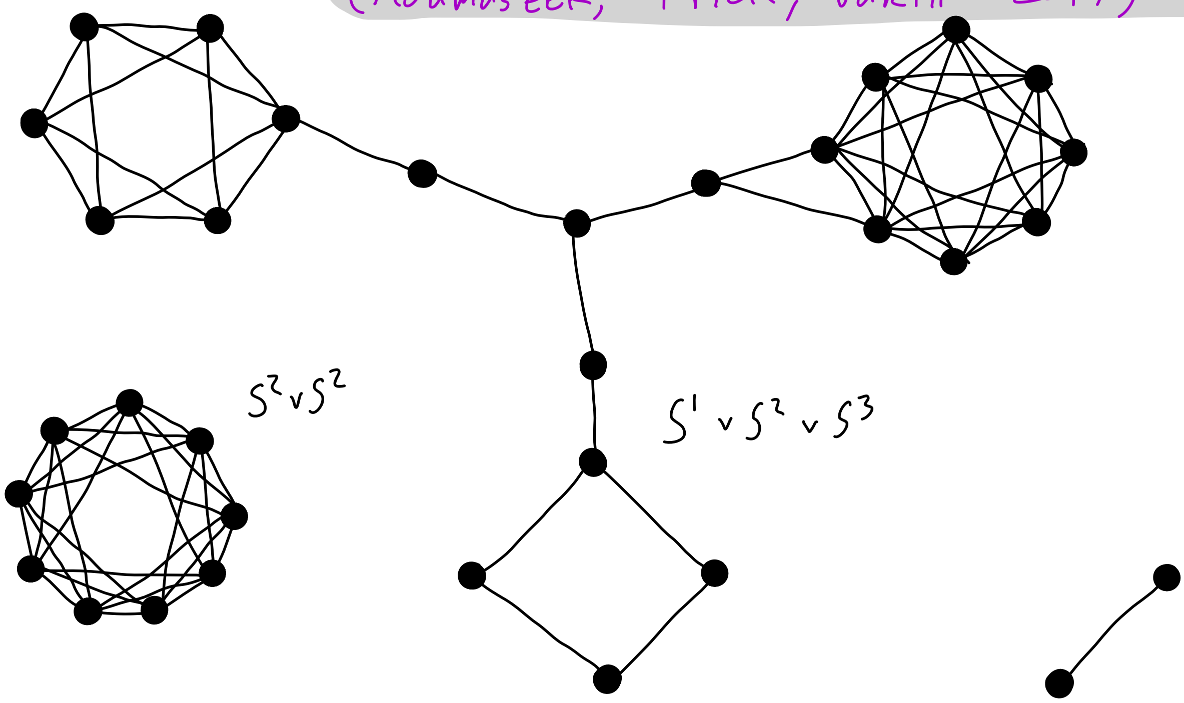
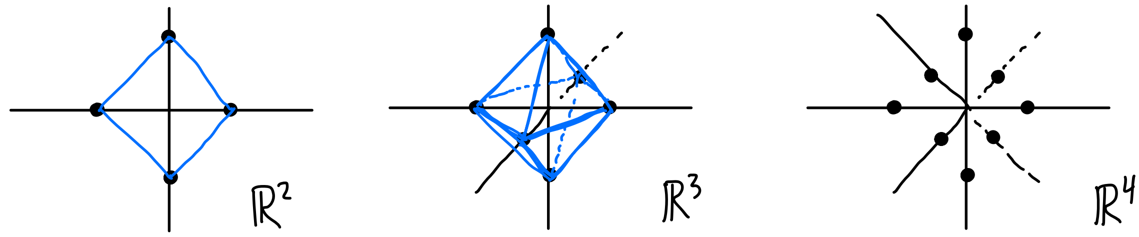


The cross-polytope in $\mathbb{R}^d$ is the convex hull of $\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$. Its boundary is S^{d-1}.



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For $X \subseteq \mathbb{R}^2$, is each component of $VR(X;r)$ a wedge of spheres?

