On Vietoris-Rips Complexes of Hypercube Graphs

Henry Adams
Joint with Michał Adamaszek

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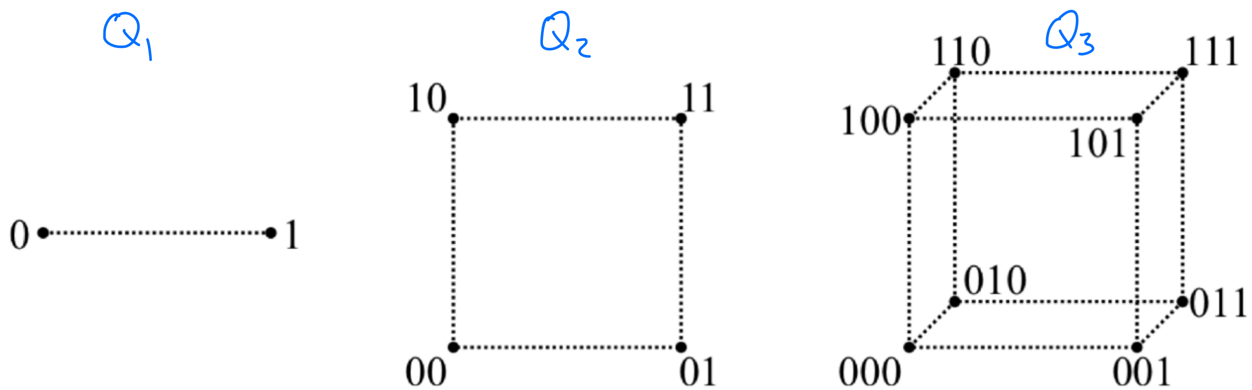

Motivation

- Medial recombination

  Emmett, Rabadán, Rosenbloom 2013

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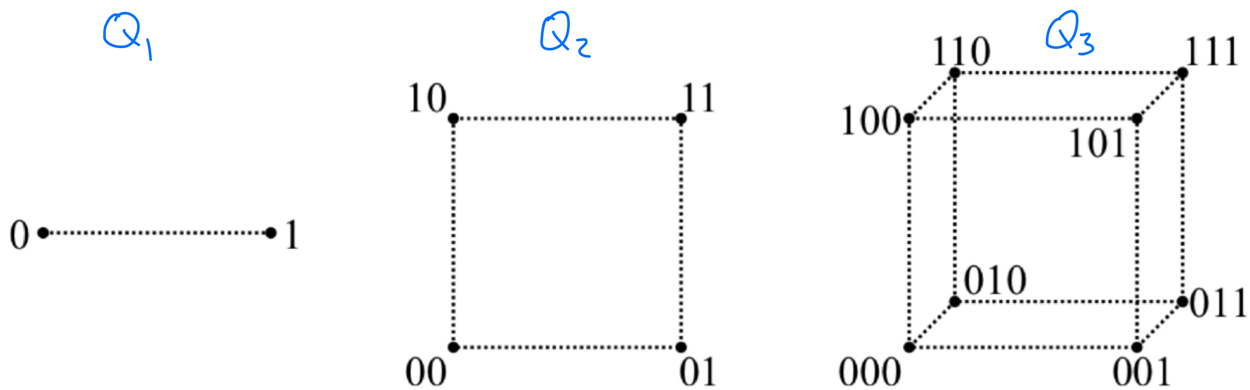

Motivation

- Medial recombination
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- Persistent homology of sum metric
  Carlsson, Filippenko 2019

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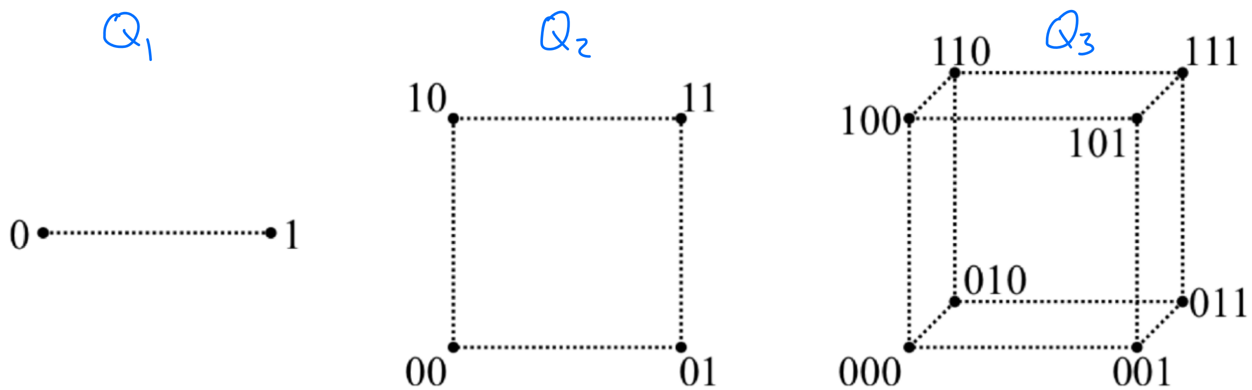
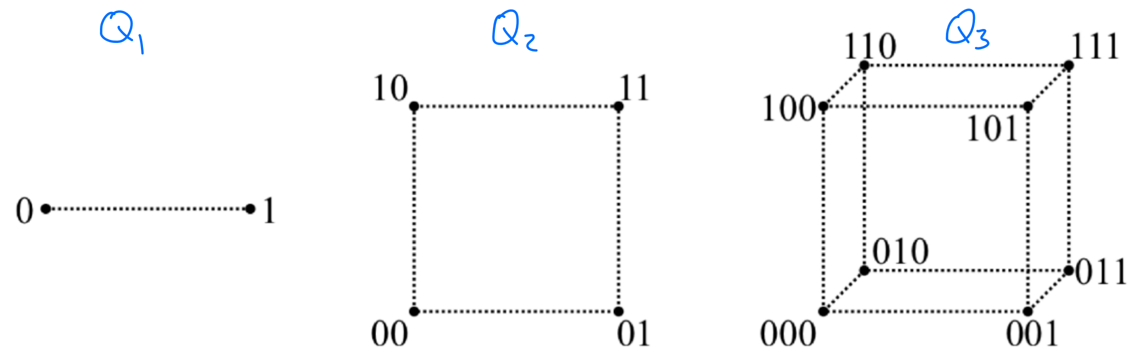


Motivation

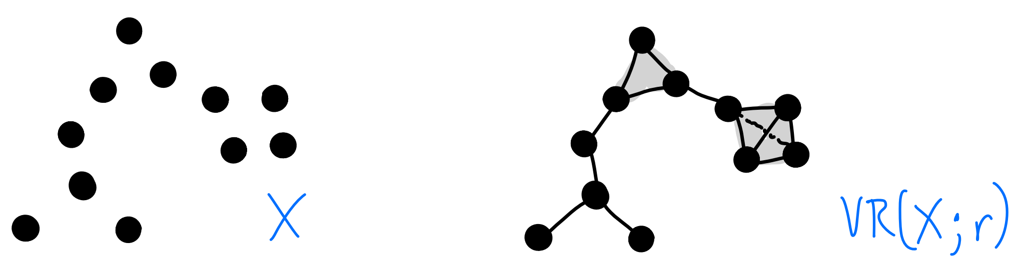
- Medial recombination
  Emmett, Rabadán, Rosenbloom 2013
- Persistent homology of sum metric
  Carlsson, Filippenko 2019
- Connections with
  - quantitative topology (filling radius)
  - combinatorial topology (Borsuk-Ulam theorems)
  - geometric group theory (Bestvina-Brady Morse theory)
  - geometric topology (thick-thin decompositions)
  - geometric measure theory (optimal transport)

Def Let Q_n be the 2^n vertices of the hypercube graph, equipped with the shortest path metric.



