Lovász' proof of the Kneser conjecture

Abstract The chromatic number of a graph is the number of vertex colors needed so that adjacent vertices have different colors. In 1955, Martin Kneser made a conjecture about the chromatic number of a certain family of graphs (now called Kneser graphs). This conjecture remained unproven for 23 years until László Lovász gave a topological proof in 1978, thus beginning the field of topological combinatorics. I will share a proof of Kneser's conjecture — essentially all proofs known to this day are topological.

Sources

Jiří Matoušek, 2003, "Using the Borsuk-Ulam Theorem"

László Lovász, 1978, "Kneser's Conjecture, Chromatic Number, and Homotopy"

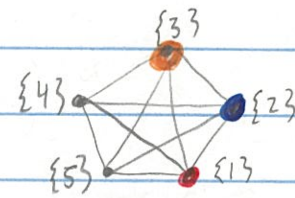
Joshua E Greene, 2002, "A New Short Proof of Kneser's Conjecture"

Kneser graphs

Def Let $KG_{n,k}$ be the graph with
- vertex set all subsets of $\{1,2,\dots,n\}$ of size k
- an edge between vertices V and V' when $V \cap V' = \phi$.

Ex $KG_{n,1}$ is complete graph

$\chi(KG_{n,1}) = n$



$KG_{5,1}$

$KG_{2k-1,k}$ has no edges

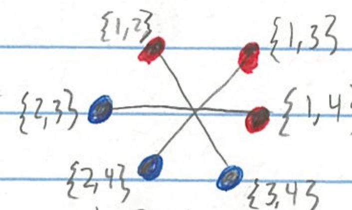
$\chi(KG_{2k-1,k}) = 1$

$KG_{3,2}$

$KG_{2k,k}$ is a matching

(every set is adjacent only to its complement

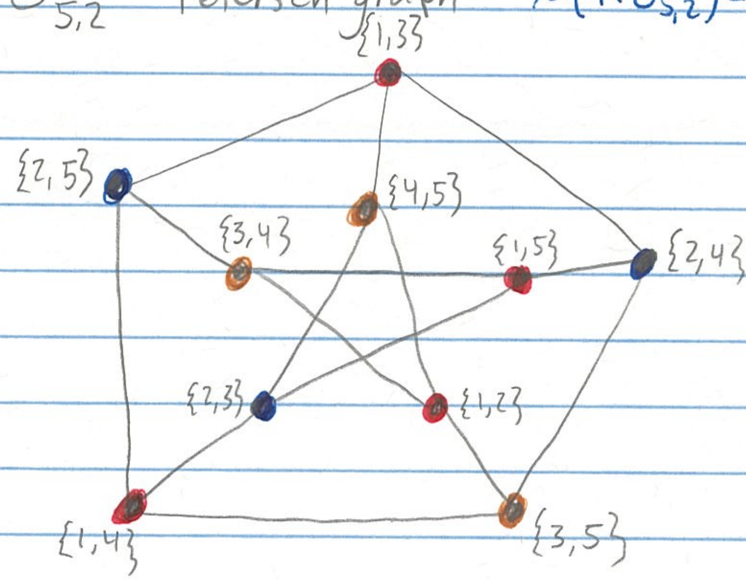
$\chi(KG_{2k,k}) = 2$ for $k \geq 1$



$KG_{4,2}$

$KG_{5,2}$ = Petersen graph $\chi(KG_{5,2}) = 3$ though there are no triangles

"(Counter)example for almost everything in graph theory"



Def The chromatic number $\chi(G)$ of a graph G is the number of vertex colors needed so that adjacent vertices have different colors.

Kneser's conjecture 1955 (proven by Lovász 1978)
For $k \geq 1$ and $n \geq 2k-1$, $\chi(KG_{n,k}) = n-2k+2$

$n-2k+2$ colors suffice
Let $\text{color}(V) = \min(\min(V), n-2k+2)$
- 1
- 2
- 3

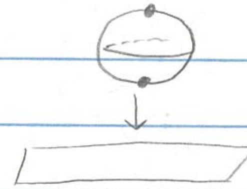
If two sets V, V' get the same color $i < n-2k+2$, then they cannot be disjoint (adjacent) since they share element i.

If two sets both get color $n-2k+2$, then they are contained in $\underbrace{\{n-2k+2, \ldots, n\}}_{2k-1 \text{ elements}}$, and hence cannot be disjoint.

