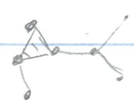
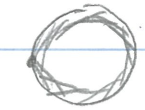
1/5/17 Florida International University

Metric reconstruction via Vietoris-Rips complexes and optimal transport

Joint with Michał Adamaszek and Florian Frick

X metric space, $r>0$

Def The <u>Vietoris-Rips simplicial complex</u> $VR(X;r)$ has
- vertex set X
- finite simplex σ when $\mathrm{diam}(\sigma)\le r$

[History] <u>Leopold Vietoris</u> (Co)homology theory for metric spaces

(Vietoris homology – counterpart to Alexander-Spanier cohomology)

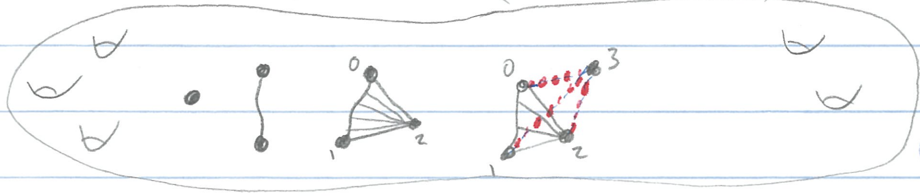
Cohomology same as Čech cohomology if compact

<u>Ilya Rips</u> Geometric group theory

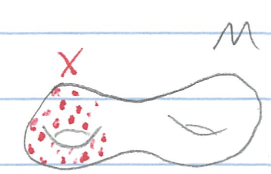
$VR(\underset{\text{w/ word metric}}{\delta\text{-hyperbolic group}};r)\simeq *$ for $r\ge 4\delta$

Thm (Hausmann '95) M compact Riemannian manifold.
Then $\exists\ r_0>0$ s.t. $VR(M;r)\simeq M\quad \forall r<r_0$

Sketch Map $VR(M;r)\to M$: (not canonical)



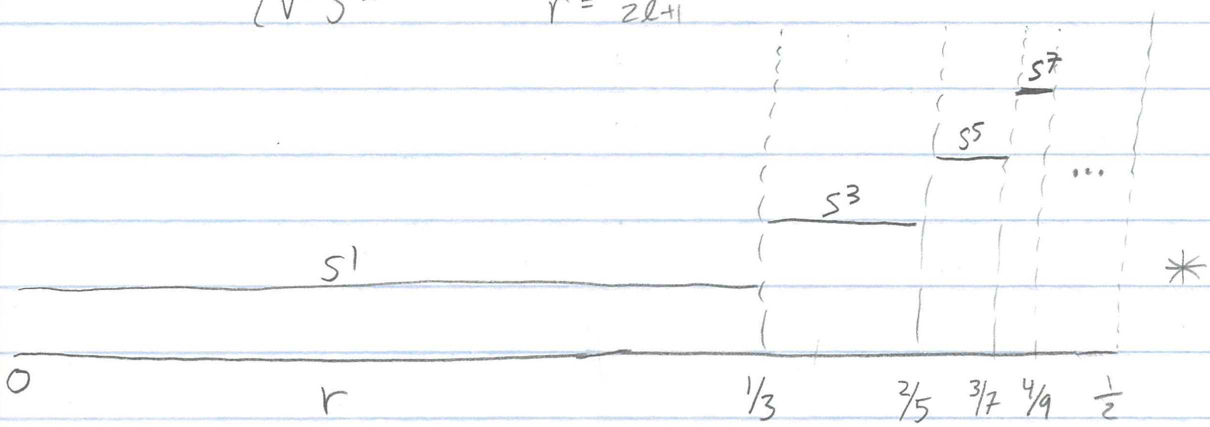
Thm (Latscher '01) M compact Riemannian manifold.
Then $\forall r<r_0\ \exists\delta>0$ s.t. if $d_{GH}(X,M)<\delta$,
then $VR(X;r)\simeq M$.



S^1 ===== circle of unit circumference w/ geodesic metric

Thm (Adamaszek, A)

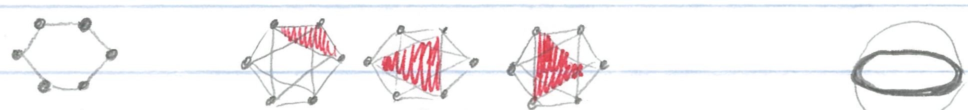
$$VR(S^1; r) \simeq \begin{cases} S^{2l+1} & \frac{l}{2l+1} < r < \frac{l+1}{2l+3} \\ \bigvee^{\infty} S^{2l} & r = \frac{l}{2l+1} \end{cases} \quad \text{for some } l \in \mathbb{N}$$



— (Only connected non-contractible manifold w/ all homotopy types known)

— (Why care? PH stability theorem)