Def The Vietoris-Rips simplicial complex $VR(X;r)$ of metric space X at scale $r \geq 0$ has

- vertex set X
- finite simplex $\sigma \subseteq X$ when diameter$(\sigma) \leq r$.



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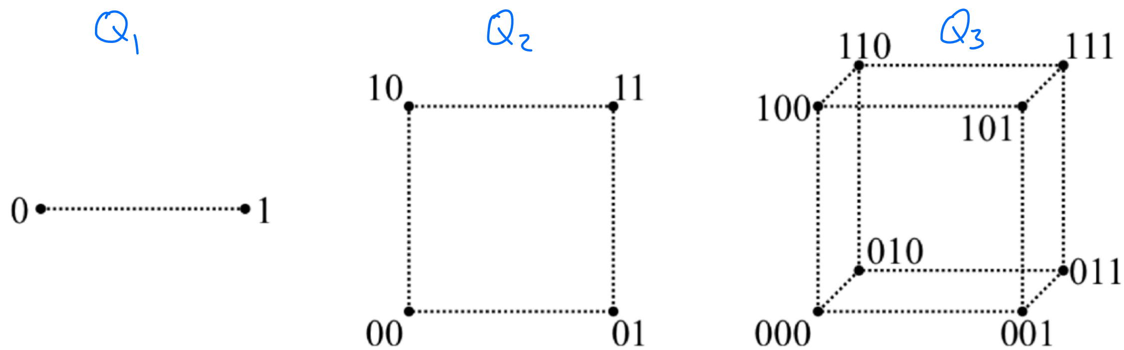

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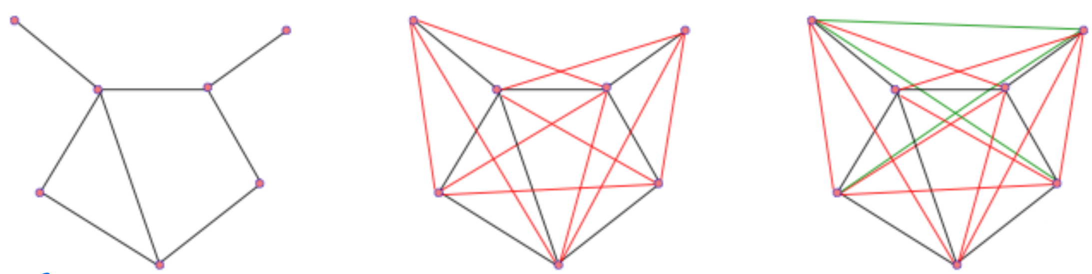
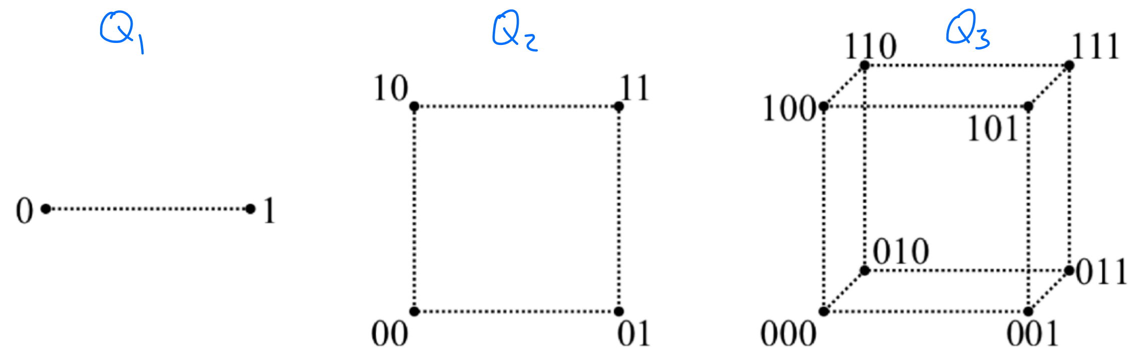


$G = VR(G;1)$ $\quad$ $VR(G;2)$ $\quad$ $VR(G;3)$

<u>Def</u> Let Q_n be the 2^n vertices of the hypercube graph, equipped with the shortest path metric.



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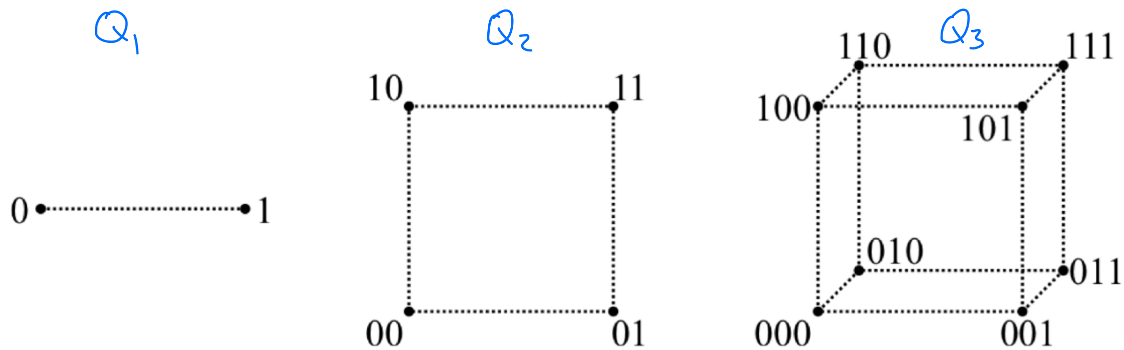