For $X \subset S^1$ (or a nearby curve) with $|X| = n$, the homotopy type of $VR(X;r)$ can be computed in time $O(n \log n)$.

For $X \subset \mathbb{R}^2$, computing the k-dimensional holes of $VR(X;r)$ takes $O(n^{3(k+2)})$.

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Q What about $\mathbb{R}^3$?

Adamaszek Stacho '12: NP-hard to compute homology of clique complex of arbitrary graph.

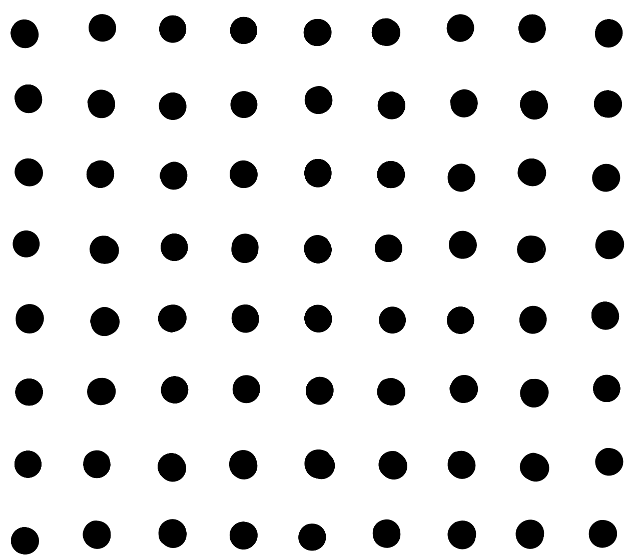
Rmk Clique complex of any finite graph can be realized as $VR(X;r)$ for $X \subset \mathbb{R}^d$ for some d.

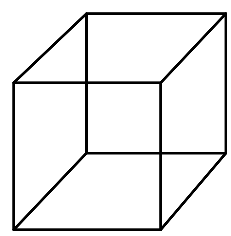
Open question # 2

Is $VR((\mathbb{Z}^n; \ell^1); r)$ contractible for $r \geq n$?

(Case $n=2,3$ by Mallery, Zaremsky)

More generally: Cayley graphs of groups, finiteness properties





Known for ℓ^∞, Known for ℓ^2 with non-optimal bound.

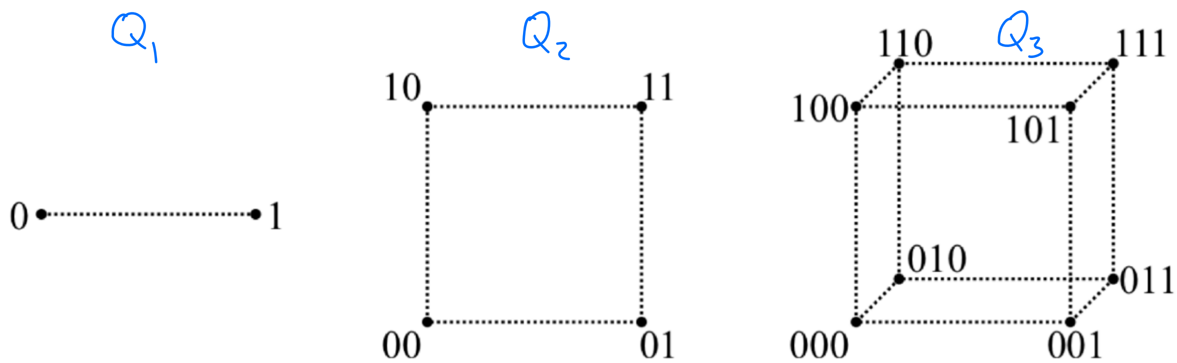
Open question #3: Hypercube graphs

What are the shapes of $VR((\{0,1\}^n;L^1);r)$ for $r<n$?

Carlsson, Fillippenko 2020

Adamaszek, Adams 2021

Def Let Q_n be the 2^n vertices of the hypercube graph, equipped with the shortest path metric.



