CIMAT, January 2020

Geometric complexes in applied topology

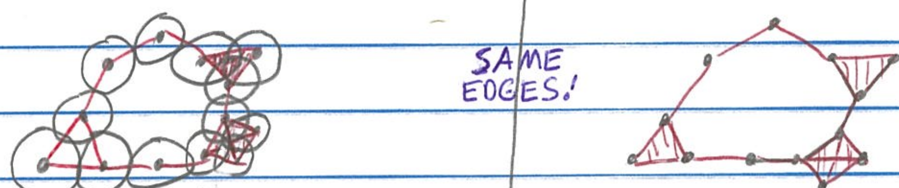
I. Vietoris-Rips complexes

II. Metric thickenings

III. Borsuk-Ulam theorems

I. Vietoris-Rips complexes

First properties



Def For $X \subseteq \mathbb{R}^n$ and $r>0$, the Čech simplicial complex $\check{C}(X;r)$ has

- vertex set X
- $\{x_0, x_1, \ldots, x_k\}$ as a simplex when $\bigcap_{i=0}^{k} \overline{B(x_i; r/2)} \neq \emptyset$.

Advantage: Nerve lemma. $\Rightarrow \check{C}(X;r) \simeq$ union of balls

Disadvantage: Hard to compute for $X \subseteq \mathbb{R}^n$ with n large.

Def For X a metric space and $r>0$, the Vietoris-Rips simplicial complex $VR(X;r)$ has

- vertex set X
- $\{x_0, x_1, \ldots, x_k\}$ as a simplex when diameter$(\{x_0, \ldots, x_k\}) \leq r$.

It's a clique or flag simplicial complex

Disadvantage: No nerve lemma.

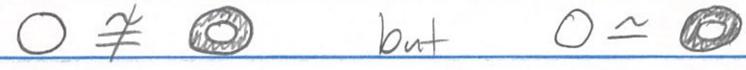
Advantage: Easier to compute. Used in practice for this reason!

"preserves dimension"

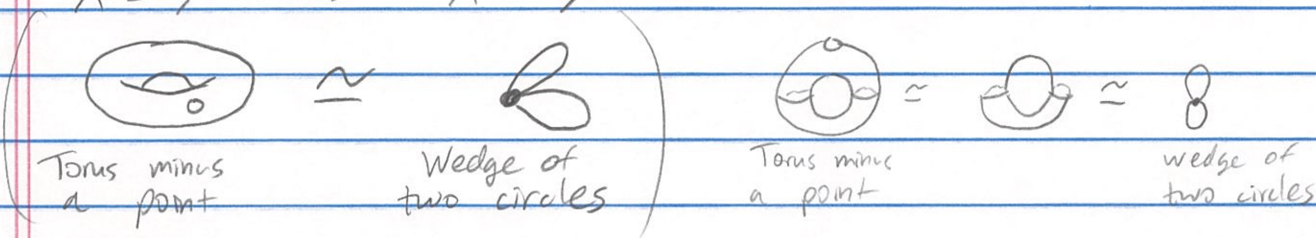
Let X and Y be topological spaces.
They're <u>homeomorphic</u> ($X \cong Y$) or
<u>homotopy equivalent</u> ($X \simeq Y$) if they
"have the same shape".

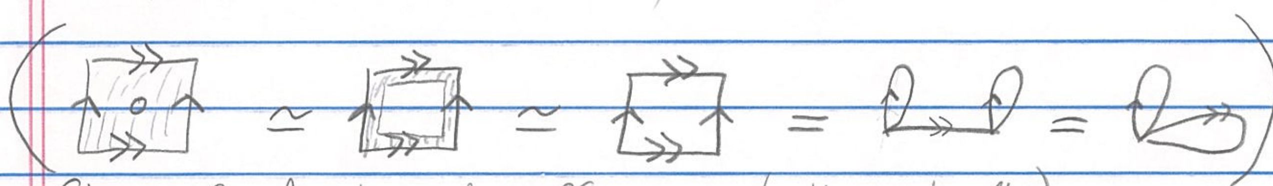
"need not preserve dimension"