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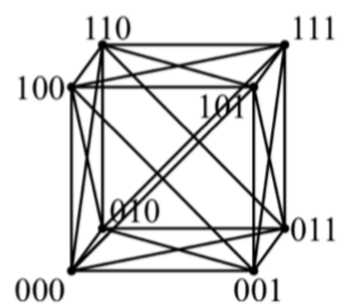
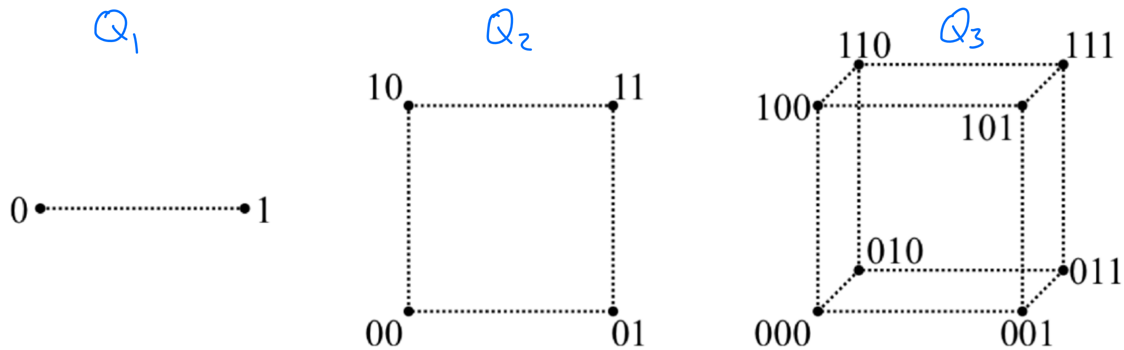
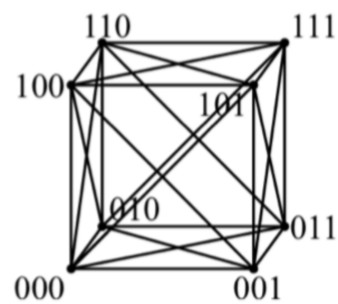
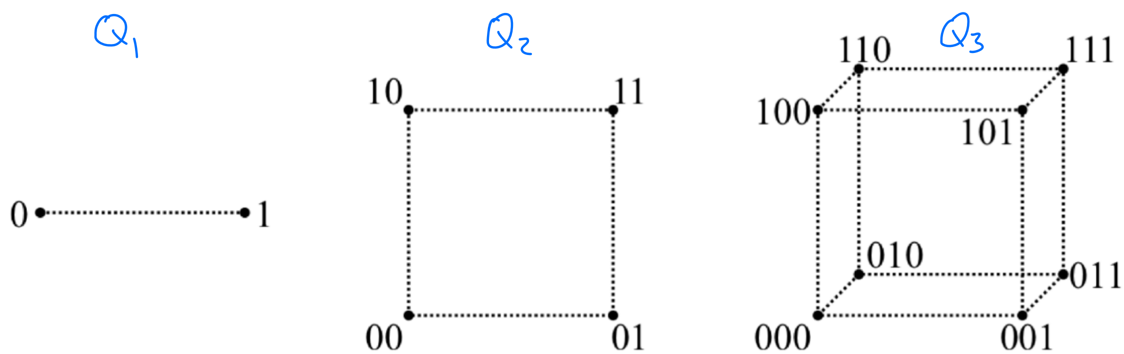




Homotopy types of $VR(Q_n; r)$

	$n=1$	2	3	4	5	6	7	8	9
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
1	$*$	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
2	$*$	$*$	S^3	$\bigvee^9 S^3$	$\bigvee^{49} S^3$	$\bigvee^{209} S^3$	$\bigvee^{769} S^3$	$\bigvee^{2561} S^3$	$\bigvee^{7937} S^3$
3	$*$	$*$	$*$	S^7					
4	$*$	$*$	$*$	$*$	S^{15}				
5	$*$	$*$	$*$	$*$	$*$	S^{31}			
6	$*$	$*$	$*$	$*$	$*$	$*$	S^{63}		
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{255}





Homotopy types of $VR(Q_n;r)$

	$n=1$	2	3	4	5	6	7	8	9
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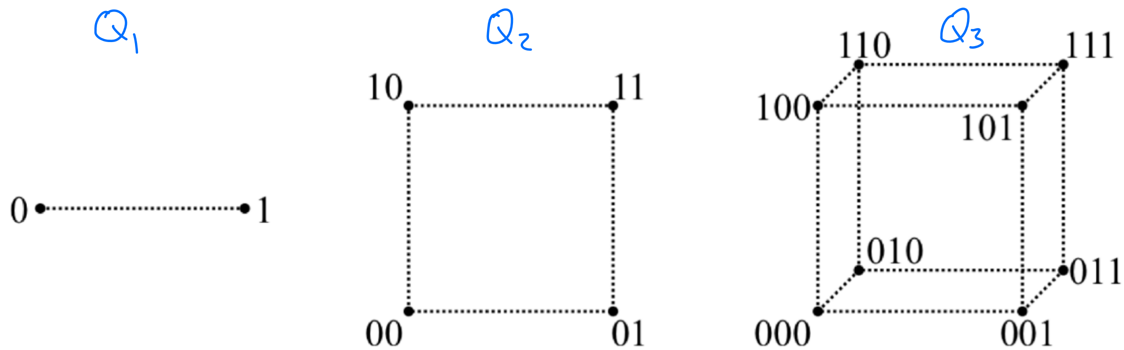
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{255}

<u>Main Theorem</u> $VR(Q_n;2) \simeq \bigvee^{c_n} S^3$ where

$c_n =$ recovers the values

$0, 0, 1, 9, 49, 209, 769, 2561, 7937, 23297, 65537, \ldots$



Homotopy types of $VR(Q_n; r)$

	$n=1$	2	3	4	5	6	7	8	9
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
1	$*$	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
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5	$*$	$*$	$*$	$*$	$*$	S^{31}			
6	$*$	$*$	$*$	$*$	$*$	$*$	S^{63}		
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{255}

<u>Main Theorem</u> $VR(Q_n; 2) \simeq \bigvee^{c_n} S^3$ where

$$c_n = \sum_{0 \le j < i < n} (j+1)(2^{n-2} - 2^{i-1})$$ recovers the values

0, 0, 1, 9, 49, 209, 769, 2561, 7937, 23297, 65537, ...

