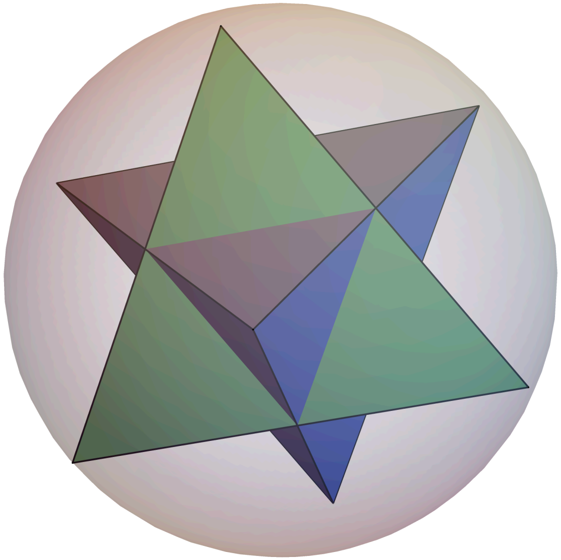


Bridging Applied and Quantitative Topology

Henry Adams

COLORADO STATE UNIVERSITY

AATRN
Applied Algebraic Topology Research Network

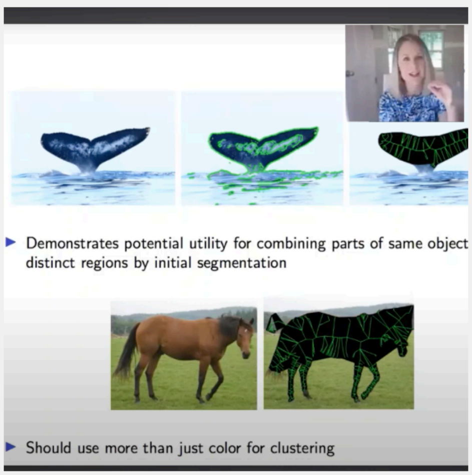
AATRN   **Home**   About   Seminar   Interviews   Interactive Events   Affiliated Activities   Participate   Ethics

AATRN

Applied Algebraic Topology Research Network

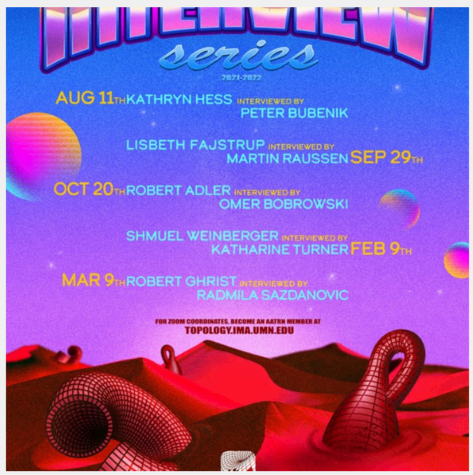
Bringing together researchers across the world to develop and use applied and computational topology.

Find out how you can participate and join AATRN (membership is free and so are all events), or check out all our content on our YouTube channel. We invite you to join the following activities, which we are currently organizing:



Seminar Series

We have a weekly Seminar Series (or sometimes a tea time) on Wednesdays.



Interview Series

See our upcoming interviews.