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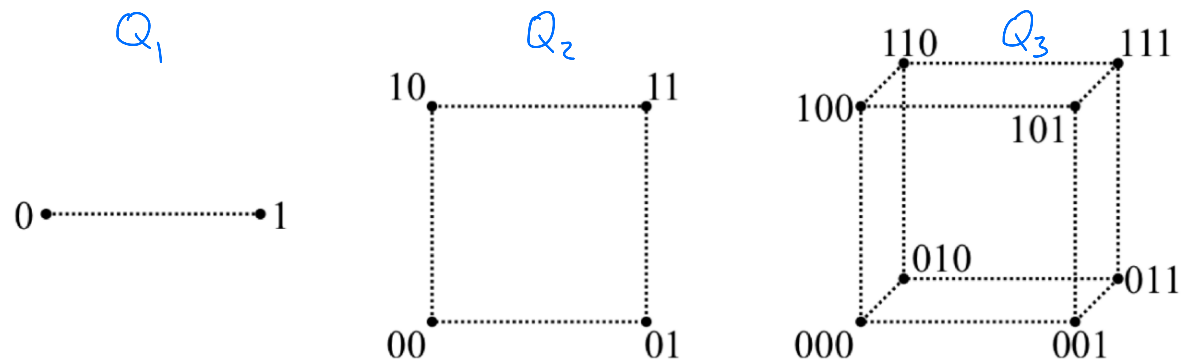
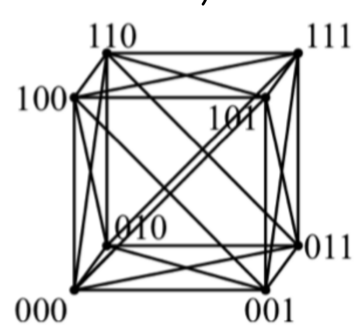


$VR(Q_3; 2) \simeq S^3$



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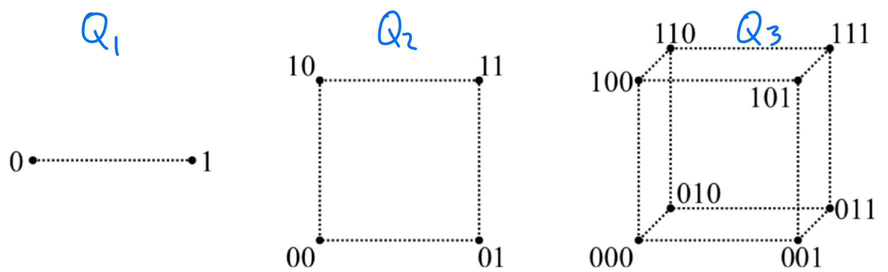

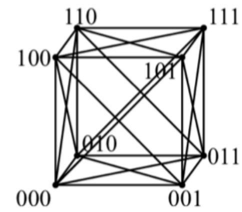

Homotopy types of $VR(Q_n; r)$

	$n=1$	2	3	4	5	6	7	8	9
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
1	$*$	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
2	$*$	$*$	S^3	$\bigvee^9 S^3$	$\bigvee^{49} S^3$	$\bigvee^{209} S^3$	$\bigvee^{769} S^3$	$\bigvee^{2561} S^3$	$\bigvee^{7937} S^3$
3	$*$	$*$	$*$	S^7	$S^4 \vee \bigvee^{10} S^7$	$\bigvee^{11} S^4 \vee \bigvee^{60} S^7$	$\bigvee^{71} S^4 \vee \bigvee^{280} S^7$	$\bigvee^{351} S^4 \vee \bigvee^{1120} S^7$	$\bigvee^{1471} S^4 \vee \bigvee^{4032} S^7$
4	$*$	$*$	$*$	$*$	S^{15}				
5	$*$	$*$	$*$	$*$	$*$	S^{31}			
6	$*$	$*$	$*$	$*$	$*$	$*$	S^{63}		
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{255}