Homotopy types of $VR(Q_n; r)$

	$n=1$	2	3	4	5	6	7	8	9
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
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7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
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$H_i(\mathrm{VR}(Q_5;3);\mathbb{Z}) \cong \mathbb{Z}$ for $i=4$, $\cong \mathbb{Z}^{10}$ for $i=7$,

$H_i(\mathrm{VR}(Q_6;3);\mathbb{Z}) \cong \mathbb{Z}^{11}$ for $i=4$, $\cong \mathbb{Z}^{60}$ for $i=7$,

$H_i(\mathrm{VR}(Q_7;3);\mathbb{Z}) \cong \mathbb{Z}^{71}$ for $i=4$, $\cong \mathbb{Z}^{280}$ for $i=7$,

Polymake

$H_i(\mathrm{VR}(Q_8;3);\frac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{351}$ for $i=4$, $\cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{1120}$ for $i=7$,

$H_i(\mathrm{VR}(Q_9;3);\frac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{1471}$ for $i=4$, $\cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{4032}$ for $i=7$,

Simon Zhang

Ripser ++

Homotopy types of $VR(Q_n; r)$

	$n=1$	2	3	4	5	6	7	8	9
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3	$*$	$*$	$*$	S^7	$S^4 \vee \bigvee^{10} S^7$	$\bigvee^{11} S^4 \vee \bigvee^{60} S^7$	$\bigvee^{71} S^4 \vee \bigvee^{280} S^7$	$\bigvee^{351} S^4 \vee \bigvee^{1120} S^7$	$\bigvee^{1471} S^4 \vee \bigvee^{4032} S^7$
4	$*$	$*$	$*$	$*$	S^{15}				
5	$*$	$*$	$*$	$*$	$*$	S^{31}			
6	$*$	$*$	$*$	$*$	$*$	$*$	S^{63}		
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
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$$
\begin{aligned}
H_i(\mathrm{VR}(Q_5;3);\mathbb{Z}) &\cong \mathbb{Z} \text{ for } i=4, && \cong \mathbb{Z}^{10} \text{ for } i=7,\\
H_i(\mathrm{VR}(Q_6;3);\mathbb{Z}) &\cong \mathbb{Z}^{11} \text{ for } i=4, && \cong \mathbb{Z}^{60} \text{ for } i=7,\\
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\end{aligned}
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Polymake

$$
\begin{aligned}
H_i(\mathrm{VR}(Q_8;3);\tfrac{\mathbb{Z}}{2\mathbb{Z}}) &\cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{351} \text{ for } i=4, && \cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{1120} \text{ for } i=7,\\
H_i(\mathrm{VR}(Q_9;3);\tfrac{\mathbb{Z}}{2\mathbb{Z}}) &\cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{1471} \text{ for } i=4, && \cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{4032} \text{ for } i=7,
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Simon Zhang

Ripser ++

Homotopy types of $VR(Q_n; r)$

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$$H_i(\mathrm{VR}(Q_5;3);\mathbb{Z}) \cong \mathbb{Z} \text{ for } i=4, \quad \cong \mathbb{Z}^{10} \text{ for } i=7,$$
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$$H_i(\mathrm{VR}(Q_7;3);\mathbb{Z}) \cong \mathbb{Z}^{71} \text{ for } i=4, \quad \cong \mathbb{Z}^{280} \text{ for } i=7,$$

Polymake

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Simon Zhang
Ripser ++

<u>Question</u> Is $VR(Q_n; r)$ always a wedge of spheres?

Homotopy types of $\mathrm{VR}(Q_n;r)$

	$n=1$	2	3	4	5	6	7	8	9
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
1	$*$	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
2	$*$	$*$	S^3	$\bigvee^9 S^3$	$\bigvee^{49} S^3$	$\bigvee^{209} S^3$	$\bigvee^{769} S^3$	$\bigvee^{2561} S^3$	$\bigvee^{7937} S^3$
3	$*$	$*$	$*$	S^7	$S^4 \vee \bigvee^{10} S^7$	$\bigvee^{11} S^4 \vee \bigvee^{60} S^7$	$\bigvee^{71} S^4 \vee \bigvee^{280} S^7$	$\bigvee^{351} S^4 \vee \bigvee^{1120} S^7$	$\bigvee^{1471} S^4 \vee \bigvee^{4032} S^7$
4	$*$	$*$	$*$	$*$	S^{15}				
5	$*$	$*$	$*$	$*$	$*$	S^{31}			
6	$*$	$*$	$*$	$*$	$*$	$*$	S^{63}		
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{255}

$$
\begin{aligned}
H_i(\mathrm{VR}(Q_5;3);\mathbb{Z}) &\cong \mathbb{Z} \text{ for } i=4, & &\cong \mathbb{Z}^{10} \text{ for } i=7,\\
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Polymake

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H_i(\mathrm{VR}(Q_8;3);\tfrac{\mathbb{Z}}{2\mathbb{Z}}) &\cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{351} \text{ for } i=4, & &\cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{1120} \text{ for } i=7,\\
H_i(\mathrm{VR}(Q_9;3);\tfrac{\mathbb{Z}}{2\mathbb{Z}}) &\cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{1471} \text{ for } i=4, & &\cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{4032} \text{ for } i=7,
\end{aligned}
$$

Simon Zhang

Ripser++

<u>Question</u> Is $\mathrm{VR}(Q_n;r)$ always a wedge of spheres?

$2^{n-4}\binom{n}{4}$ is the # of Q_4 subgraphs in Q_n

Homotopy types of $VR(Q_n; r)$

	$n=1$	2	3	4	5	6	7	8	9
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
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4	$*$	$*$	$*$	$*$	S^{15}				
5	$*$	$*$	$*$	$*$	$*$	S^{31}			
6	$*$	$*$	$*$	$*$	$*$	$*$	S^{63}		
7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{127}	
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	S^{255}

$$H_i(\mathrm{VR}(Q_5;3);\mathbb{Z}) \cong \mathbb{Z} \text{ for } i=4, \quad \cong \mathbb{Z}^{10} \text{ for } i=7,$$
$$H_i(\mathrm{VR}(Q_6;3);\mathbb{Z}) \cong \mathbb{Z}^{11} \text{ for } i=4, \quad \cong \mathbb{Z}^{60} \text{ for } i=7,$$
$$H_i(\mathrm{VR}(Q_7;3);\mathbb{Z}) \cong \mathbb{Z}^{71} \text{ for } i=4, \quad \cong \mathbb{Z}^{280} \text{ for } i=7,$$

Polymake

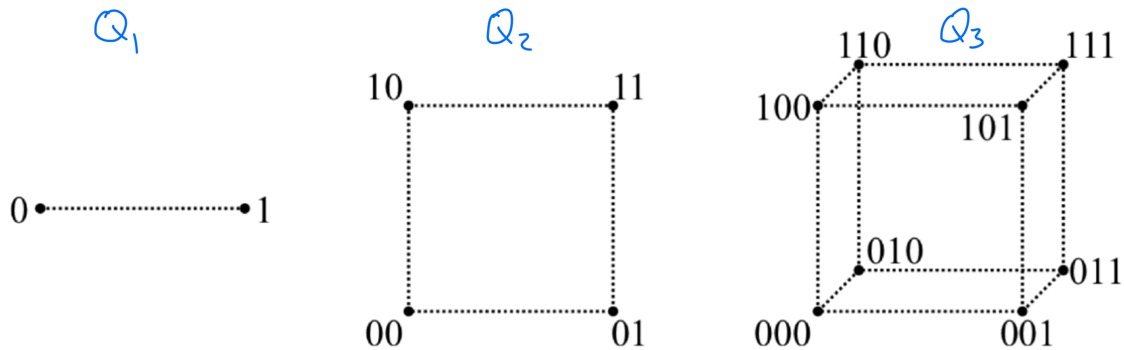
$$H_i(\mathrm{VR}(Q_8;3);\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{351} \text{ for } i=4, \quad \cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{1120} \text{ for } i=7,$$
$$H_i(\mathrm{VR}(Q_9;3);\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{1471} \text{ for } i=4, \quad \cong (\tfrac{\mathbb{Z}}{2\mathbb{Z}})^{4032} \text{ for } i=7,$$

Simon Zhang Ripser++

<u>Question</u> Is $VR(Q_n;r)$ always a wedge of spheres?