Tutorial-a-thon

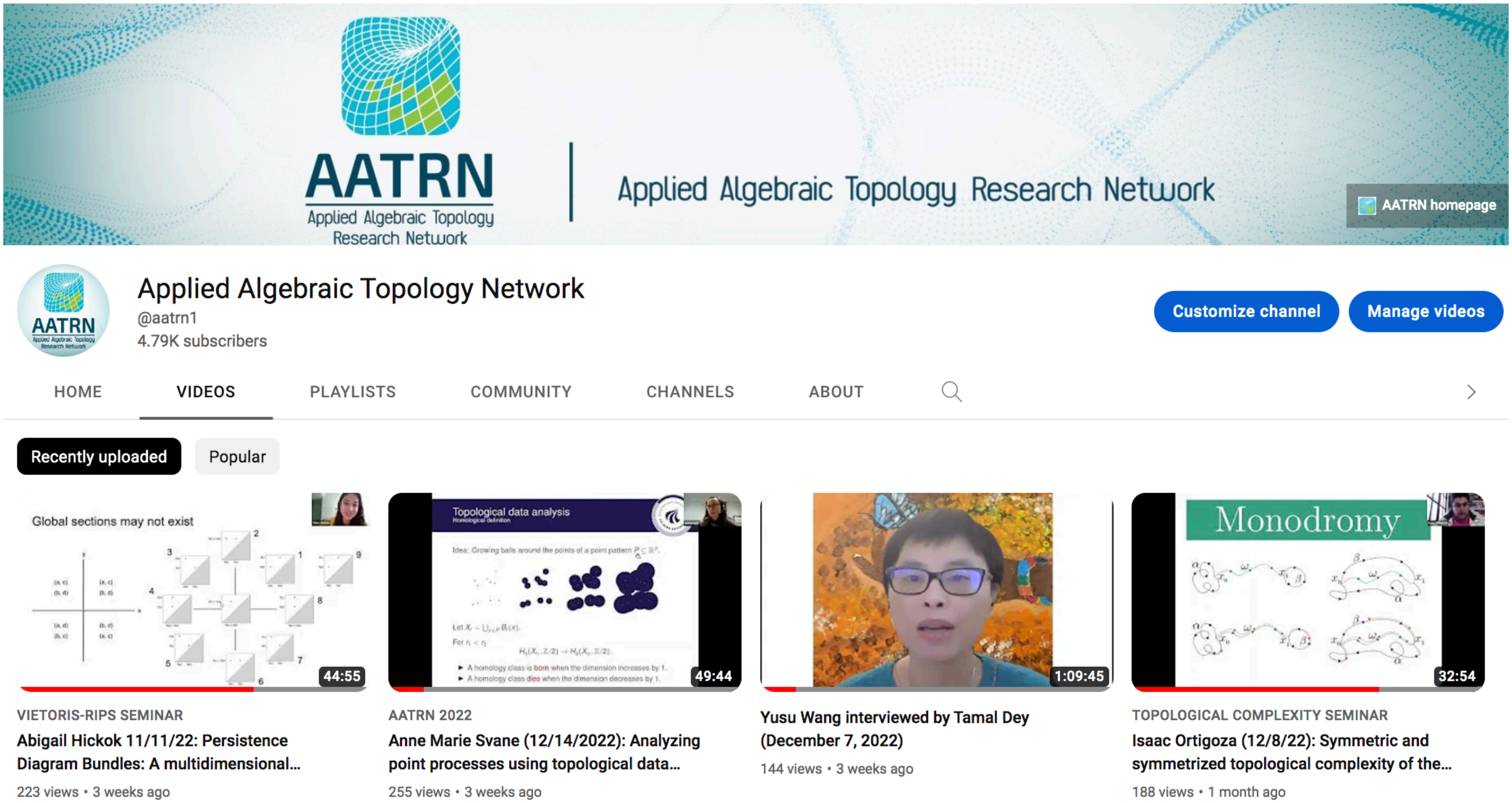
Our next tutorial-a-thon is in 2023. Stay tuned!

AATRN: www.aatrn.net , 1-2 online events per week.
Research talks, interviews, poster sessions, tutorial-a-thons, tea times.
YouTube: 4,750 subscribers, 475 videos, 24 hours watched per day.
We are now accepting contributed videos from you!

AATRN: www.aatrn.net , 1-2 online events per week.
Research talks, interviews, poster sessions, tutorial-a-thons, tea times.
YouTube: 4,750 subscribers, 475 videos, 24 hours watched per day.
We are now accepting contributed videos from you!