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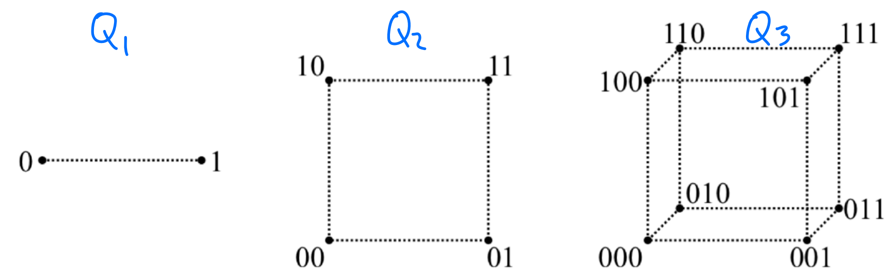

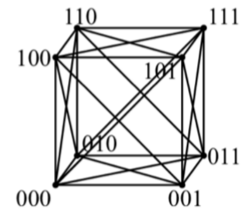

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8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{255}

$H_i(\mathrm{VR}(Q_5;3);\mathbb{Z}) \cong \mathbb{Z}$ for $i=4$, $\cong \mathbb{Z}^{10}$ for $i=7$,
$H_i(\mathrm{VR}(Q_6;3);\mathbb{Z}) \cong \mathbb{Z}^{11}$ for $i=4$, $\cong \mathbb{Z}^{60}$ for $i=7$,
$H_i(\mathrm{VR}(Q_7;3);\mathbb{Z}) \cong \mathbb{Z}^{71}$ for $i=4$, $\cong \mathbb{Z}^{280}$ for $i=7$,

Polymake, 2014

$H_i(\mathrm{VR}(Q_8;3);\frac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{351}$ for $i=4$, $\cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{1120}$ for $i=7$,
$H_i(\mathrm{VR}(Q_9;3);\frac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{1471}$ for $i=4$, $\cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{4032}$ for $i=7$,

Ripser ++, 2020
Zhang, Xiao, Wang
Bauer

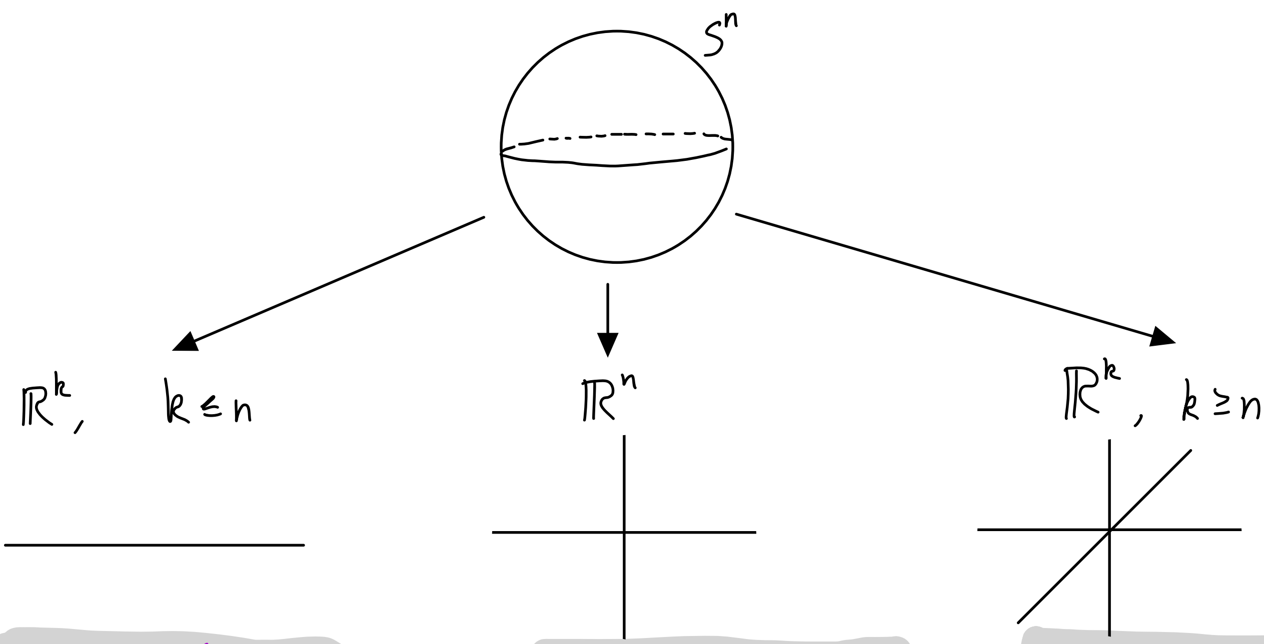
4-dim'l skeleton of n-cube polytope is $\left(\sum_{i=4}^{n-1} 2^{i-4}\binom{i}{4}\right)$ - fold wedge sum of 4-spheres