4-dim'l skeleton of n-cube polytope is $\simeq$ to $\left(\sum_{i=4}^{n-1} 2^{i-4}\binom{i}{4}\right)$-fold wedge sum of 4-spheres

$2^{n-4}\binom{n}{4}$ is the # of Q_4 subgraphs in Q_n



Homotopy types of $VR(Q_n; r)$

	$n=1$	2	3	4	5	6	7	8	9
$r=0$	S^0	$\bigvee^3 S^0$	$\bigvee^7 S^0$	$\bigvee^{15} S^0$	$\bigvee^{31} S^0$	$\bigvee^{63} S^0$	$\bigvee^{127} S^0$	$\bigvee^{255} S^0$	$\bigvee^{511} S^0$
1	$*$	S^1	$\bigvee^5 S^1$	$\bigvee^{17} S^1$	$\bigvee^{49} S^1$	$\bigvee^{129} S^1$	$\bigvee^{321} S^1$	$\bigvee^{769} S^1$	$\bigvee^{1793} S^1$
2	$*$	$*$	S^3	$\bigvee^9 S^3$	$\bigvee^{49} S^3$	$\bigvee^{209} S^3$	$\bigvee^{769} S^3$	$\bigvee^{2561} S^3$	$\bigvee^{7937} S^3$
3	$*$	$*$	$*$	S^7	$S^4 \vee \bigvee^{10} S^7$	$\bigvee^{11} S^4 \vee \bigvee^{60} S^7$	$\bigvee^{71} S^4 \vee \bigvee^{280} S^7$	$\bigvee^{351} S^4 \vee \bigvee^{1120} S^7$	$\bigvee^{1471} S^4 \vee \bigvee^{4032} S^7$
4	$*$	$*$	$*$	$*$	S^{15}				
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6	$*$	$*$	$*$	$*$	$*$	$*$	S^{63}		
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<u>Main Theorem</u> $VR(Q_n; 2) \simeq \bigvee^{c_n} S^3$

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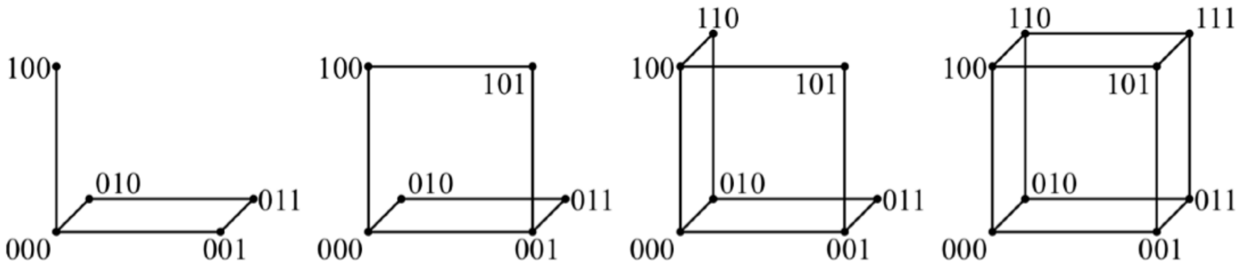
Idea 1:

<u>Main Theorem</u> $VR(Q_n;2) \simeq \bigvee^{c_n} S^3$

Idea 1: Add vertices one at a time



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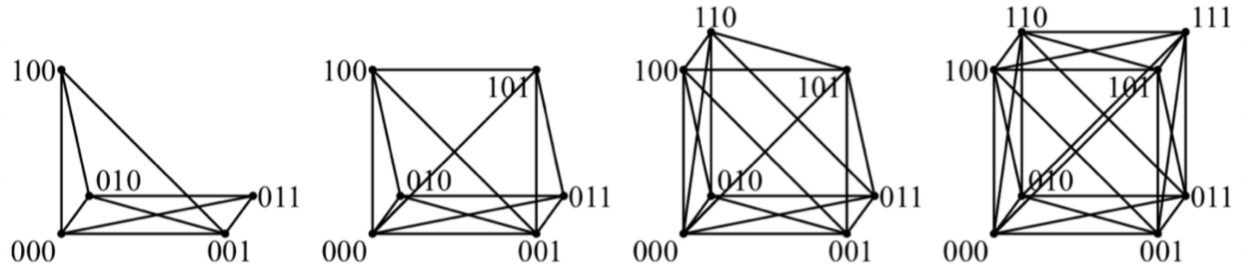

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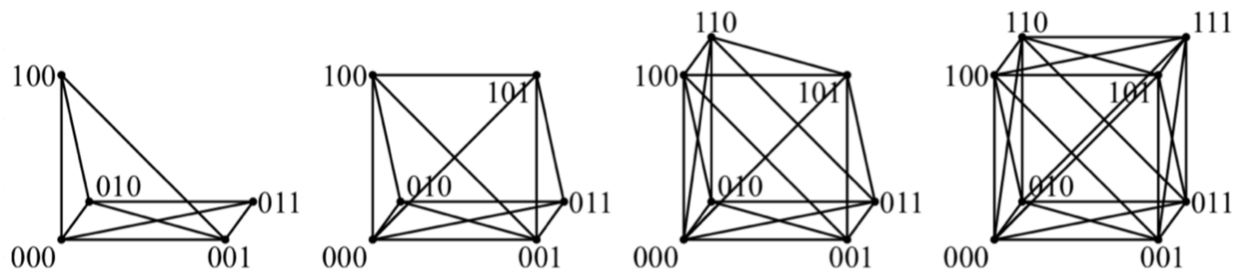
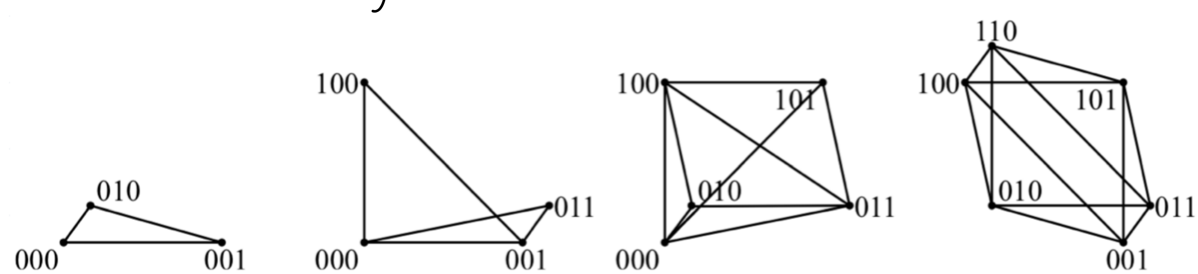


