Metric Reconstruction Via Optimal Transport

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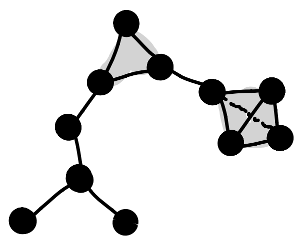
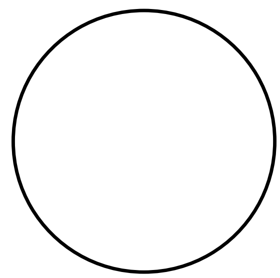
Main point A simplicial complex whose vertex set is a metric space should often be equipped with an optimal transport metric instead of the simplicial complex topology.

X metric space, $r \geq 0$.

Def The *Vietoris-Rips simplicial complex* has

- vertex set X
- finite simplex $\sigma \subseteq X$ when $\text{diameter}(\sigma) \leq r$.





History

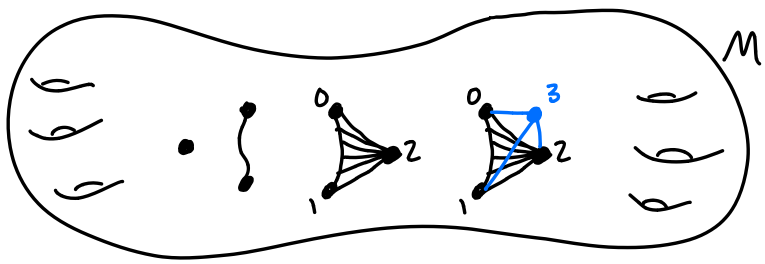
- Cohomology theory for metric spaces
- Geometric group theory
- Applied topology

Thm (Hausmann 1995)

M compact Riemannian manifold.
Then $\exists\ r_0 > 0$ such that $VR(M;r) \simeq M \ \forall r < r_0$.
↑ depends on curvature of M

Proof Sketch

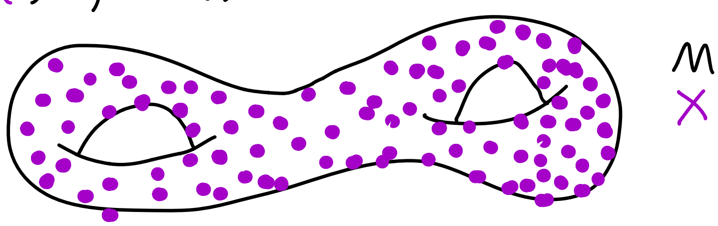
$$VR(M;r)$$
$$\downarrow$$
$$M$$



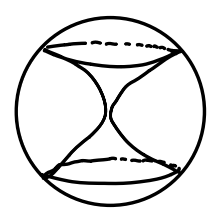
- Not canonical
- $M \hookrightarrow VR(M;r)$ not continuous.

Thm (Latschev 2001)

M, r_0 as above.
$\forall r < r_0\ \exists \delta > 0$ such that if $d_{GH}(X, M) < \delta$,
then $VR(X;r) \simeq M$.

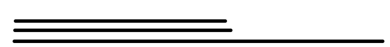


Ex Cyclo-octane molecule C_8H_{16}
(Martin, Thompson, Coutsias, Watson 2010)