<u>Ex</u> 

<u>Ex</u> 

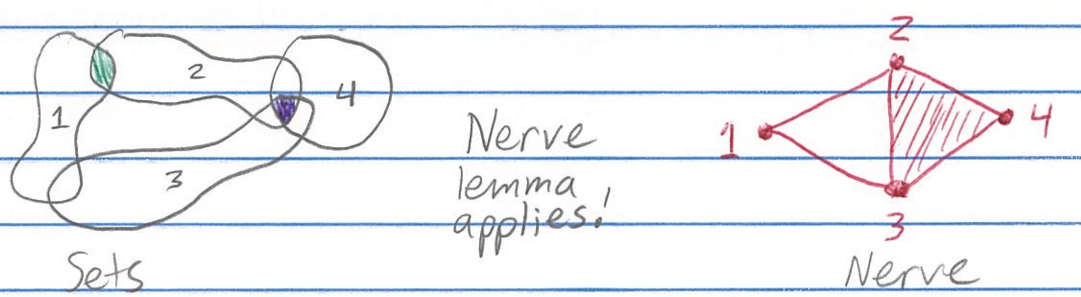
<u>Prop</u> $X \cong Y \implies X \simeq Y$

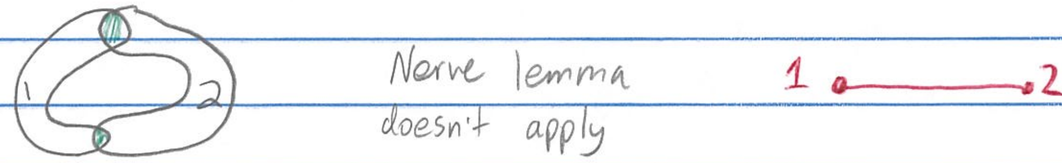
<u>Ex</u> 



<u>Ex</u> Story of donut and coffee cup (with no handle).

<u>Nerve lemma</u> If you have a collection of "nice enough" sets, in which all intersections are either empty or contractible, then the <u>nerve</u> of the sets is homotopy equivalent to their union.

<u>Ex</u> 



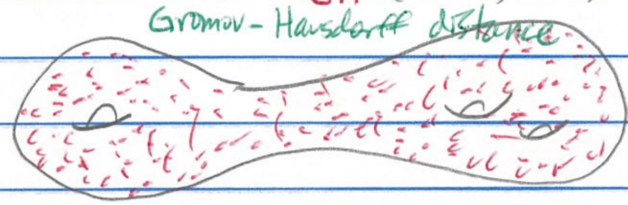
(Since intersections of balls are empty or contractible, the nerve lemma says the Čech complex (a nerve of balls) is $\simeq$ to the union of balls.)

$\boxed{\text{Reconstruction results for Vietoris-Rips complexes}}$

Thm (Latschev 2001) For M a compact Riemannian manifold and $r>0$ sufficiently small, $\exists\, \delta>0$ s.t. if $d_{GH}(X,M)<\delta$, then $VR(X;r)\simeq M$.

Gromov-Hausdorff distance

X

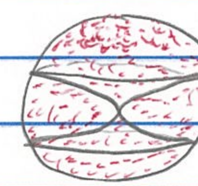


M

Ex Cyclo-octane molecule C_8H_{16}

Algebraic variety of degree 2

(Martin, Thompson, Coutsias, Watson 2010)

 $\subseteq \mathbb{R}^{24}$

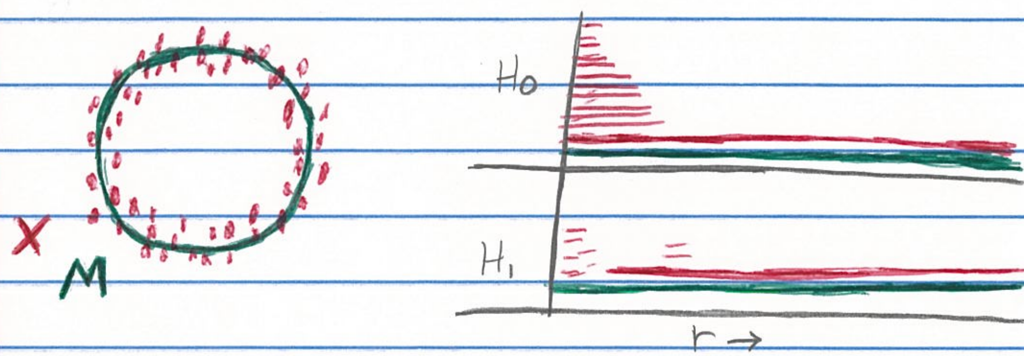
$24 = 8 \cdot 3$

carbons — x,y,z

"Hourglass" is Klein bottle

Stability theorem
(Chazal, de Silva, Oudot 2013)
Nearby metric spaces have nearby persistent homology (PH) barcodes

Ex



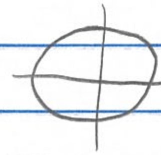
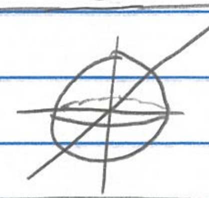
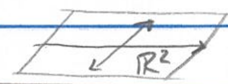
PH of VR(X;r) and VR(M;r)

Consequence As a dataset X converges to a manifold M (say as more samples are drawn), the PH of VR(X;r) converges to that of VR(M;r).

