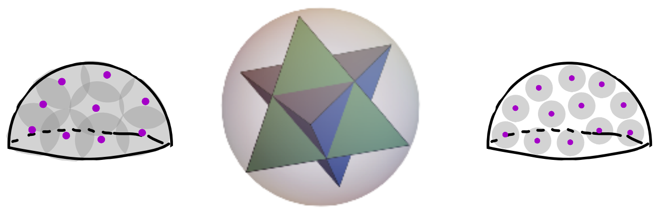
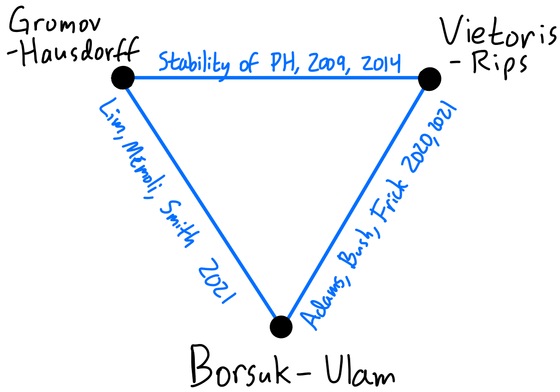
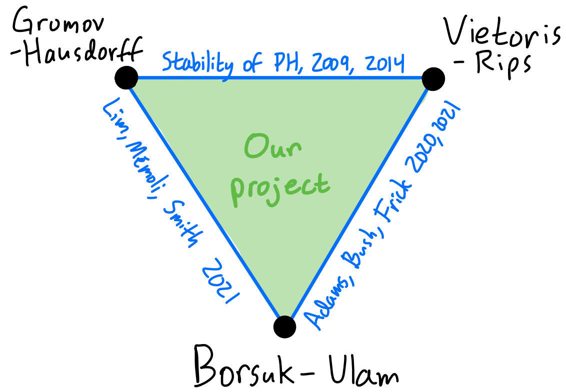
Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes



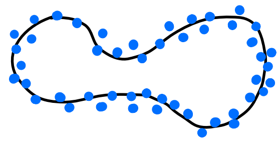
Joint with CSU, OSU, CMU, Berlin

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes



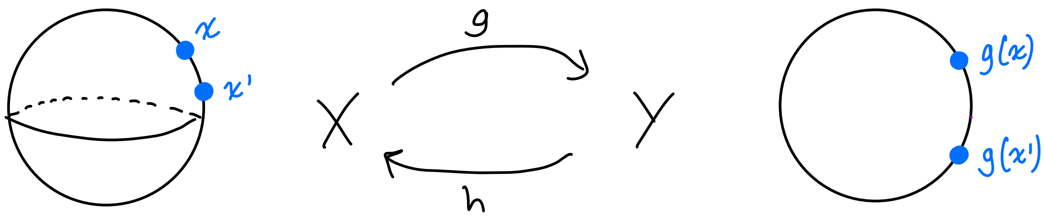


Gromov–Hausdorff distances



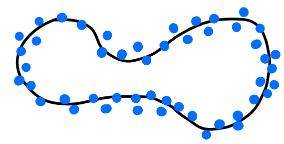
X, Y compact metric spaces

Def $2\cdot d_{GH}(X,Y) = \inf_{\substack{g: X\to Y\\ h: Y\to X}} \max\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h)\}.$



$$\mathrm{dis}(g) = \sup_{x,x'\in X} \left| d(x,x') - d(g(x), g(x')) \right|$$

Gromov–Hausdorff distances



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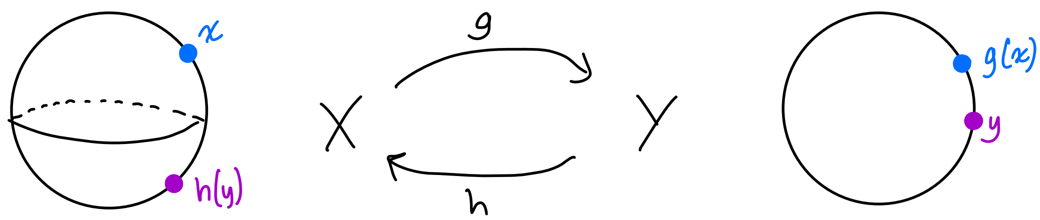

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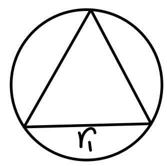
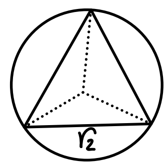
Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5	Symmetric matrix				0	$\geq r_5$	
S^6	Non-zero entries in $(\pi/2, \pi)$					0	$\geq r_6$





For $n<k$, $2 \cdot d_{GH}(S^n, S^k) \geq \max\{r_n, \ \pi - \mathrm{cov}_{k+1}(S^n)\}$.

Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam.

Lim, Mémoli, Smith, 2021

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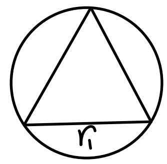

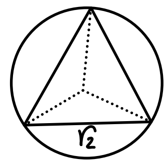

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Theorem (Oct, 2021) For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}}$$

Lim, Mémoli, Smith, 2021

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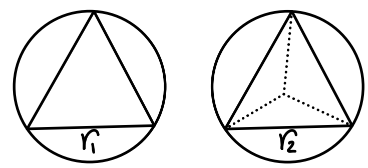
Symmetric matrix

Non-zero entries in $(\pi/2, \pi)$



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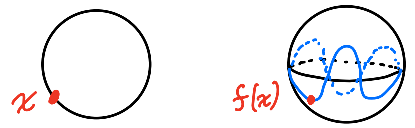
No cases of $>$ are currently known

A., Bush, Frick, 2021
Equality for $n=1$, k odd

<u>Def</u> $\mathrm{cov}_k(X) :=$ infimum r s.t. k balls of radius $\frac{r}{2}$ cover X.

Borsuk-Ulam theorems

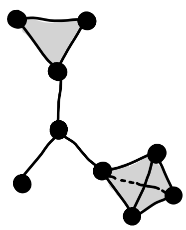


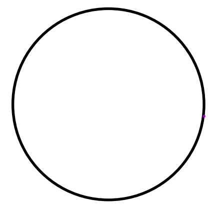
Def A map $f: S^k \to S^n$ is odd if $f(-x) = -f(x)$ $\forall x \in X$

Borsuk-Ulam: There is no cont. odd $S^k \to S^n$ for $k > n$.

Vietoris – Rips simplicial complexes

<u>Def</u> X metric space, $r \geq 0$. <u>Vietoris-Rips complex $VR(X;r)$</u> has vertex set X, all simplices of diameter $\leq r$.





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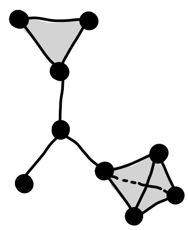
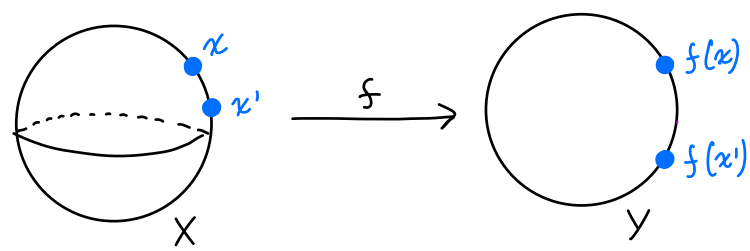


Map $f: X \to Y$ induces a (cont.) simplicial map $f: VR(X,r) \longrightarrow VR(Y; dis(f)+r)$.

$$x \longmapsto f(x)$$



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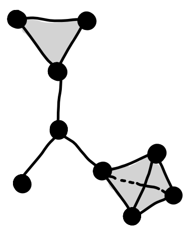

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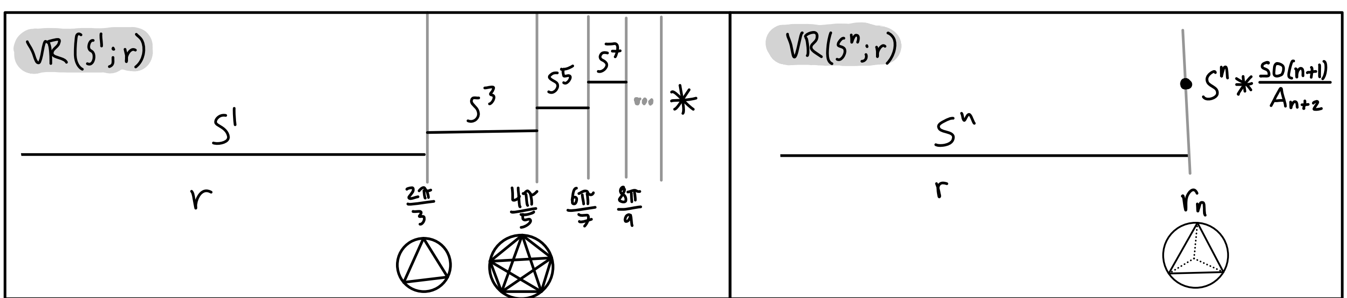