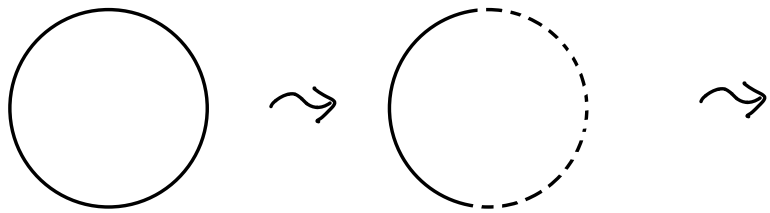
Stability

$PH_1(VR(M;r))$



$PH_1(VR(X;r))$

Metric Reconstruction



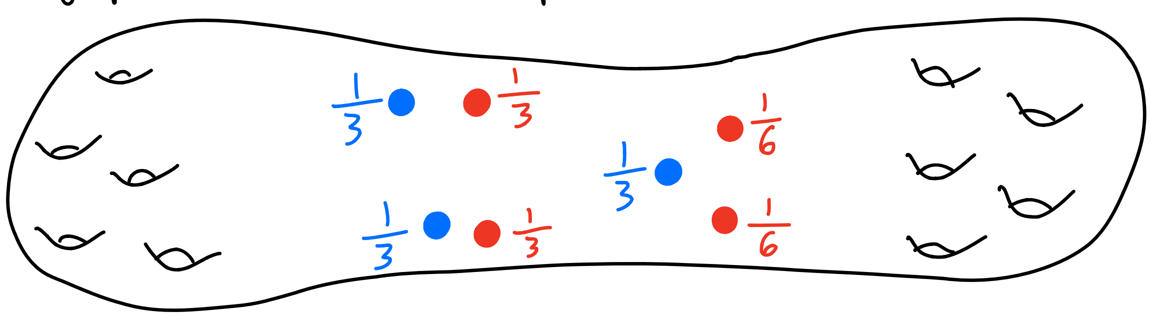
metric space M $\qquad$ $X \subseteq M$ $\qquad$ $\rightsquigarrow$ $VR(X;r) \overset{?}{\simeq} M$

metrizable $\iff$ locally finite

Def X metric space, $r \geq 0$.
The Vietoris–Rips metric thickening is

$$VR^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; \begin{array}{ll} x_i \in X, & \mathrm{diam}(\{x_0, \dots, x_k\}) \leq r \\ \lambda_i \geq 0 & \sum \lambda_i = 1 \end{array} \right\},$$

equipped with the p-Wasserstein metric.



Prop $VR^m(X;r)$ is an r-thickening of X.

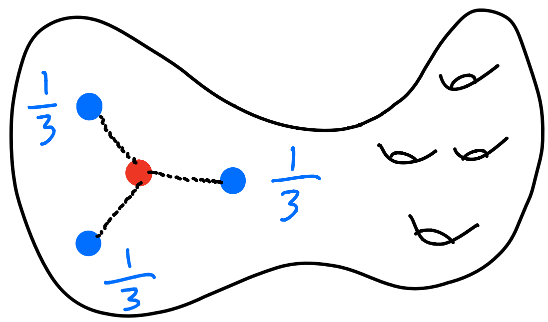
Rmk Can do this for any simplicial complex K whose vertex set is a metric space, yielding the metric thickening K^m.

Thm M complete Riemannian manifold, $r_0 \geq 0$ small enough so that measures of diameter $\leq r_0$ have unique Karcher means. Then $VR^m(M;r) \simeq M$ for $r \leq r_0$.

Pf Sketch

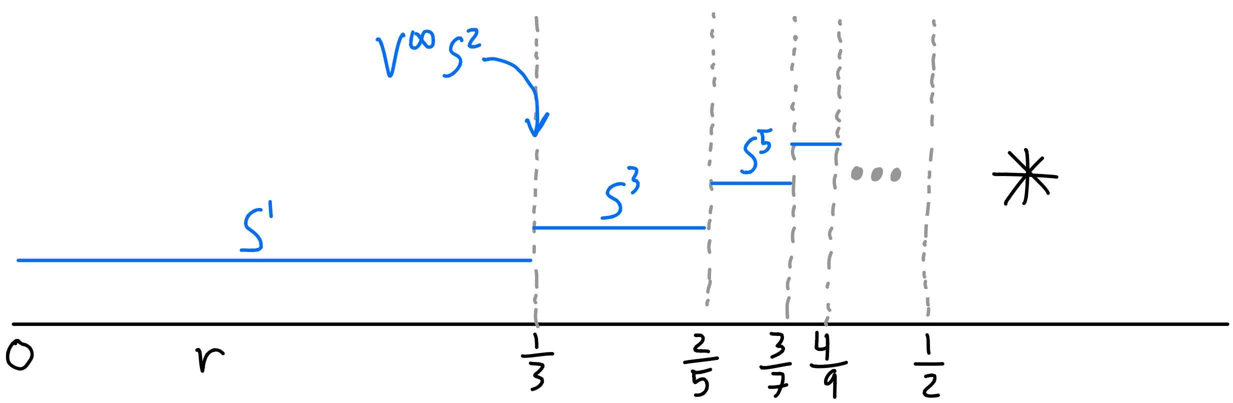
$$\begin{tikzcd} VR^m(M;r) \arrow[d, bend right] \\ M \arrow[u, bend right] \end{tikzcd}$$