Idea 2: Study links



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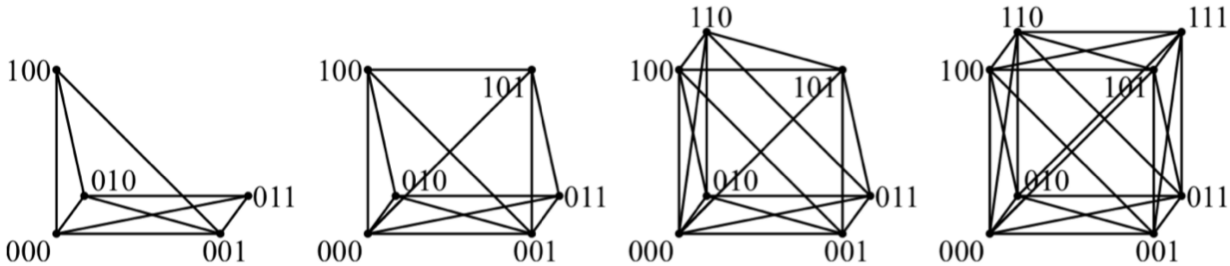
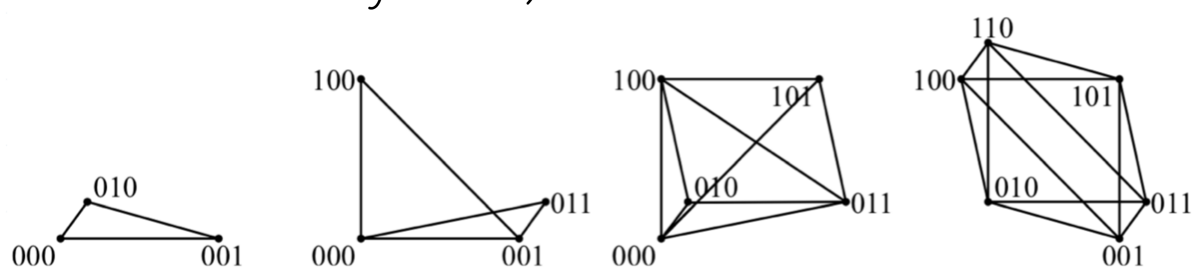


Idea 2: Study links, show links $\simeq \bigvee^{?} S^2$



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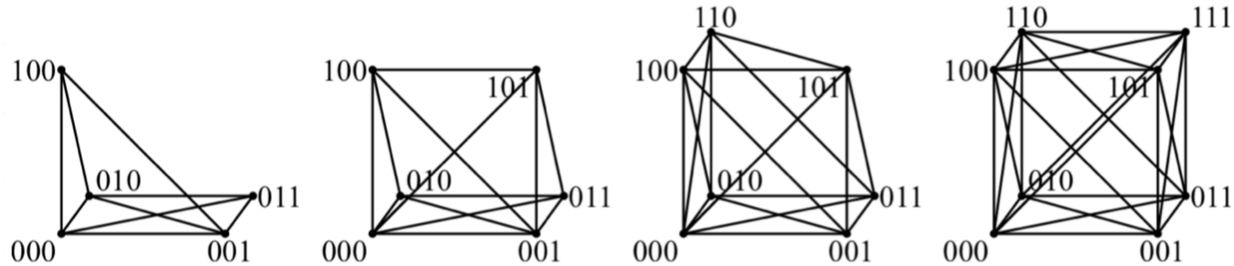

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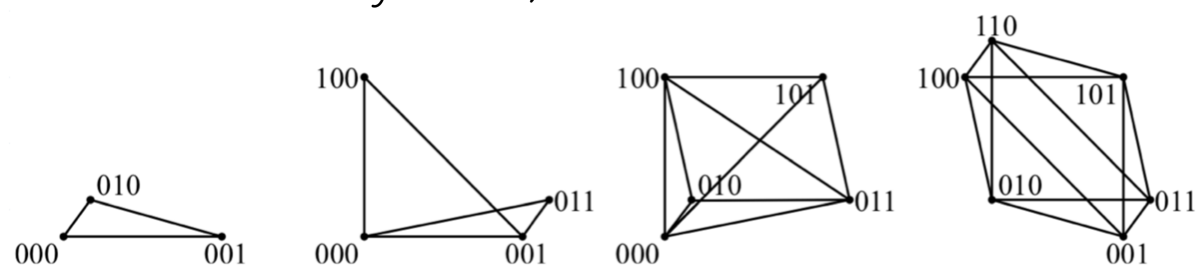
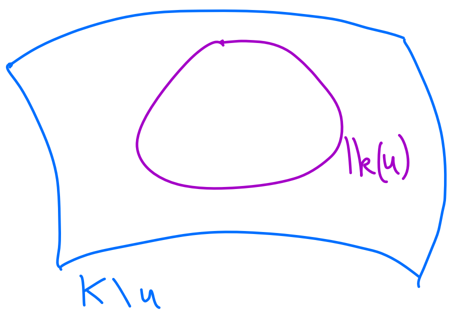


Idea 3: Induct on # vertices, using splitting lemma.

Splitting lemma



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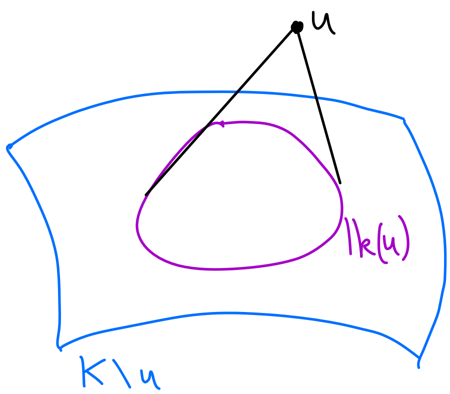

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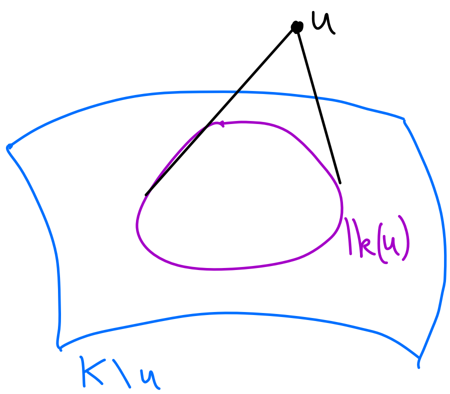