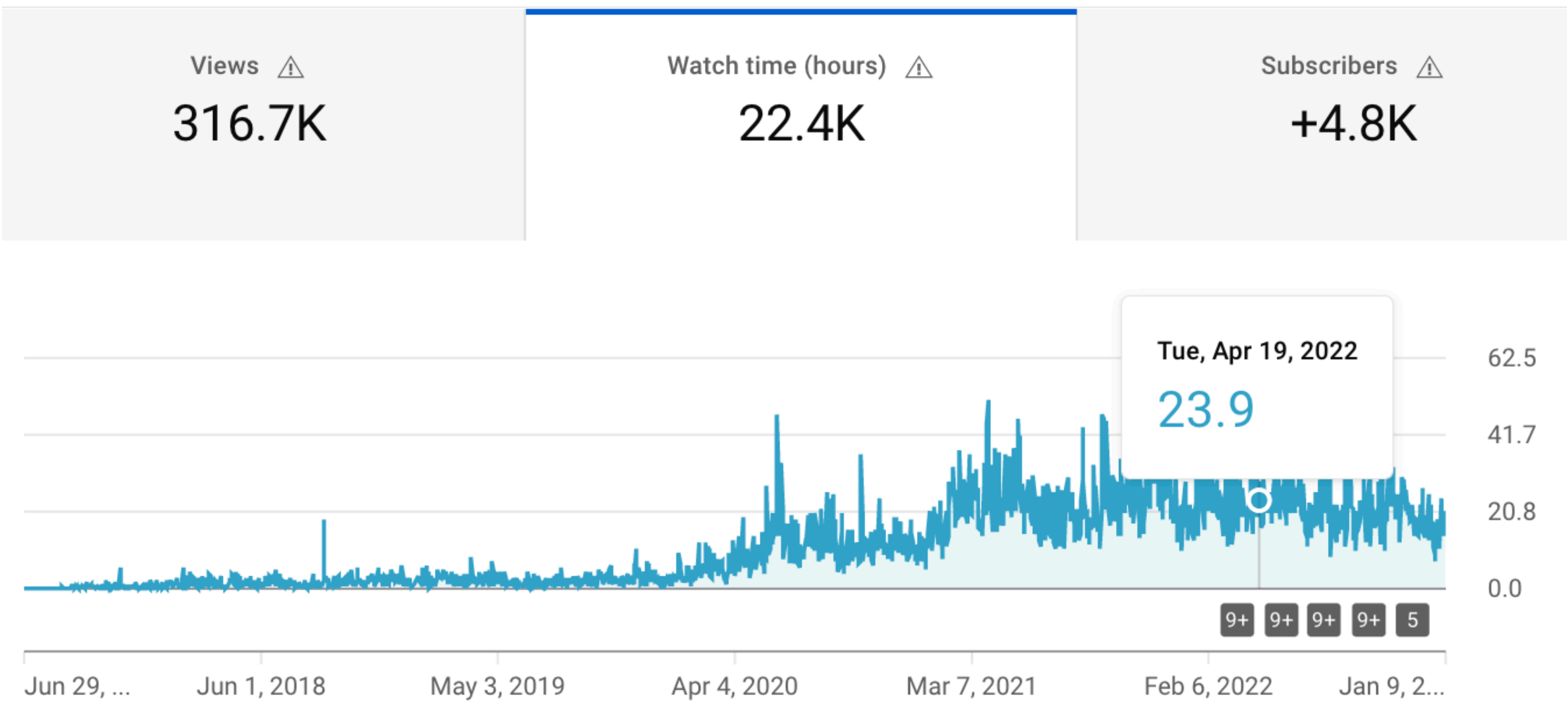
AATRN: www.aatrn.net , 1-2 online events per week.
Research talks, interviews, poster sessions, tutorial-a-thons, tea times.
YouTube: 4,750 subscribers, 475 videos, 24 hours watched per day.
We are now accepting contributed videos from you!