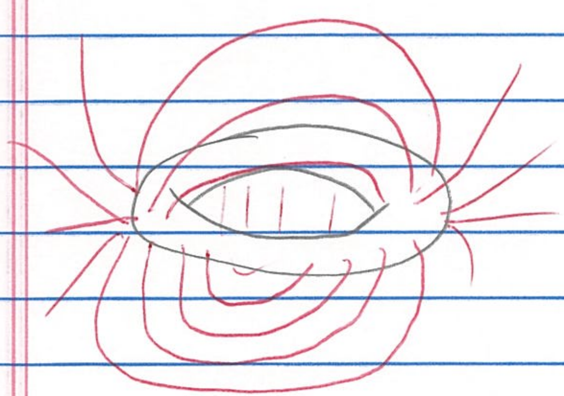
Question What is the PH of VR(M;r) for M a simple manifold, such as a circle, n-sphere, or torus?

$\boxed{\text{Background on spheres, cross-polytopes, and wedge sums}}$

Def The <u>$(n+1)$-ball</u> is $D^{n+1} = \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 := \sqrt{x_1^2 + \dots + x_n^2} \leq 1 \}$.
The <u>n-sphere</u> is $S^n = \partial B^{n+1} = \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \}$.

	S^0	S^1	S^2	S^3
picture 1	 -1, 1			??
picture 2	$\cdot\,\infty$  $\mathbb{R}^0$	$\cdot\,\infty$  $\mathbb{R}^1$	$\cdot\,\infty$  $\mathbb{R}^2$	$\cdot\,\infty$  $\mathbb{R}^3$

Rmk S^3 is the union of two solid tori:



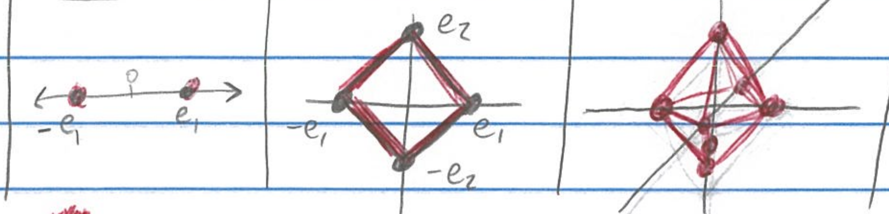
$$S^3 = \partial D^4 \cong \partial(D^2 \times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$$
$$= \underbrace{(S^1 \times D^2)}_{\text{solid torus 1}} \cup \underbrace{(D^2 \times S^1)}_{\text{solid torus 2}}$$

This is called the "genus 1 Heegard splitting" of S^3.
Also $S^{2\ell+1} = (S^{2\ell-1} \times D^2) \cup (D^{2\ell} \times S^1)$
← show steps

Def The $(n+1)$-dimensional cross-polytope is

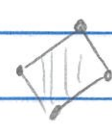
$$C^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\|_1 := |x_1| + \dots + |x_n| \le 1\}$$
$$= \text{Convex Hull}\left(\{\pm e_1, \pm e_2, \dots, \pm e_{n+1}\}\right)$$
$$\pm\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad \pm\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \quad \pm\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

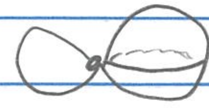
Its boundary is $\partial C^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\|_1 = 1\}$.

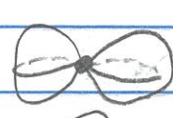
	$\partial C^1 = S^0$	$\partial C^2 \cong S^1$	$\partial C^3 \cong S^2$
picture			

Note $\partial C^{n+1} \cong \partial D^{n+1} = S^n$

Def Given two spaces X and Y, their <u>wedge sum</u> $X \vee Y$ is formed by gluing X and Y together at a single point.