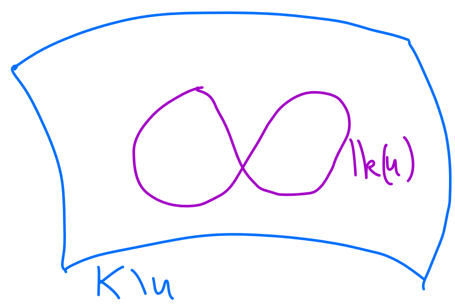

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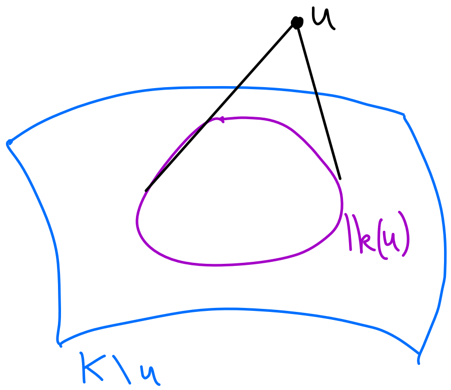
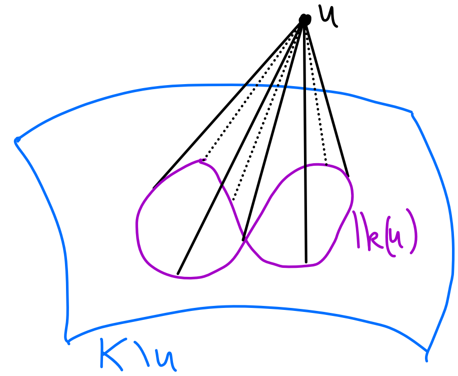
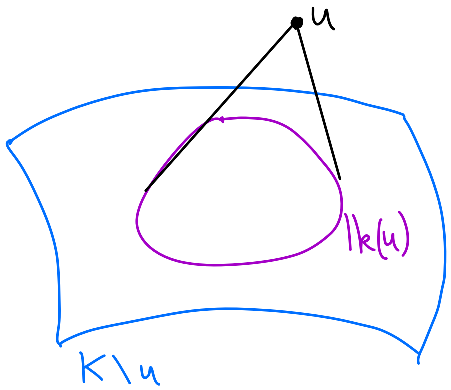
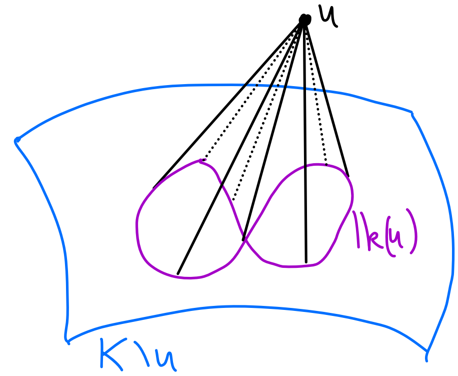




<u>Splitting lemma</u> Let K be a simplicial complex and $u \in K$ a vertex.
If $\mathrm{lk}(u) \hookrightarrow K \setminus u$ is a null-homotopy, then $K \simeq (K \setminus u) \vee \Sigma\, \mathrm{lk}(u)$.





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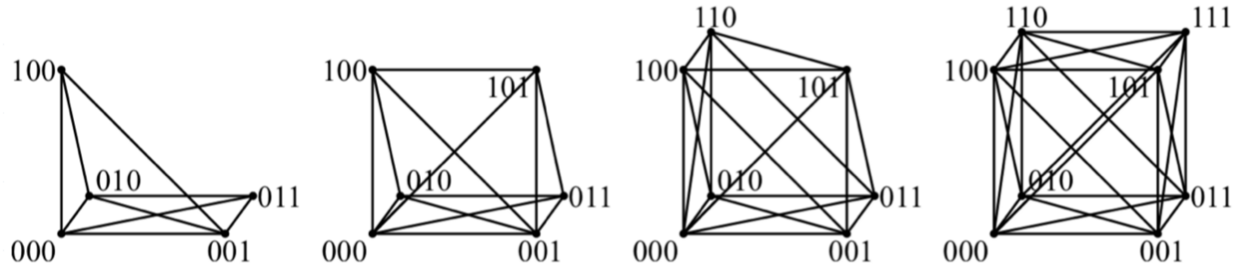

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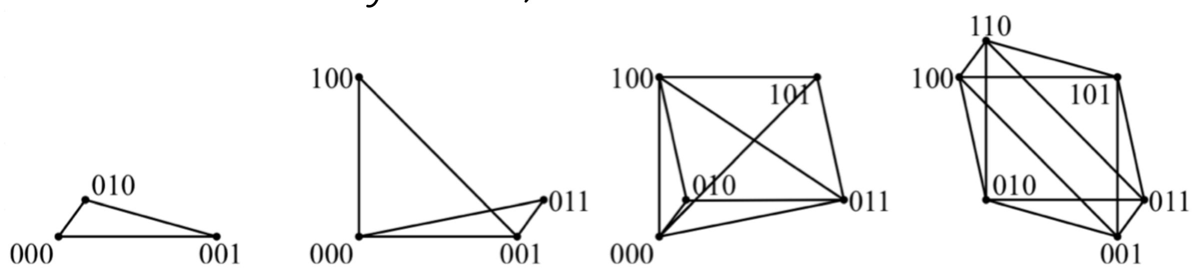

Idea 3: Induct on # vertices, using splitting lemma.

Hypothesis: $lk(u) \hookrightarrow K \setminus u$ is a null-homotopy.
Conclusion: $K \simeq (K \setminus u) \vee \Sigma\, lk(u)$

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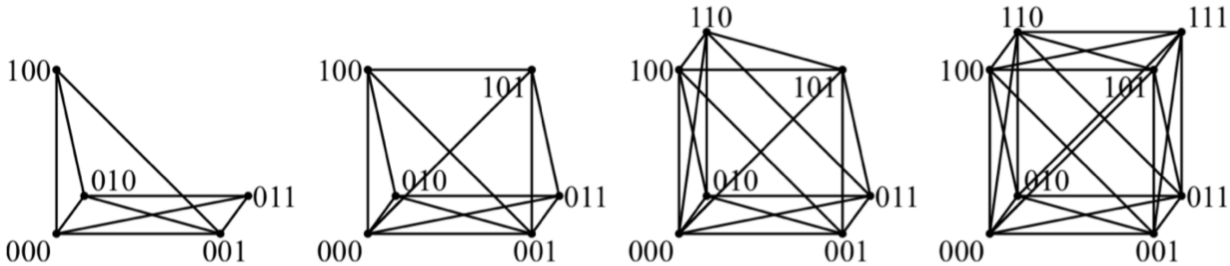