$2^{n-4}\binom{n}{4}$ is # of Q_4 subgraphs in Q_n.

Conjecture: Is $\mathrm{rank}\left(H_{2^r-1}(VR(Q_n;r))\right) = 2^{n-r-1}\binom{n}{r+1}$ for $r \geq 3$?

Q_{r+1} in Q_n

Open Question #4: Borsuk-Ulam theorems



Almgren (1965):
Geometric measure theory
Gromov (2003):
Filling radius, characteristic classes

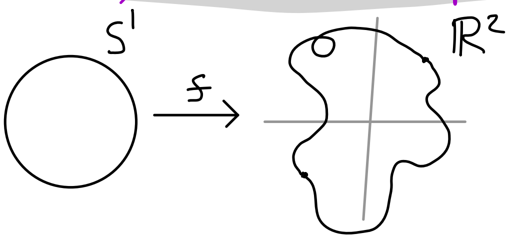
Borsuk-Ulam (1933)

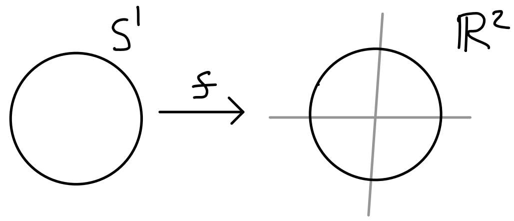
A, Bush, Frick (2020)

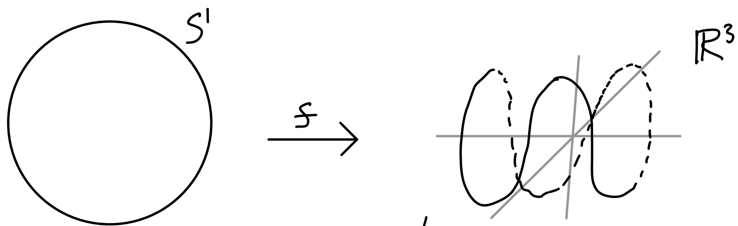
For $f: S^n \to \mathbb{R}^n$, there exists $x \in S^n$ with $f(x) = f(-x)$.

Borsuk-Ulam theorems for $f: S^n \to \mathbb{R}^k$ with $k \geq n$?

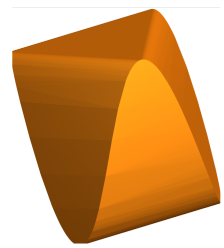
A, Bush, Frick, 2020, "Metric thickenings, Borsuk-Ulam theorems, and orbitopes"







$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta)$



<u>Thm</u> For $f: S^1 \to \mathbb{R}^{2k+1}$, $\exists\ X \subset S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.

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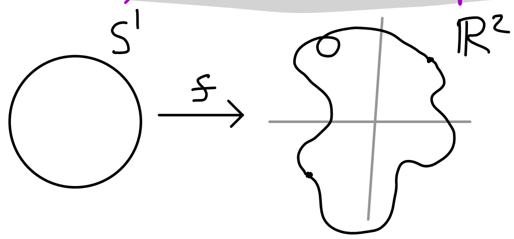