Ex $X =$  $Y =$  $X \vee Y =$ 

Ex $S^1 \vee S^2 =$ 

Ex $S^2 \vee S^2 =$ 

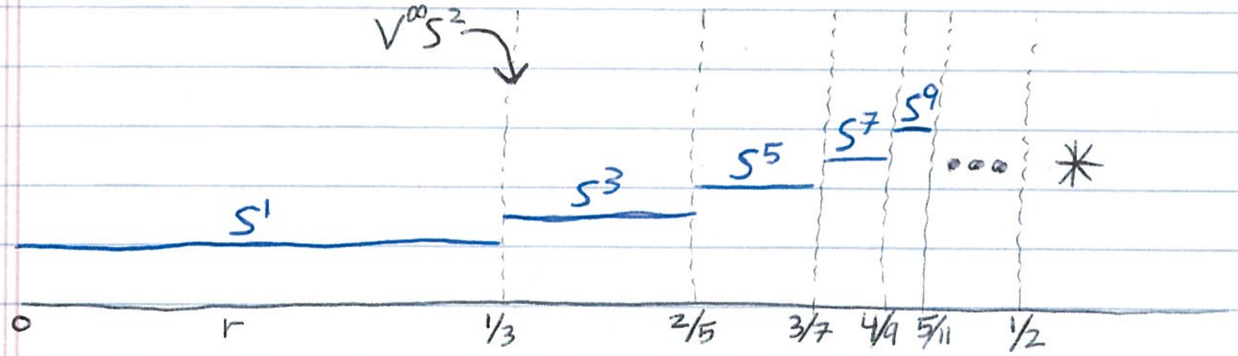
Ex $S^2 \vee S^2 \vee S^2 =$ 

Ex $\vee^\infty S^2 := S^2 \vee S^2 \vee S^2 \vee \dots =$ 

S^1 is circle with geodesic metric, unit circumference.

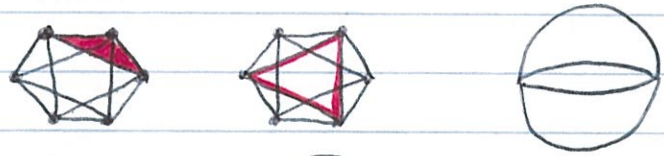
Thm Adamaszek Adams 2017

$$\mathrm{VR}(S^1;r) \simeq \begin{cases} S^{2\ell+1} & \text{if } \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \bigvee^{\infty} S^{2\ell} & \text{if } r = \frac{\ell}{2\ell+1} \end{cases} \quad \text{for } \ell \in \mathbb{N}.$$



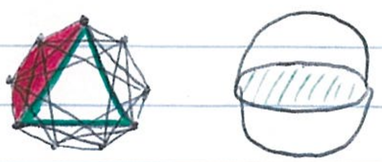
Only connected non-contractible manifold with all VR homotopy types known.
Why care? Stability theorem.

$\mathrm{VR}(\cdots;\frac{1}{3})$



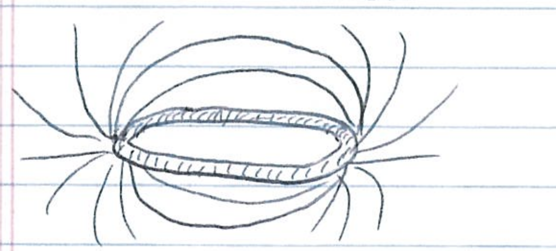
S^2

$\mathrm{VR}(\cdots;\frac{1}{3})$



$S^2 \vee S^2$

$\mathrm{VR}(S^1;\frac{1}{3})$



$$S^3 = S^1 \times D^2 \underset{S^1\times S^1}{\cup} D^2 \times S^1$$

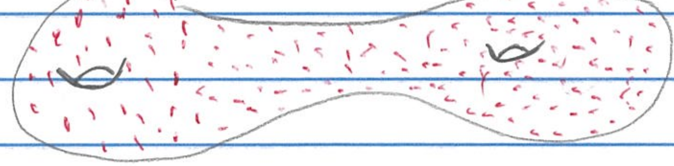
$$S^{2\ell+1} = S^{2\ell-1} \times D^2 \underset{S^{2\ell-1}\times S^1}{\cup} D^{2\ell} \times S^1$$

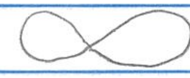
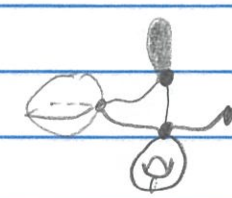
- Proof combinatorial (colimits, homotopy colimits)
- $\check{C}(S^1;r)$ regime when Nerve lemma fails
- How do we "fix" $\bigvee^{\infty} S^2$?