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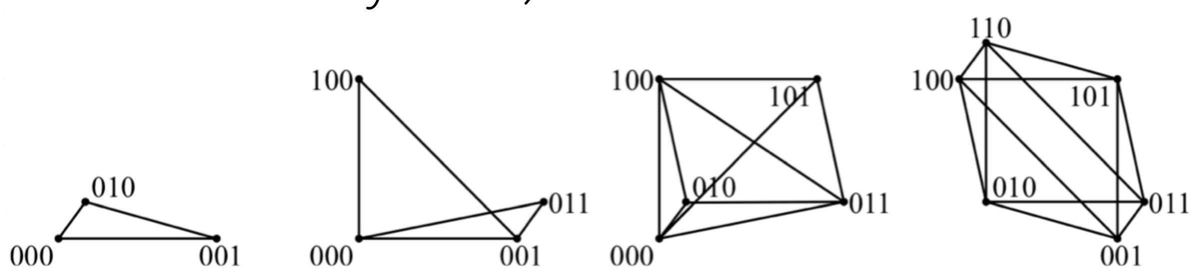

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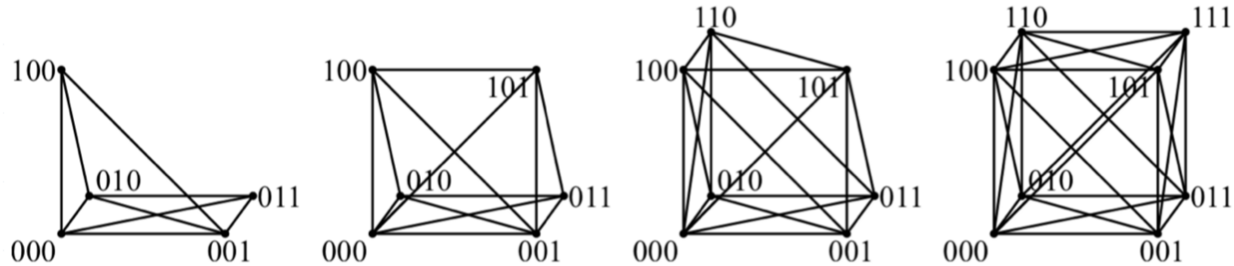

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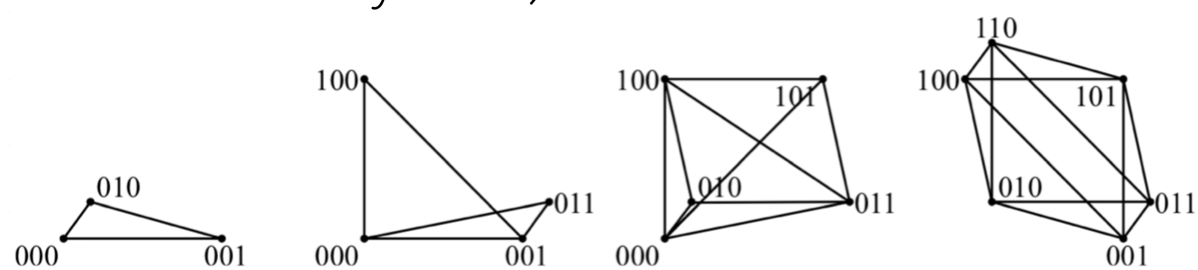

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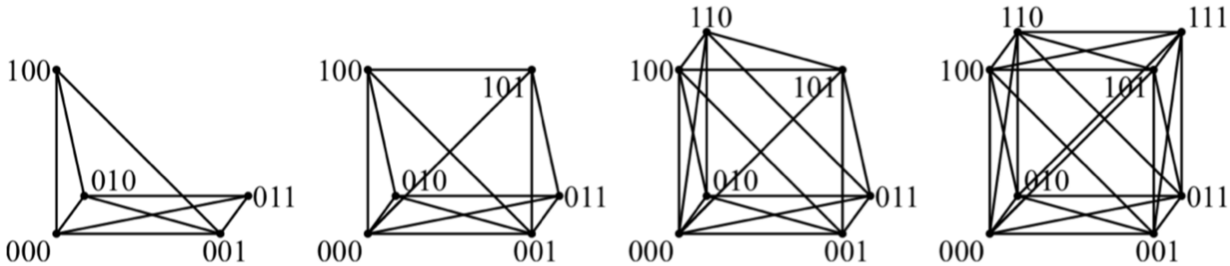

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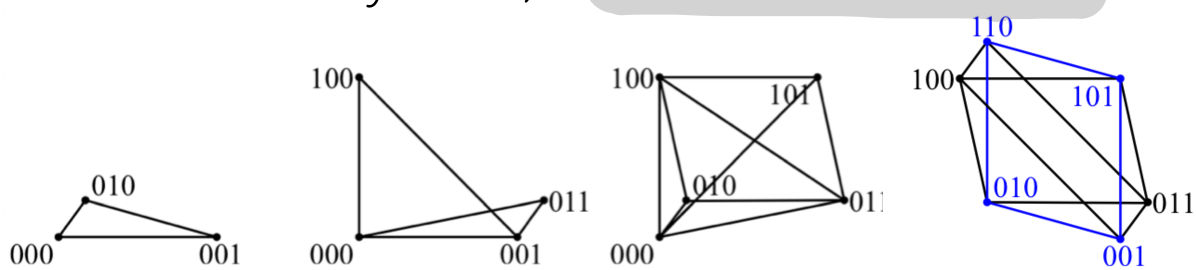

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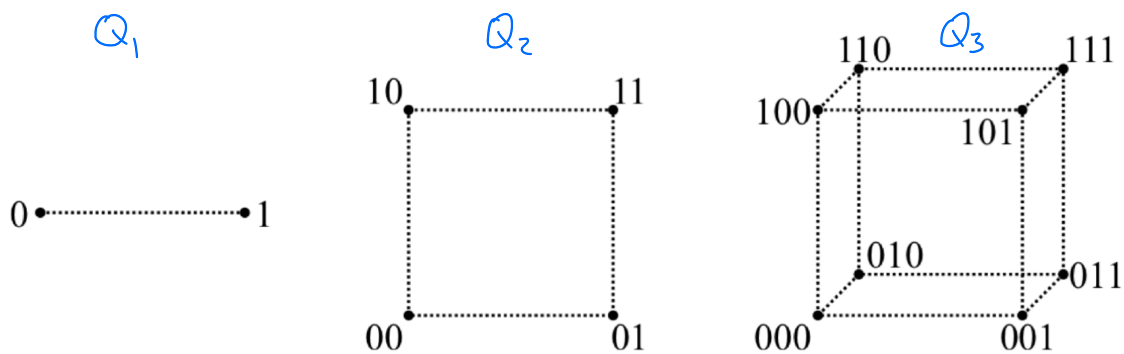

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Question Is $VR(Q_n; r)$ always a wedge of spheres?

Thanks to the organizers

Massimo Ferri
Vidit Nanda
Jie Wu
Guowei Wei
Kelin Xia

for a fantastic workshop!