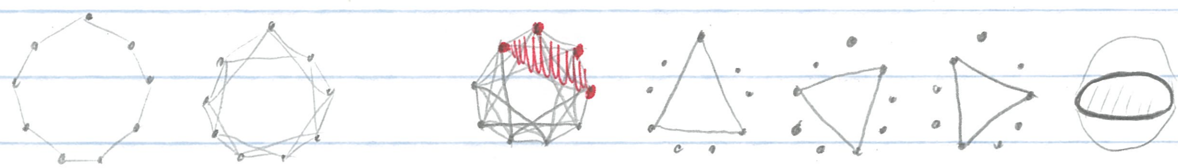
VR(; r)



S^1 $\quad$ $\frac{1}{6} \le r < \frac{1}{3}$

S^2 $\quad$ $r = \frac{1}{3}$

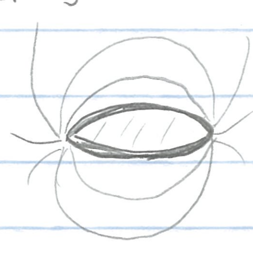
VR(; r)



S^1 $\quad$ $\frac{1}{9} \le r < \frac{2}{9}$

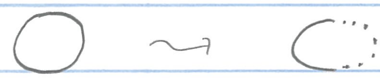
S^1 $\quad$ $\frac{2}{9} \le r < \frac{1}{3}$

$\vee^2 S^2$ $\quad$ $r = \frac{1}{3}$



Mention $\check{C}(S^1; r)$, regime when Nerve Lemma fails

Metric reconstruction



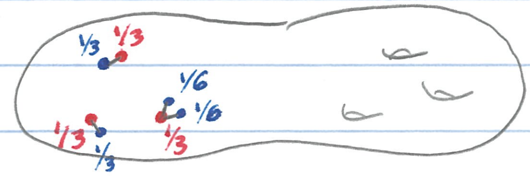
$\rightsquigarrow VR(X;r) \overset{?}{\simeq} M$

metric space M — $X \subseteq M$ — Not a metric space. Maybe not metrizable (if not locally finite)

Def X metric space, $r > 0$.
Metric space $VR^m(X;r)$ is $|VR(X;r)|$ as a set, equipped w/ the 1-Wasserstein metric.

Explicitly $\sum_{i=0}^{k} a_i x_i \in VR^m(X;r)$ $\quad a_i \geq 0 \quad \sum a_i = 1$
$\quad x_i \in X \quad diam([x_0, \ldots, x_k]) \leq r$

$$d\left(\sum_{i=0}^{k} a_i x_i, \sum_{j=0}^{k'} a'_j x'_j\right) = \inf_{\substack{\text{matchings } \{p_{i,j}\} \\ p_{i,j} \geq 0 \\ \sum_j p_{i,j} = a_i \\ \sum_i p_{i,j} = a'_j}} \sum_{i,j} p_{i,j}\, d(x_i, x'_j)$$



(A point is a probability measure w/ finite support & diam $\leq r$
A matching is a joint pdf w/ given marginals)

Prop $VR^m(X;r)$ is an r-thickening of X
(extends metric, and $d(X, VR^m(X;r)) \leq r$)
(Gromov studied in case X discrete)

Thm (Adamaszek, A, Frick)

M complete Riemannian manifold. Let $r_0 > 0$ satisfy

- balls of radius r_0 geodesically convex
- $r_0 < \frac{\pi}{4}\Delta^{-1/2}$ where sectional curvatures $\le \Delta$.

Then $VR^m(M;r) \simeq M$ for $r < r_0$.

Sketch

$VR^m(M;r) \to M$ (canonical)

$\sum a_i x_i \longmapsto$ Karcher or Frechét mean (center of mass)

Linear homotopies (inside vector space: dual space of continuous f^{ns})

Rmk

$$VR^m(S^1;\tfrac{1}{3}) = VR^m(S^1;\tfrac{1}{3}) \setminus \begin{smallmatrix}\text{interiors of}\\ \text{regular } \Delta^2\end{smallmatrix} \cup \Delta^2 \times S^1$$
$$\simeq S^1 \times D^2 \cup D^2 \times S^1$$
$$= S^1 * S^1$$
$$= S^3$$

Thm (Adamaszek, A, Frick)

$$VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ \Sigma^{n+1} \frac{SO(n+1)}{A_{n+1}} & r = r_n \end{cases}$$

r_n = diameter of inscribed regular Δ^{n+1}

$r_1 = \frac{1}{3}$

A_{n+1} = rotational symmetries of Δ^{n+1}

Sketch

$$VR^m(S^n;r_n) = VR^m(S^n;r_n) \setminus \begin{smallmatrix}\text{interiors of}\\ \text{regular } \Delta^{n+1}\end{smallmatrix} \cup \Delta^{n+1} \times \frac{SO(n+1)}{A_{n+1}}$$
$$\simeq S^n \times C\left(\frac{SO(n+1)}{A_{n+1}}\right) \cup C(S^n) \times \frac{SO(n+1)}{A_{n+1}}$$
$$= S^n * \frac{SO(n+1)}{A_{n+1}}$$
$$= \Sigma^{n+1} \frac{SO(n+1)}{A_{n+1}}$$

Open Larger r? (Strongly self-dual polytopes)

Other manifolds? VR(tori w/ L^∞ metric) Flat metric?