<u>Answer 1</u> $<$ instead of $\leq$ <u>Answer 2</u> Metric thickenings (next time)

Review from last time

- Nerve lemma for Čech complexes
- For X a metric space and $r>0$, $VR(X) = \{\sigma \overset{\text{finite}}{\subseteq} X \mid \text{diam}(\sigma) \le r\}$.
- Thm (Latschev 2001) For M a compact Riemannian manifold and $r>0$ sufficiently small, $\exists\, \delta>0$ s.t. if $d_{GH}(X,M)<\delta$, then $VR(X;r) \simeq M$



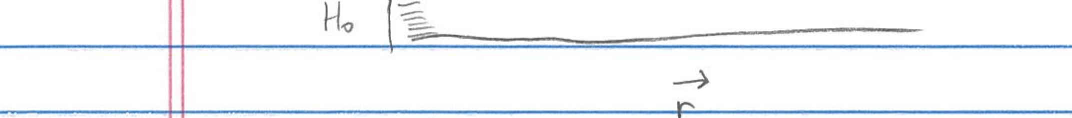
— Versions for M non-manifold  

— Theoretical guarantees rarely hold in practice.

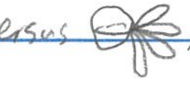
— No good way to choose r

(Need r small depending on M (unknown), large depending on δ).

- In practice, folks estimate $H_*(M)$ from $PH_*(VR(X;r))$



Any reasonable model should have $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}^4$, $H_2 \cong \mathbb{Z}$.

PH doesn't tell you  versus  versus , etc!

- Mathematical opportunity: As $X \to M$, $\underbrace{PH_*(VR(X;r))}_{} \to \underbrace{PH_*(VR(M;r))}_{}$.

Data scientist practitioners compute!

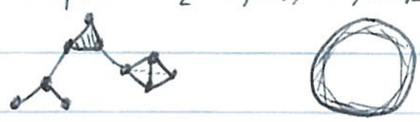
Mathematicians don't yet understand except for $M=S^1$

II. Metric reconstruction via optimal transport

Joint with Michał Adamaszek and Florian Frick

X metric space, $r \geq 0$.

Def The Vietoris–Rips simplicial complex $VR(X;r)$ has
- vertex set X
- simplex $\{x_0, x_1, \ldots, x_k\} \subseteq X$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



Clique or flag simplicial complex

History Leopold Vietoris (111 years old)
- cohomology theory for metric spaces
- recovers Čech cohomology if X compact
- Vietoris homology is counterpart to Alexander–Spanier cohomology

Ilya Rips
- Geometric group theory
- $VR\left(\begin{smallmatrix}\delta\text{-hyperbolic group} \\ \text{word metric}\end{smallmatrix}; r\right) \simeq *$ for $r \geq 4\delta$

Thm (Hausmann 1995)
M compact Riemannian manifold.
Then $\exists r_0 > 0$ such that $VR(M;r) \simeq M$ $\forall r < r_0$.

Sketch $VR(M;r)$
$\downarrow$
M

Not canonical
$M \hookrightarrow VR(M;r)$ not continuous

Thm (Latscher 2001) M, r_0 as above.
$\forall r < r_0$ $\exists \delta > 0$ such that if $d_{GH}(X,M) < \delta$, then $VR(X;r) \simeq M$.

Ex Cyclo-octane molecule C_8H_{16}
(Martin, Thompson, Coutsias, Watson 2010)

Stability $d_B(PH(VR(X;-)), PH(VR(Y;-))) \leq 2 \cdot d_{GH}(X,Y)$ (Think $Y=M$)

Metric reconstruction

 $\rightsquigarrow$  $\rightsquigarrow$ $VR(X;r) \overset{?}{\simeq} M$

metric space M $\qquad$ $X \subseteq M$

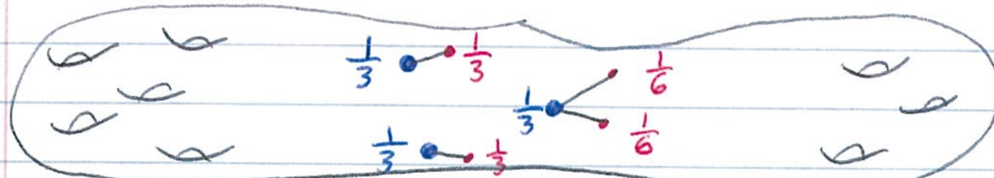
metrizable $\Leftrightarrow$ locally finite

Def X metric space, $r \geq 0$. The Vietoris-Rips thickening is

$$VR^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; \begin{array}{l} k \in \mathbb{N},\quad x_i \in X,\quad \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r, \\ \lambda_i \geq 0 \quad \sum \lambda_i = 1 \end{array} \right\},$$

equipped with the 1-Wasserstein metric.