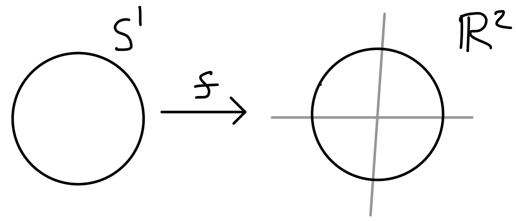

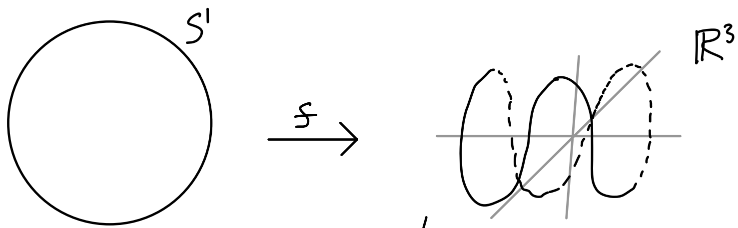

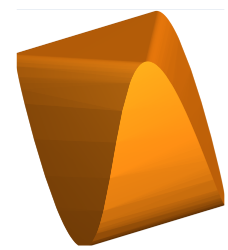

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Sharpness of diameter bound

$$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$$
$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \dots)$$

Borsuk-Ulam theorems for $f: S^n \to \mathbb{R}^k$ with $k \geq n$?

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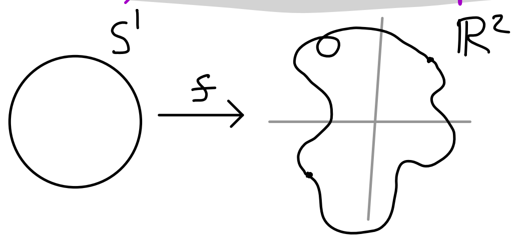

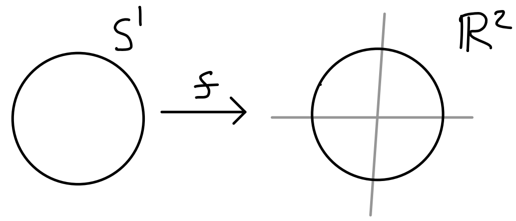

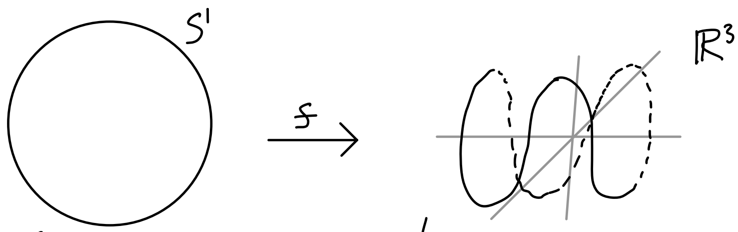

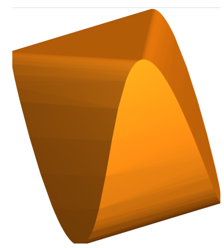

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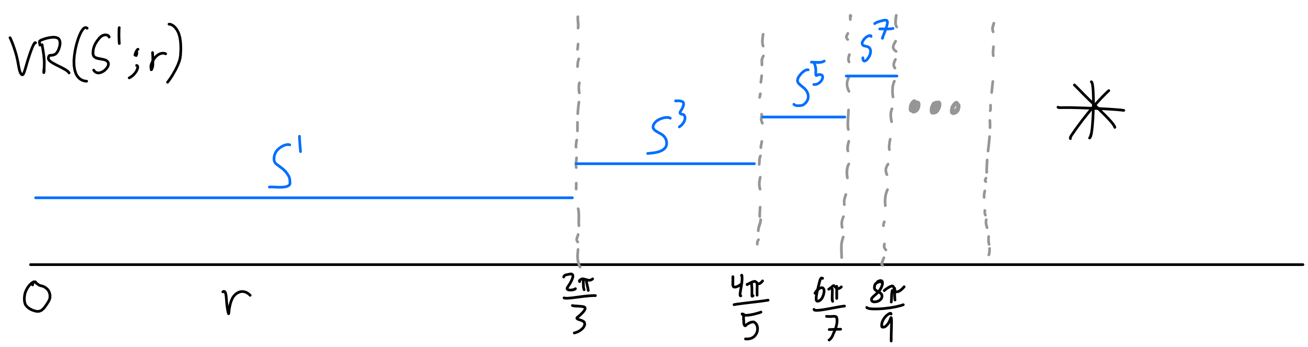
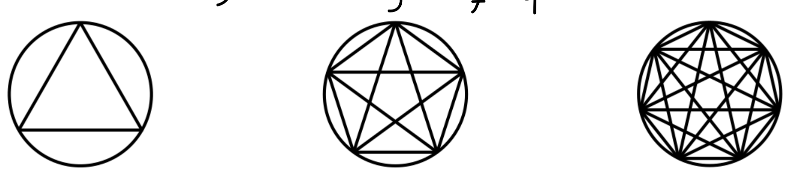
<u>Proof</u>

