Title: Neighborly polytopes and the sparsity-promoting $\ell^1$ norm

References: Donoho technical report, 2005
Donoho & Tanner, PNAS, 2005

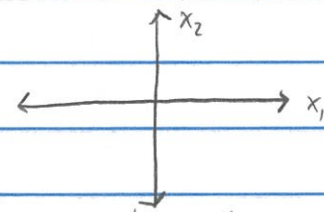
Fix $A \in \mathbb{R}^{d \times n}$, $y \in \mathbb{R}^d$ with $d < n$. Let $x \in \mathbb{R}^n$.

(0) $\min \|x\|_0$ subject to $y = Ax$

\# nonzero entries — underdetermined

This is NP-hard; the $\ell^0$ ball is not convex

Combinatorial optimization problems Knapsack, satisfiability as special cases.



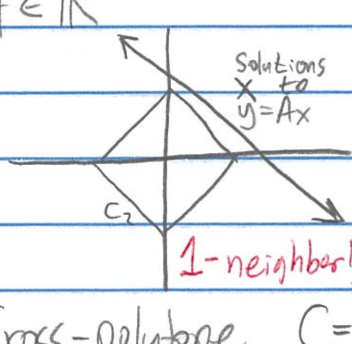

$\{x \in \mathbb{R}^n \mid \|x\|_0 \le 1\}$

$\mathbb{R}^3$ 1, 2

Consider the convex relaxation

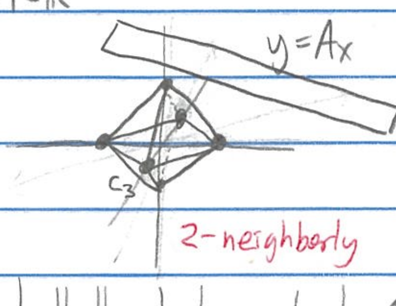
(1) $\min \|x\|_1$ subject to $y = Ax$

<u>Pics</u> $A \in \mathbb{R}^{1\times 2}$



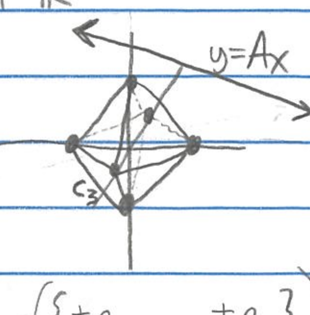

1-neighborly

$A \in \mathbb{R}^{1\times 3}$




2-neighborly

$A \in \mathbb{R}^{2\times 3}$




Cross-polytope $C_n = \{x \in \mathbb{R}^n \mid \|x\|_1 = |x_1| + \dots + |x_n| \le 1\} = \mathrm{Conv}(\{\pm e_1, \dots, \pm e_n\})$

Inflate ball until first intersection.
Solution is often sparse!

- Platonic solid in all dimension
- Boundary is minimal homolog generator for clique complex

"Equivalence" of $\ell^0$ and $\ell^1$ optimization

Important Corollary:

The overwhelming majority of $A \in \mathbb{R}^{n \times d}$, $y \in \mathbb{R}^d$ (A a random orthogonal projection) with $n$ large (larger than pictures on prior page) and $d = \lfloor 0.7\, n \rfloor$ have the property that if $x$ is a solution to (0) with less than $0.49\underline{d}$ nonzeros, then $x$ is also the unique solution to (1).