Think of x_i as δ_{x_i}, a Dirac δ-measure.



$$d\left(\sum_{i=0}^{k} \lambda_i x_i,\ \sum_{j=0}^{k'} \lambda_j' x_j'\right) = \inf_{\{p_{i,j} \geq 0\},\ \sum_j p_{i,j} = \lambda_i,\ \sum_i p_{i,j} = \lambda_j'} \sum p_{i,j}\, d(x_i, x_j')$$

A matching or transport plan is a joint p.d.f. with given marginals.

Prop $VR^m(X;r)$ is an r-thickening of X.

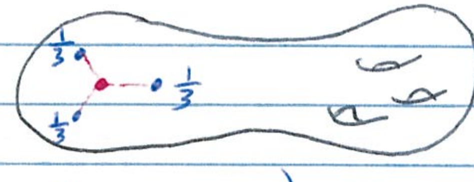
Extends metric on X, and $d(X, VR^m(X;r)) \leq r$.

Gromov studied in the case X discrete.

Thm M complete Riemannian manifold, $r_0 \geq 0$ satisfies
- balls of radius r_0 geodesically convex
- $r_0 < \frac{\pi}{4} K^{-1/2}$ (K sectional curvatures)

Then $VR^m(M;r) \simeq M$ for $r < r_0$.

Pf Sketch $VR^m(M;r) \rightleftarrows M$ $\quad \sum \lambda_i x_i \mapsto$ Karcher or Frechét mean



Linear homotopies (dual space of measures)

Rmk $VR^m(S^1; \frac{1}{3}) \simeq S^3$

Thm $VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ \Sigma^{n+1} \frac{SO(n+1)}{A_{n+2}} & r = r_n \end{cases}$

r_n = diameter of inscribed regular Δ^{n+1}

A_{n+2} = alternating group (rotational symmetries of Δ^{n+1})

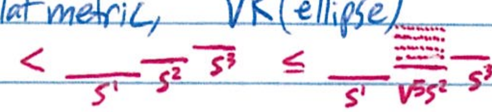
Pf Sketch $VR^m(S^n;r_n) = VR^m(S^n;r_n) \setminus$ interiors of regular Δ^{n+1} $\cup$ $\Delta^{n+1} \times \frac{SO(n+1)}{A_{n+2}}$

$\simeq S^n \times C\left(\frac{SO(n+1)}{A_{n+2}}\right) \cup C(S^n) \times \frac{SO(n+1)}{A_{n+2}}$

$= S^n * \frac{SO(n+1)}{A_{n+2}}$

$= \Sigma^{n+1} \frac{SO(n+1)}{A_{n+2}}$.

Questions
- Larger r? Lovász' strongly-self-dual polytopes
- Other manifolds? VR(L^∞ tori), flat metric, VR(ellipse)
- Čech

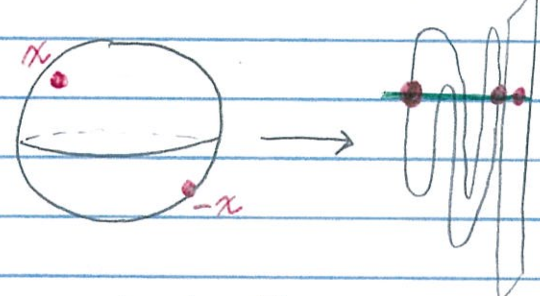


- Morse theory, Morse-Bott theory
- Barvinok-Novik orbitopes, Borsuk-Ulam type theorems
- $VR_< \simeq VR^m_<$?
- Applied topology, AATRN seminar, YouTube

III. Borsuk–Ulam theorems

Joint with Johnathan Bush and Florian Frick

Borsuk–Ulam Thm For $f: S^n \to \mathbb{R}^n$, $\exists x \in S^n$ with $f(x) = f(-x)$
≈ 1933



"Exactly equatorial S^0 in single fiber"

Waist Inequality Thm For $f: S^n \to \mathbb{R}^q$ with $q \le n$,
$\exists y \in \mathbb{R}^q$ with $\mathrm{Vol}_{n-q}\, f^{-1}(y) \geq \mathrm{Vol}_{n-q}\, S^{n-q}$
$\uparrow$ equatorial $S^{n-q} \subseteq S^n$

Constant 1 is sharp Let $f: S^n \to \mathbb{R}^q$ be the restriction of a projection $\mathbb{R}^{n+1} \to \mathbb{R}^q$.
Fibers of f are $(n-q)$-spheres (or points), the largest of which is equatorial.

Proof for $q=1$ is easy A special case of the isoperimetric inequality on spheres (1919) says if $U \subseteq S^n$ has $\mathrm{vol}_n(U) = \frac{1}{2}\mathrm{vol}_n(S^n)$, then $\mathrm{vol}_{n-1}(\partial U) \geq \mathrm{vol}_{n-1}(S^{n-1})$.