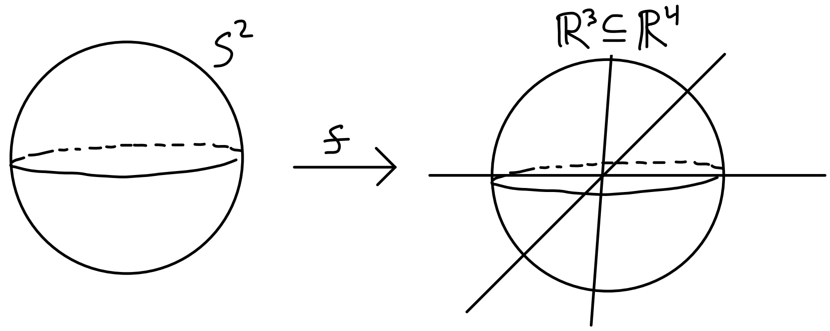
$S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$ induces

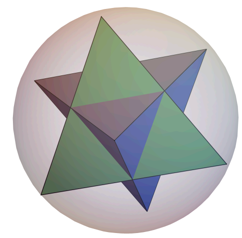
$VR(S^1; r) \xrightarrow{f} \mathbb{R}^{2k+1}$





<u>Thm</u> For $f: S^n \to \mathbb{R}^{n+2}$, $\exists$ $X \subset S^n$ of diameter at most r_n such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.



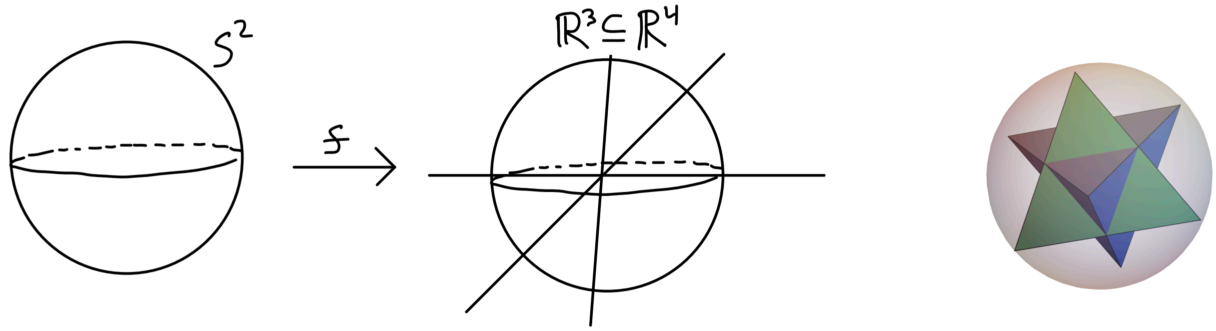


<u>Proof</u>

$$S^n \xrightarrow{\quad f \quad} \mathbb{R}^{n+2}$$

$$S^n * \frac{SO(n+1)}{A_{n+2}} \simeq VR^m(S^n; r) \xrightarrow{\quad f \quad} \mathbb{R}^{n+2}$$