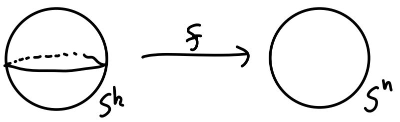
$$c_{1,2k+1} = c_{1,2k} = \frac{2\pi k}{2k+1} \qquad c_{n,n+2} = c_{n,n+1} = r_n$$

$$\underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n;r)\}}_{c_{n,k}}$$

Dubins & Schwarz '81
Lim, Mémoli, Smith, 2021

(Discont.) odd maps $f: S^{n+1} \to S^n$ have $\mathrm{dis}(f) \geq r_n$.

<u>Generalization</u> (Discont.) odd maps $f: S^k \to S^n$ for $k>n$ have $\mathrm{dis}(f) \geq c_{n,k}$.

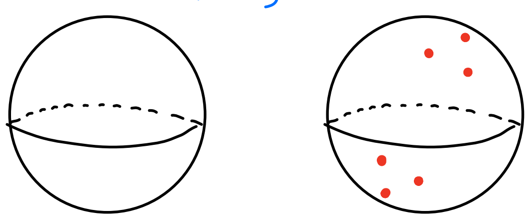


<u>Proof</u>

For $\varepsilon>0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.
Produce a cont. odd map

$$S^k \xrightarrow[\text{of unity}]{\text{partition}} VR(X;\varepsilon) \longrightarrow VR(S^n; \mathrm{dis}(f)+\varepsilon).$$

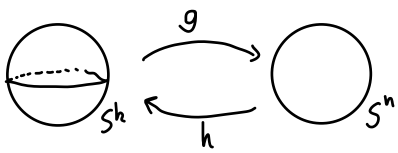
$$x \longmapsto f(x)$$



Hence $\mathrm{dis}(f)+\varepsilon \geq c_{n,k}$ $\forall \varepsilon>0$, so $\mathrm{dis}(f) \geq c_{n,k}$.

Theorem (Oct, 2021) For $n<k$,

$$2\cdot d_{GH}(S^n,S^k)\geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k\to VR(S^n;r)\}}_{c_{n,k}}.$$



Proof of Theorem follows Lim, Mémoli, Smith, 2021

$$2\cdot d_{GH}(S^n,S^k) = \inf_{\substack{g:S^k\to S^n\\ h:S^n\to S^k}} \max\{dis(g),\ dis(h),\ codis(g,h)\}$$

$$\geq \inf_{g:S^k\to S^n} dis(g)$$

$$= \inf_{\text{odd } g:S^k\to S^n} dis(g) \quad \text{by Lemma 5.5}$$

$$\geq c_{n,k}.$$

<u>Question</u> Tight upper bounds on $d_{GH}(S^n, S^k)$ via maps?

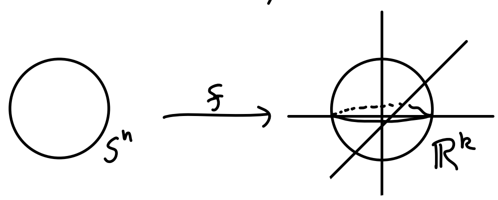
<u>Question</u> Bounds on $d_{GH}(X,Y)$ for more general families of G-equivariant metric spaces X, Y ?

<u>Question</u> Relate the p-Gromov-Wasserstein distance $d_{p\text{-}GW}$ to p-Vietoris-Rips thickenings VR_p ?

<u>Question</u> How does the generalization of Dubins & Schwarz relate to Tverberg ?

Aside A., Bush, Frick 2021 show:

If $f: S^n \to \mathbb{R}^k$ is odd for $k > n$, then there is a set $X \subset S^n$ of diameter $\leq c_{n,k}$ with $\vec{0} \in \operatorname{conv}(f(X))$.



Proof

$$S^n \longrightarrow \mathbb{R}^k \text{ induces}$$
$$VR(S^n; c_{n,k}) \longrightarrow \mathbb{R}^k$$