Choose $y \in \mathbb{R}$ so that $\mathrm{vol}_n(\{x \in S^n \mid f(x) \leq y\}) = \frac{1}{2}\mathrm{vol}_n(S^n)$.
Then the boundary $f^{-1}(y)$ satisfies $\mathrm{vol}_{n-1} f^{-1}(y) \geq \mathrm{vol}_{n-1}(S^{n-1})$.

Proof for $q \geq 2$ is hard

- fibers no longer divide S^n into regions
- not clear which $y \in \mathbb{R}^q$ to look at
- an arbitrarily large "fraction" of the fibers can be arbitrarily small, which can't happen for $q = 1$.

Minimax proof Almgren 1965, geometric measure theory, minimal surfaces, 100-200 pages

Short proof non-sharp constant Gromov 1983, isoperimetric inequality

Borsuk-Ulam proof Gromov 2003, sharp constant, even more topology (hard generalizations of Borsuk-Ulam using characteristic classes) than other proofs (degree theory).

<u>Rmk</u> The waist inequality is more closely connected to topology than is its cousin, the isoperimetric inequality. First rigorous proof: 1800's

One small reason why is that the waist inequality implies <u>topological invariance of dimension</u>

$$\mathbb{R}^q \cong \mathbb{R}^{q'} \iff q = q'$$

<u>Typical proof</u> $S^{q-1} \simeq \mathbb{R}^q \setminus pt \cong \mathbb{R}^{q'} \setminus pt \simeq S^{q'-1}$

$$\Longrightarrow H_*(S^{q-1}) \cong H_*(S^{q'-1})$$

$$\Longrightarrow q = q'.$$

<u>Waist inequality proof</u> Let $q' > q$.

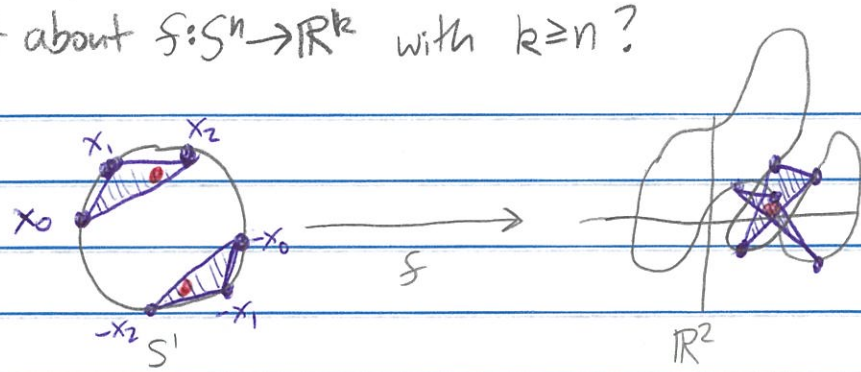
$$\begin{tikzcd} S^n \arrow[r, "L \text{ linear}"'] \arrow[rr, bend right, "f = h \circ L"'] & \mathbb{R}^{q'} \arrow[r, "\cong", "h"'] & \mathbb{R}^q \end{tikzcd}$$

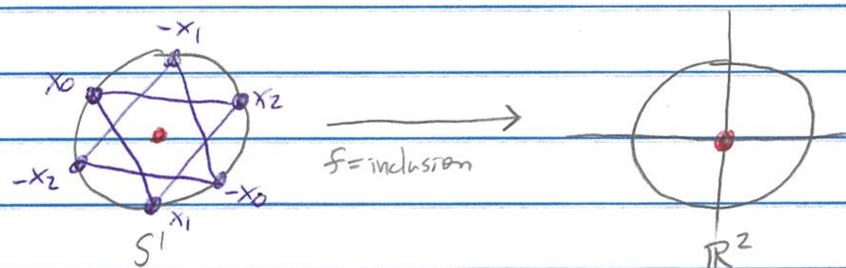
L linear $\Rightarrow$ fibers of L are $(n-q')$-dim'l spheres.
h injective $\Rightarrow$ fibers of f are also $(n-q')$-dim'l spheres with $\mathrm{vol}_{n-q} = 0$ (since $n-q' < n-q$).
This contradicts the waist inequality.

<u>Quantitative topology</u> is a "hot" area.
Considers not only dimensions, but also sizes (volumes, diameters, lengths, radii).
Many of its foundational tools invented by Gromov (waist inequality).
Feels related to persistent homology & applied topology.

Q What about $f: S^n \to \mathbb{R}^k$ with $k \geq n$?