AATRN: www.aatrn.net , 1-2 online events per week.
Research talks, interviews, poster sessions, tutorial-a-thons, tea times.
YouTube: 4,750 subscribers, 475 videos, 24 hours watched per day.
We are now accepting contributed videos from you!



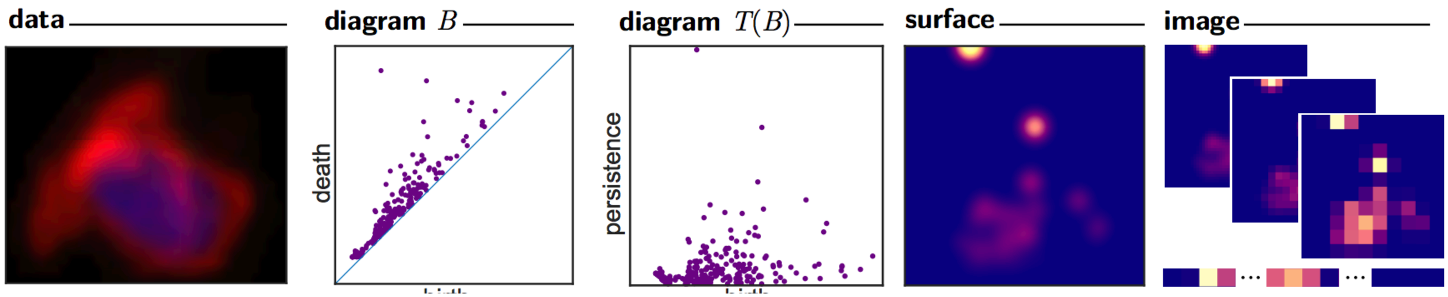
AATRN: www.aatrn.net , 1-2 online events per week.
Research talks, interviews, poster sessions, tutorial-a-thons, tea times.
YouTube: 4,750 subscribers, 475 videos, 24 hours watched per day.
We are now accepting contributed videos from you!

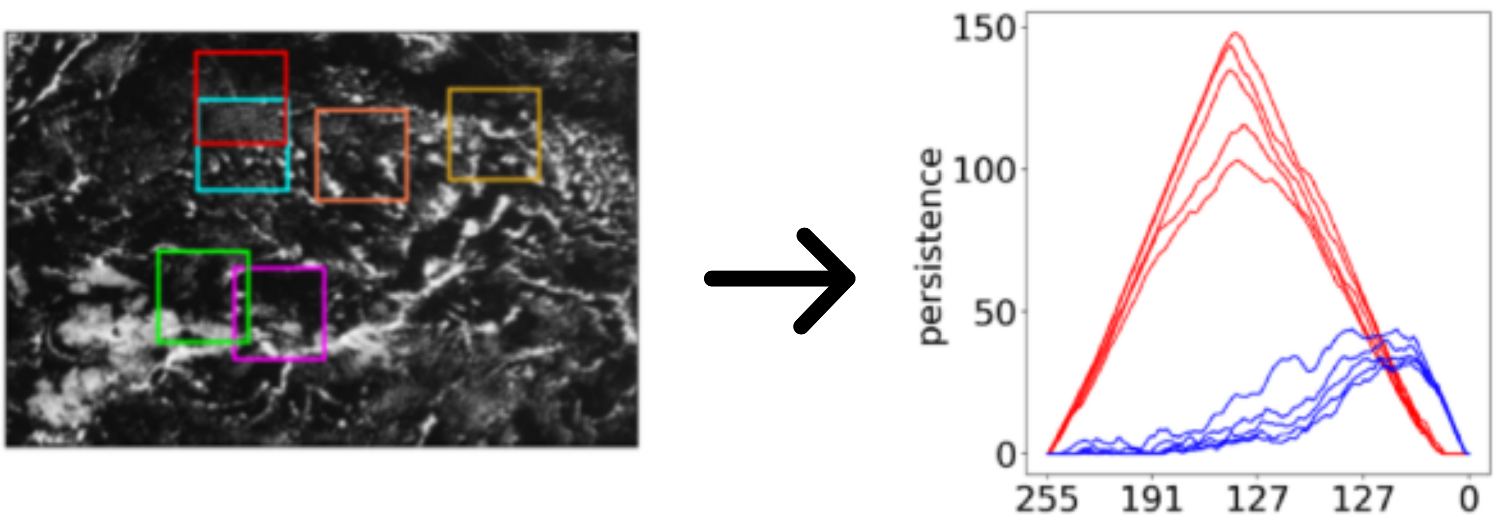
Your channel has gotten 316,747 views so far