KIP:

Rmk Fractional chromatic numbers
$\chi_f(G) = \inf\{\frac{a}{b} \mid \text{vertices of } G \text{ covered} \geq b \text{ times by } a \text{ independent sets}\}$
$$\chi_f(G) \leq \chi(G)$$
(Rare) wide gap:
$$\frac{n}{k} = \chi_f(KG_{n,k}) \leq \chi(KG_{n,k}) = n-2k+2$$

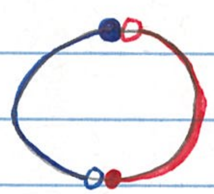
Ex $3 \approx \frac{3k-1}{k} = \chi_f(KG_{3k-1,1}) \leq \chi(KG_{3k-1,k}) = k+1$



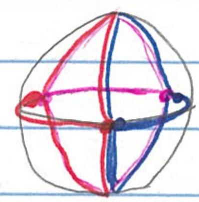
Borsuk-Ulam Theorem (3 equivalent versions)

- For every continuous $f: S^d \to \mathbb{R}^d$ $\exists\, x \in S^d$ with $f(x) = f(-x)$ (Temperature & pressure on earth)
- There is no continuous $f: S^d \to S^{d-1}$ with $f(-x) = -f(x)$ for all $x \in S^d$ ($\mathbb{Z}/2$-spaces. No $\mathbb{Z}/2$-equivariant map) (Equivariant cohomology)
- Whenever S^d is covered by $d+1$ sets $A_1, A_2, \dots, A_{d+1}$, each open or closed, there is some $x \in S^d$ with $x, -x \in A_i$ for some i.

S^1



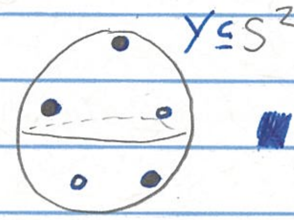
S^2



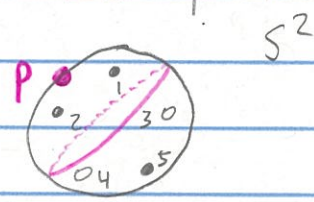
<u>$n-2k+2$ colors are needed</u> (simplified proof by Greene, 2002)

Let $d = n-2k+1$
Fix $Y \subseteq S^d$ with $|Y| = n$ and no hyperplane through origin in $\mathbb{R}^{d+1}$ containing $\geq d+1$ points in Y

<u>Ex</u> $KG_{5,2}$
$d = 2$

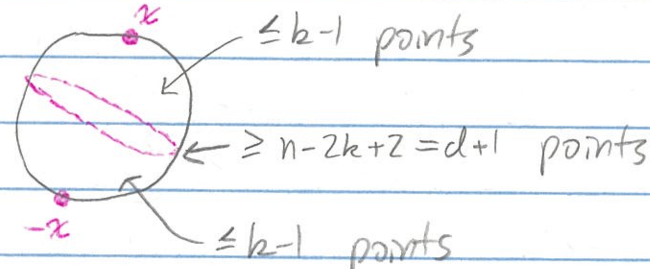


Suppose for a contradiction we have a coloring of $KG_{n,k}$ by at most d colors.
Define $A_1, \ldots A_d \subseteq S^d$ via:
$p \in A_i$ if some i-colored k-tuple is in the open hemisphere centered at p



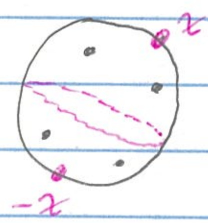
$p \in A_1$ if the tuple $\{1,2\}$ has color 1.
Let $A_{d+1} = S^d \setminus (A_1 \cup \ldots \cup A_d)$.
Note $A_1, \ldots, A_d$ open, and A_{d+1} closed.
By Borsuk-Ulam, $\exists x \in S^d$ with $x, -x \in A_i$ for some i.

Note $i \neq d+1$ since then $x, -x \in A_{d+1}$ would imply



which contradicts general position.

So $i \in \{1, \ldots, d\}$, which means we have two disjoint k-tuples of color i.



Hence there is no coloring with at most $d = n-2k+1$ colors.

Rmk Lovász' proof gives a topological bound on the chromatic number of any graph, using the homotopy connectivity of a simplicial complex built on that graph.