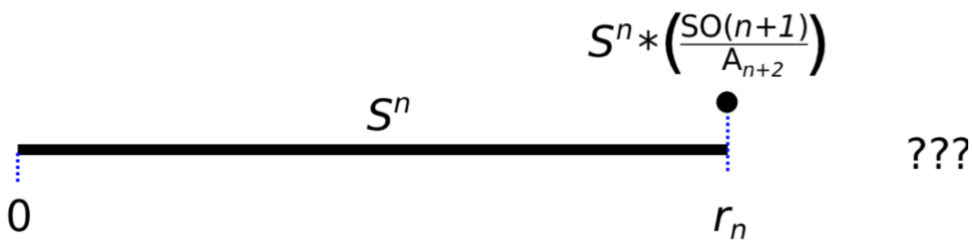
Thm For $f: S^n \to \mathbb{R}^{n+2}$, $\exists$ $X \subset S^n$ of diameter at most r_n such that $\operatorname{conv}(f(X)) \cap \operatorname{conv}(f(-X)) \neq \emptyset$.

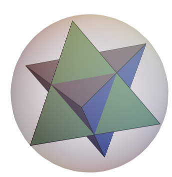


Proof

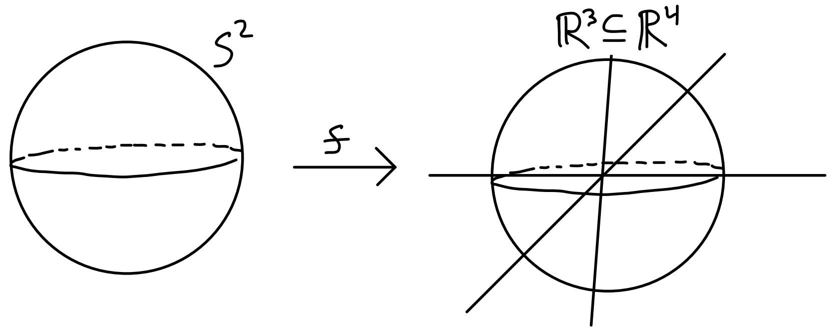
$$S^n \xrightarrow{f} \mathbb{R}^{n+2}$$

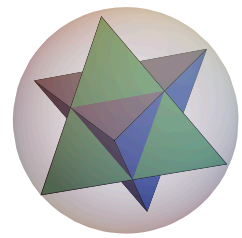
$$S^n * \frac{SO(n+1)}{A_{n+2}} \simeq VR^m(S^n; r) \xrightarrow{f} \mathbb{R}^{n+2}$$





<u>Thm</u> For $f: S^n \to \mathbb{R}^{n+2}$, $\exists$ $X \subset S^n$ of diameter at most r_n such that $\operatorname{conv}(f(X)) \cap \operatorname{conv}(f(-X)) \neq \emptyset$.

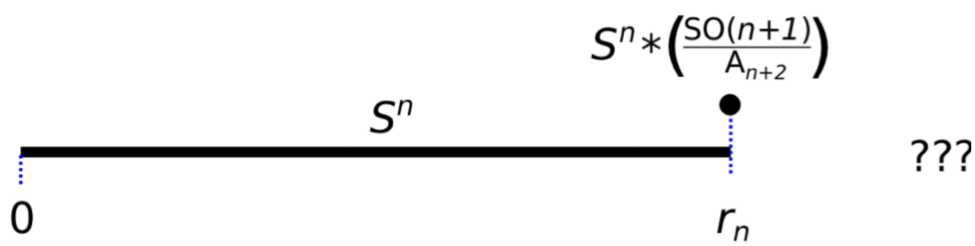


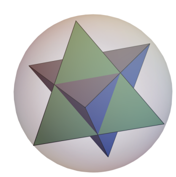


<u>Proof</u>

$$S^n \xrightarrow{\quad f \quad} \mathbb{R}^{n+2}$$

$$S^n * \frac{SO(n+1)}{A_{n+2}} \simeq VR^m(S^n; r) \xrightarrow{\quad f \quad} \mathbb{R}^{n+2}$$





<u>Open Question</u> What about $S^n \to \mathbb{R}^k$ for $k > n$?

Open question #5

What are the homotopy types of Vietoris-Rips complexes of manifolds, specifically spheres, ellipsoids, tori, $\mathbb{R}P^n$, $\mathbb{C}P^n$?