AATRN: www.aatrn.net , 1-2 online events per week.
Research talks, interviews, poster sessions, tutorial-a-thons, tea times.
YouTube: 4,750 subscribers, 475 videos, 24 hours watched per day.
We are now accepting contributed videos from you!

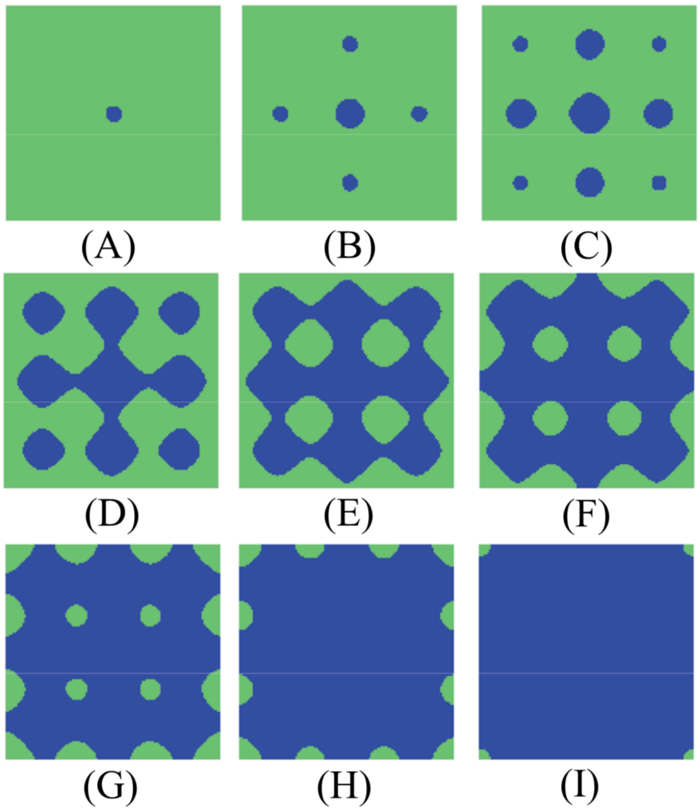
Research Themes

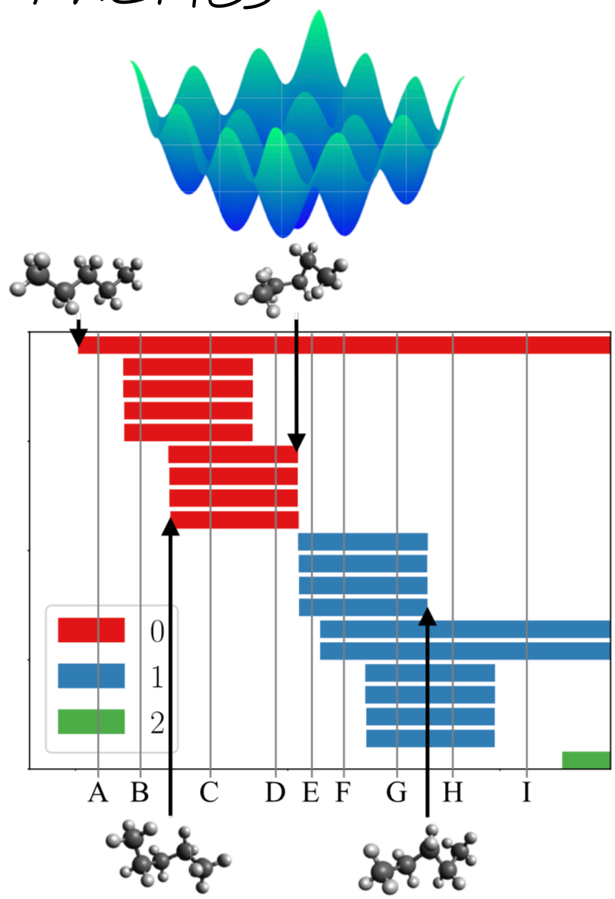