Main Theorem $A \in \mathbb{R}^{d \times n}$ with $d < n$. Then

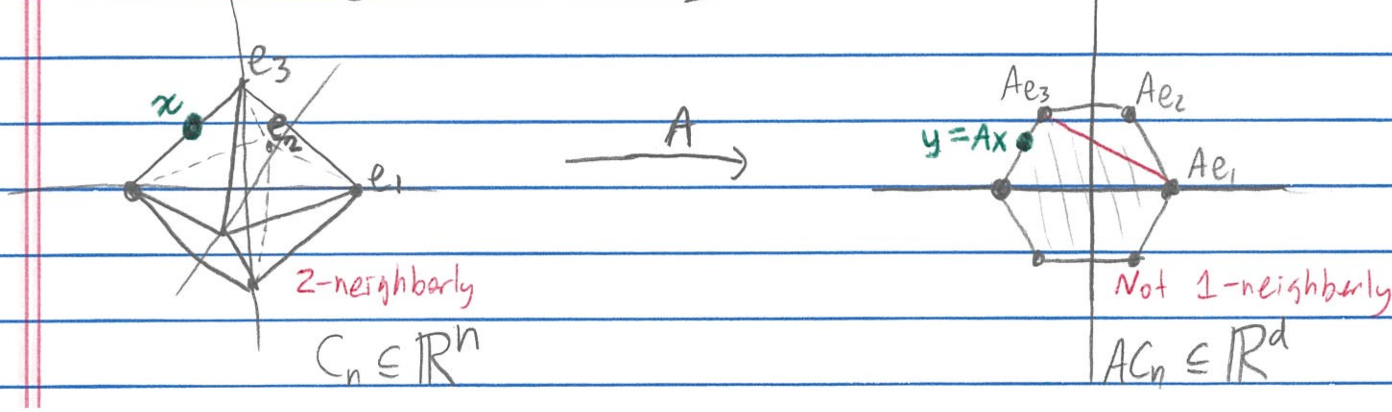
- the polytope $AC_n$ has $2n$ vertices and is $k$-neighborly

$\Longleftrightarrow$

- whenever $y = Ax$ has a solution $x$ with at most $k+1$ nonzeros, $x$ is the unique solution to (1).

Here $AC_n = \mathrm{Conv}(\{\pm Ae_1, \ldots, \pm Ae_n\}) = \mathrm{Conv}(\{\pm \text{ each column of } A\})$

Ex $A = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & \sqrt{3}/2 & \sqrt{3}/2 \end{bmatrix}$



Note $C_n$ and $AC_n$ are centrally symmetric polytopes, meaning $C_n = -C_n$ and $AC_n = -AC_n$, i.e. reflecting through the origin leaves them unchanged.

Def A centrally symmetric polytope is $k$-neighborly if any collection of $k+1$ vertices not including an antipodal pair form a face.

Ex The cross-polytope $C_n$ in $\mathbb{R}^n$ is $(n-1)$-neighborly.
Ex $AC_n$ will often be $k$-neighborly for $k$ relatively large, especially if $A$ is a special matrix (Fourier, partial Vandermonde, augmented Hadamard, incoherent dictionary, signal processing, error correcting codes).
(The corollary compares the expected # of faces of $AC_n$ to those of $C_n$.)

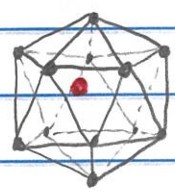
Rmk Theorem allows us to construct neighborly polytopes from known nice matrices, and to construct nice matrices from know neighborly polytopes.

Rmk Convex hull of symmetric points from $(\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$ is known to be "neighborly". Connected to MDS of circle (Kassab, Blumstein), Vietoris-Rips of circle, and Borsuk-Ulam theorems into higher-dimensional codomains (Bush, Frick).

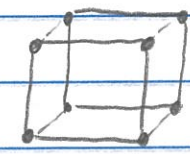
Rmk Face numbers of polytopes!

## Proof of ($\Longrightarrow$) in Main Theorem

<u>Fact</u> k-neighborly polytopes for $k \geq 1$ are <u>simplicial</u>
(all faces are simplices).



Simplicial
Not 1-neighborly



Not simplicial

<u>Lemma 1</u> If $y$ is a point in a face of a simplicial polytope, then $y$ has a unique representation as a convex combination of vertices, which all belong to the face.

<u>Lemma 2</u> If $AC_n$ has $2n$ vertices and is k-neighborly, then $F$ is an $i$-face of $C_n$
$\Longleftrightarrow$ $AF$ is an $i$-face of $AC_n$
for all $0 \leq i \leq k$

## Proof of ($\Longrightarrow$) in Main Theorem

Suppose $x \in \mathbb{R}^n$ has at most $k+1$ nonzeros.
So $x$ is in a $k$-face of a scaled $C_n$.
By Lemma 2, $y = Ax$ is in a $k$-face of a scaled $AC_n$;
so $x$ is a solution to (1).
Furthermore, by Lemma 1 $x$ is the unique solution to (1).
(So (1) "magically" finds the solution to (0)!)