Ex





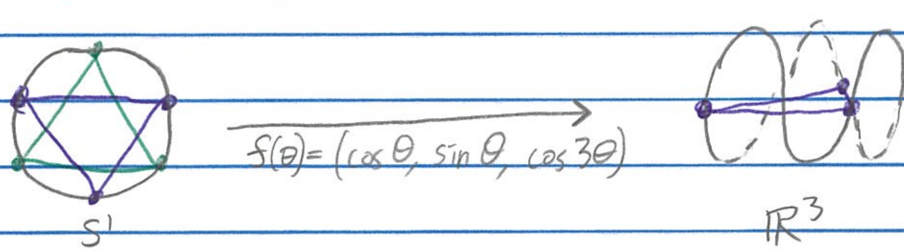
Bound into $\mathbb{R}^{2k}$ also works into $\mathbb{R}^{2k+1}$

critical scales for $VR(S^1;r)$

Thm (A, Bush, Frick, 2020) For $\boxed{f: S^1 \to \mathbb{R}^{2k+1}}$, there exists a set $\{x_0, x_1, \ldots\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

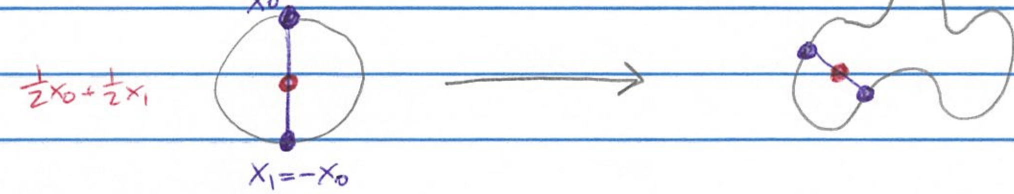
Convex combinations in $\mathbb{R}^{2k+1}$. Same coefficients!

Ex



Rmk In the prior example into $\mathbb{R}^2$, we had a circle's worth of solutions (always true?). In this example into $\mathbb{R}^3$, that reduces down to a single solution.

Rmk Interesting diameter bounds are $< \frac{1}{2}$.



$f: S^1 \longrightarrow \mathbb{R}^n$

n	0	1	2	3	4	5	6	7
diameter bound	0	0	1/3	1/3	2/5	2/5	3/7	3/7

Rmk The bounds on diameter are optimal in all dimensions. To see that they can't be improved, consider a truncation of the map $f(\theta) = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$.

Proof sketch

$S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$ induces

$$S^{2k+1} \simeq VR^m\left(S^1; \tfrac{k}{2k+1}\right) \xrightarrow{f} \mathbb{R}^{2k+1}$$

$$\sum \lambda_i \delta_{x_i} \longmapsto \sum \lambda_i f(x_i)$$

sum of vectors in $\mathbb{R}^{2k+1}$

Apply the standard Borsuk-Ulam theorem to get antipodal points in $S^{2k+1} \simeq VR^m(S^1; \frac{k}{2k+1})$ that collide in $\mathbb{R}^{2k+1}$. □

$\sum \lambda_i x_i$ and $\sum \lambda_i (-x_i)$

Diameter at most $\frac{k}{2k+1}$

Rmk As the homotopy connectivity of $VR^m(S^1;r)$ increases, so does the allowable dimension of the codomain for Borsuk-Ulam type theorems!

[same bound into $\mathbb{R}^{n+1}$]

Thm (A, Bush, Frick 2020) For $f: S^n \longrightarrow \mathbb{R}^{n+2}$, there exists a set $\{x_0, x_1, \ldots\}$ of diameter at most r_n such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof sketch

$S^n \xrightarrow{f} \mathbb{R}^{n+1}$ induces

Intuition: "$S^{n+2} \subseteq$" $VR^m(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$

Rigorous: Borsuk-Ulam applies not only to $S^{n+2} \longrightarrow \mathbb{R}^{n+2}$, but also to any $Y \longrightarrow \mathbb{R}^{n+2}$ with Y $(n+1)$-connected. $\square$

Diameter bound sharp: Consider $S^n \hookrightarrow \mathbb{R}^{n+1} \subseteq \mathbb{R}^{n+2}$.

Rmk Also have versions for $f: S^{2n-1} \rightarrow \mathbb{R}^{2kn+2n-1}$ with non-sharp diameter bound ($\frac{k}{2k+1}$ for great circles of length 1).

Rmk Michael Crabb, Aberdeen:

- Fiber bundles, characteristic classes
- Stiefel-Whitney class, Euler class
- Cohomology of quotients of classical groups (Braun, Browder, 1965)