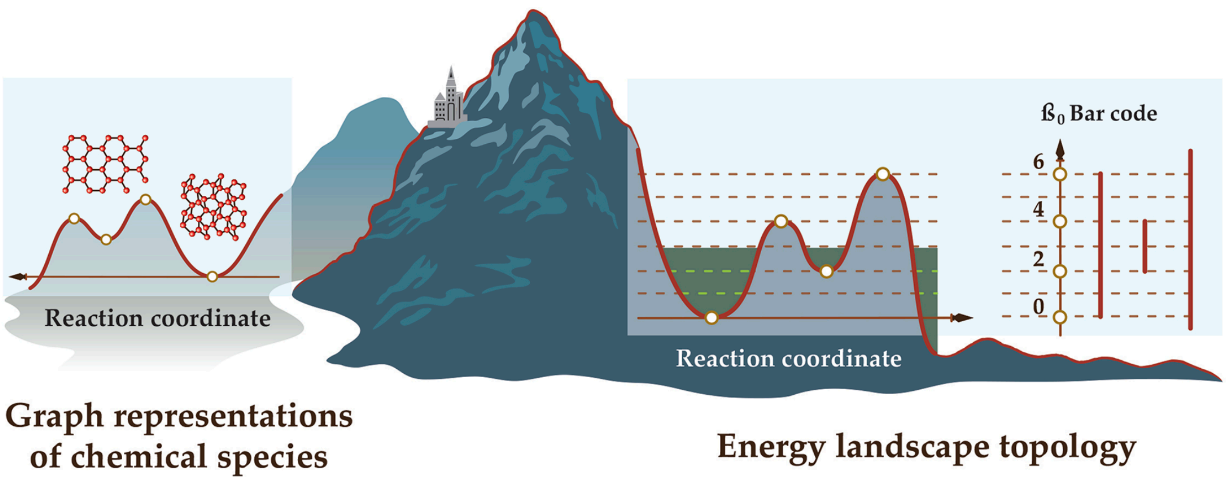


Data Science

Research Themes

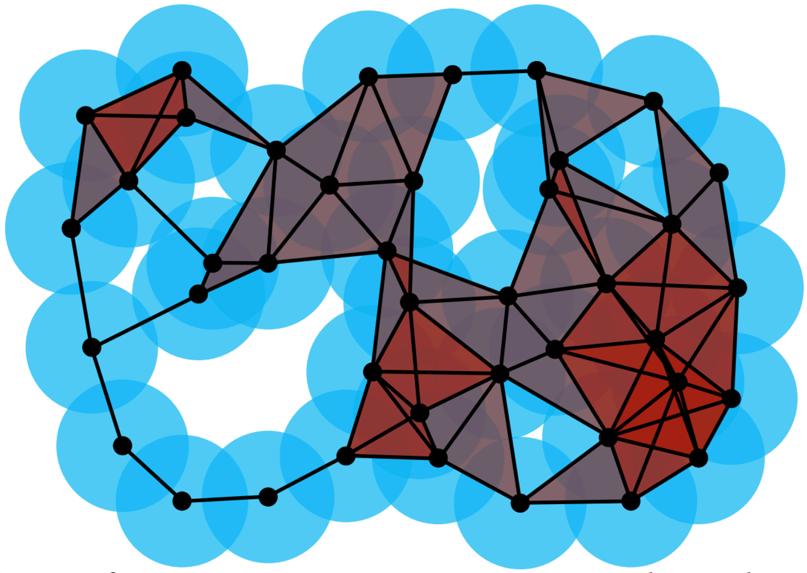


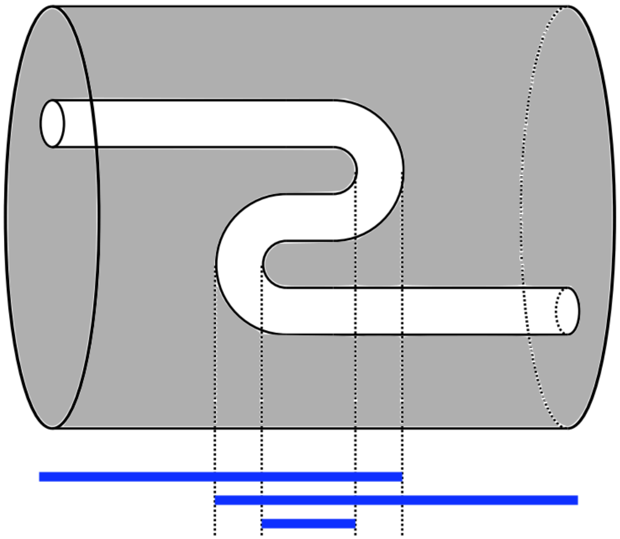


