An Introduction to Matroids

Hassler Whitney, 1935, unifying abstract treatment of independence in linear algebra, graph theory, combinatorics, algebra (independent field extensions).

Following "What is a matroid?" by James Oxley.

Def A matroid $M=(E,\mathcal{I})$ is a finite set E and a collection $\mathcal{I}$ of ("independent") subsets of E s.t.

nonempty (i) $\emptyset \in \mathcal{I}$

hereditary (ii) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$

augmentation (iii) if $X, Y \in \mathcal{I}$ and $|X| = |Y|+1$, then $\exists\, x \in X \setminus Y$ s.t. $Y \cup \{x\} \in \mathcal{I}$.

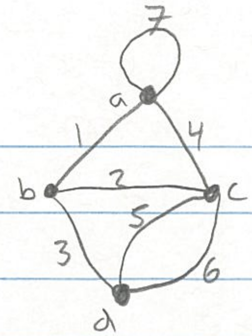
Rmk All maximal sets in $\mathcal{I}$ have the same size, the <u>rank</u>. (pure simplicial complex)

Ex Matrix

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$E = \{1,2,\dots,7\}$. $\mathcal{I}$ is the collection of linearly independent sets of vectors over $\mathbb{R}$, i.e. any collection of ≤ 3 vectors in E not containing $\{7\}$, $\{5,6\}$, $\{1,2,4\}$, $\{2,3,5\}$, or $\{2,3,6\}$.

This <u>vector matroid</u> is denoted $M[A]$ or $M[A,\mathbb{R}]$.
(works for any matrix and field)

Ex Graph $G =$ 

$E = \{1, 2, \ldots, 7\}$. I is the collection of cycle-free subsets, i.e. those subsets not contaming $\{7\}$, $\{5,6\}$, $\{1,2,4\}$, $\{2,3,5\}$, $\{2,3,6\}$, $\{1,3,4,5\}$, or $\{1,3,4,6\}$.

This <u>cycle matroid</u> is denoted $M(G)$. Works for any graph: maximal independent sets are spanning trees or forrests.

"Accidentally," $M[A] = M(G)$ here.

Rmk Matroids can also be defined in terms of bases, circuits (minimal dependent sets), rank functions, or closure operations.

$$\left(cl(X) = \{e \in E : \rho(X \cup \{e\}) = \rho(X)\}\right)$$

Representability Questions

Thm For G a graph and $\mathbb{F}$ a field, $M(G) = M[A_G, \mathbb{F}]$ where A_G is the "signed adjacency matrix" of G.

Ex For G above,

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \end{bmatrix} \end{array}.$$

($+1$ always above -1)

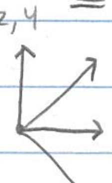
Question Which matroids are representable via a matrix (i.e. $M = M[A, \mathbb{F}]$) over

- some field $\mathbb{F}$?
- $\mathbb{F}_2$ (binary) or $\mathbb{F}_3$ (ternary)?
- all fields (regular)?

Ex The above theorem says that cycle matroids are regular.

Def For $|E| = n$ and $0 \le r \le n$, the <u>uniform matroid</u> $U_{r,n} = (E, \mathcal{I})$ has $\mathcal{I} = \{\text{all subsets of } E \text{ of size} \le r\}$.

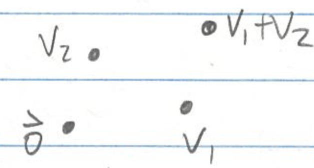
Ex $U_{2,4} = M\left[\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}, \mathbb{F}_3\right]$ is ternary.



Could be any field besides $\mathbb{F}_2$

Ex $U_{2,4}$ is not binary.

Pf Suppose for a contradiction $U_{2,4} = M[A, \mathbb{F}_2]$.
$U_{2,4}$ has rank 2
$\Rightarrow$ $\text{span}_{\mathbb{F}_2}(A)$ is 2-dimensional, i.e., has three nonzero vectors

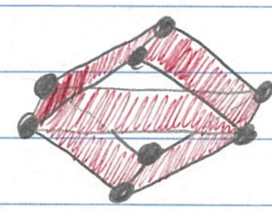


So A cannot have 4 distinct nonzero columns, a contradiction.

Ex Neither $U_{2,5}$ nor $U_{3,5}$ are ternary.

Ex $U_{3,6}$ is representable over $\mathbb{F}_4 = \{0, 1, w, w+1\}$ with $w^2 = w+1$, $2=0$.

Ex Vámos matroid



$E = 8$ vertices. Rank 4.
Of the $\binom{8}{4} = 70$ 4-element subsets, all but the 5 drawn in red are independent.
Not representable over any field.

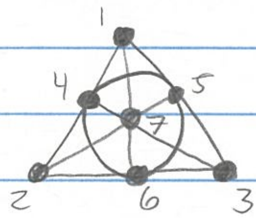
Thm A matroid is regular $\Leftrightarrow$ it is binary and ternary.

Thm A matroid is binary $\Leftrightarrow$ it has no minor isomorphic to $U_{2,4}$.

This is analogous to the theorem that a graph is planar $\Leftrightarrow$ it has no minor isomorphic to K_5 or to $K_{3,3}$.

Thm A matroid is ternary $\Longleftrightarrow$ it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, F_7, or F_7^*.
(dual)

Fano matroid F_7

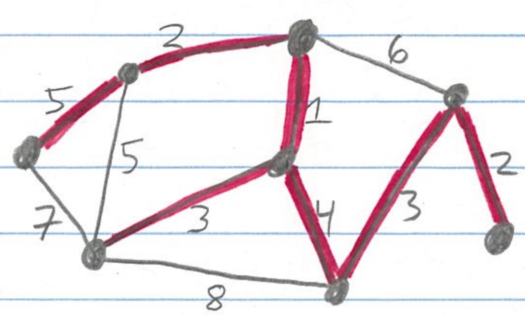


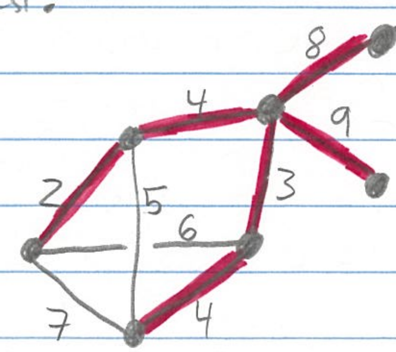
$E = \{1, 2, \ldots, 7\}$
$\mathcal{I}$ = subsets of size ≤ 3 not containing a line.

Combinatorial Optimization

Matroid $M = (E, \mathcal{I})$.
Given a function $w: E \to \mathbb{R}_{\geq 0}$, how do we find some $X \in \mathcal{I}$ with $w(X) := \sum_{e \in X} w(e)$ maximal?

Ex Given a graph with weighted edges, Kruskal's algorithm is a greedy algorithm that finds a minimal weight spanning tree or spanning forrest.





The greedy algorithm for solving combinatorial optimization problems in matroids

Step 1 Initialize $X = \emptyset$ $(\in I)$

Step 2 While $\exists$ $e \notin X$ for which $X \cup \{e\} \in I$, choose such an e with $w(e)$ maximal, and replace X with $X \cup \{e\}$.

Thm An equivalent definition of a matroid is obtaned by replacing (iii) with (iii'):

(iii') For all positive real weight functions $w: E \to \mathbb{R}_{>0}$, the greedy algorithm produces a maximal weight member of I.

So matroids can be described as combinatorial objects on which the greedy algorithm works!