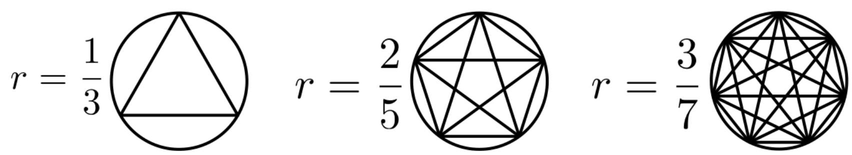
$$\sum \lambda_i x_i$$
$$\downarrow$$
Karcher or Frechét mean



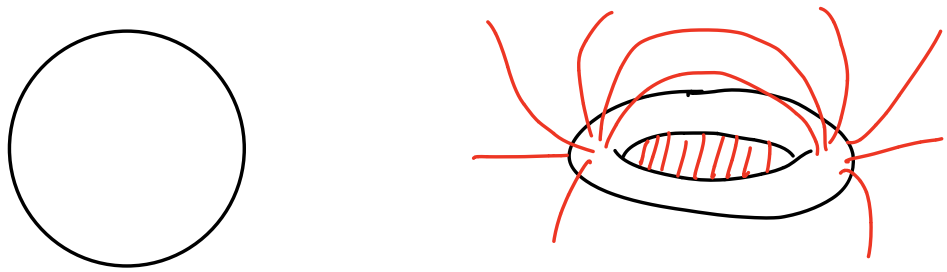
S^1 is circle with geodesic metric, unit circumference.

<u>Thm</u> $VR(S^1;r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ \bigvee^{\infty} S^{2k} & \text{if } r = \frac{k}{2k+1} \end{cases}$ $\quad k \in \mathbb{N}$



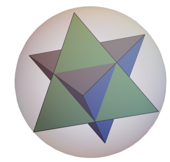


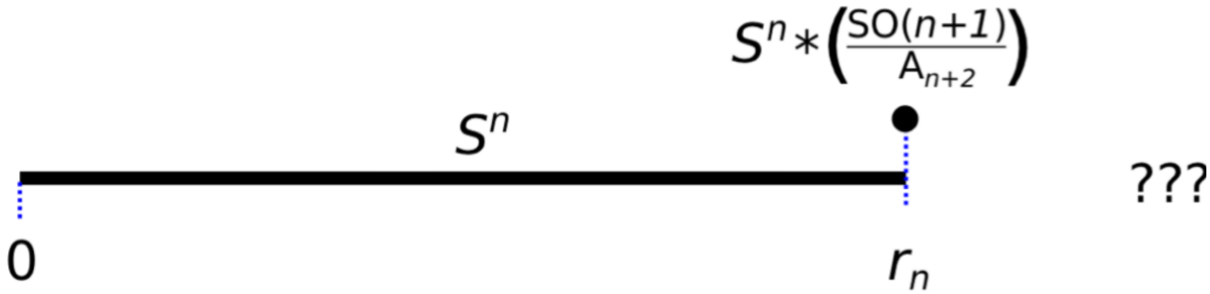
By contrast, $VR^m(S^1;\frac{1}{3}) \simeq S^3$. <u>Why?</u>

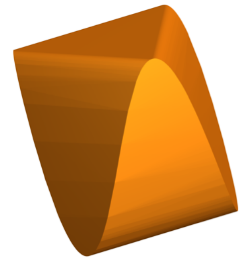


More generally,

Thm $VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$







Questions

(1) $VR^m(S^n;r)$ for larger r?

(2) $\check{C}ech^m(S^n;r)$?

(3) Other manifolds ?

(4) $VR^m_<(X;r) \simeq VR_<(X;r)$?

(5) Stability of $VR^m(X;r)$ for X infinite ?

(6) Morse and Morse-Bott theories (Mirth PhD thesis)

(7) Borsuk-Ulam theorems $S^n \to \mathbb{R}^k$ for $k \geq n$

(A., Bush, Frick, Mathematika 2020)

(8) Categorical framework – infinite support ?

(A., Bush, Mirth, Applied Category Theory 2020)

(9) Connections to quantitative topology, especially Kuratowski embeddings and filling radii.

(Lim, Mémoli, Okutan 2020, Katz 1983–1991)