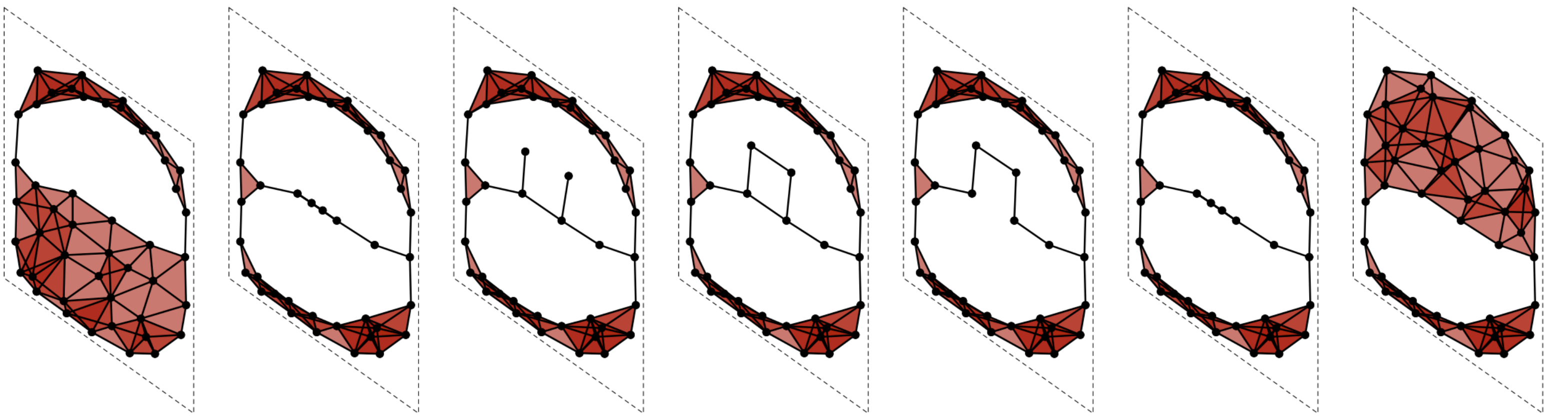


Computational Chemistry

Research Themes

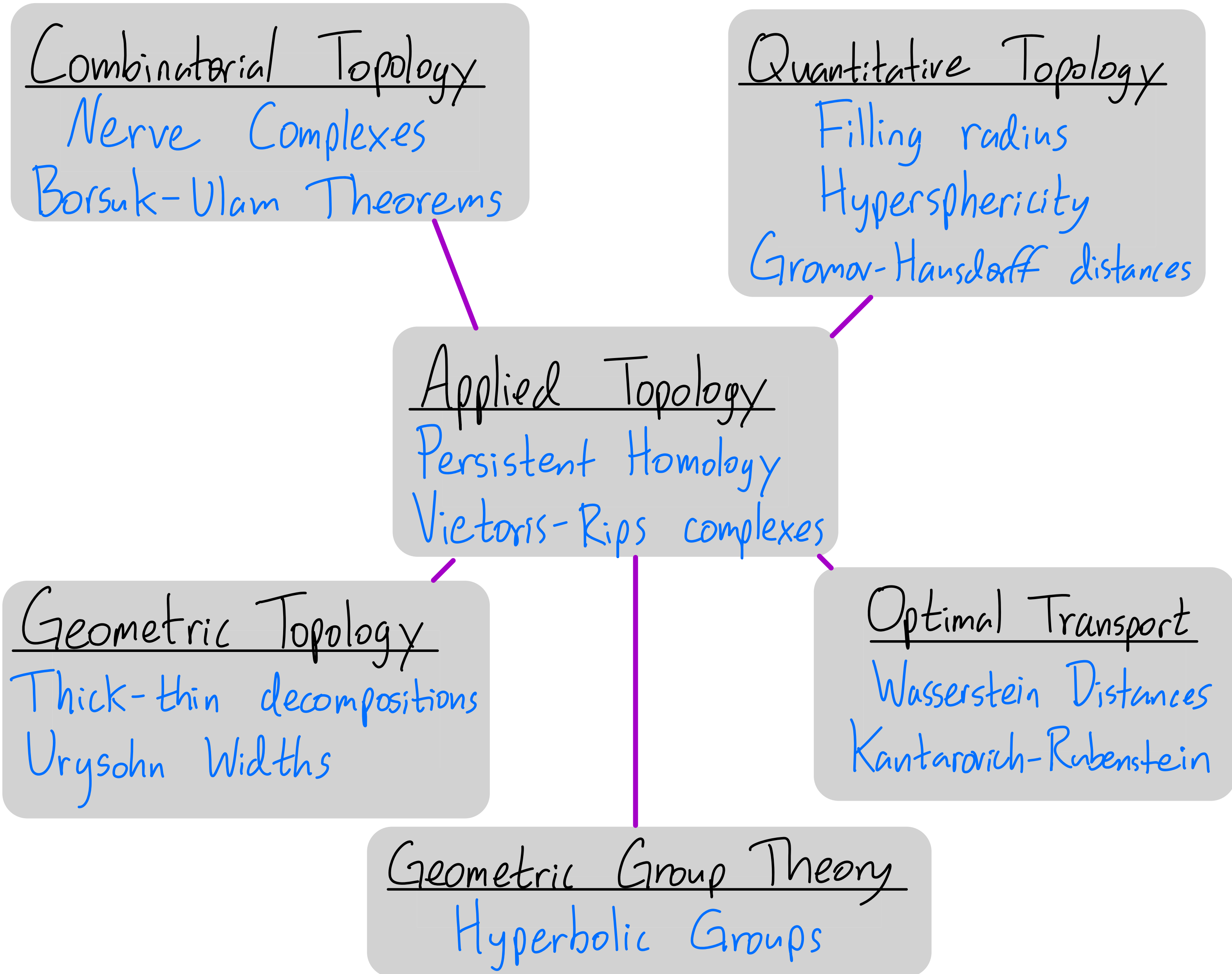






Sensor Networks

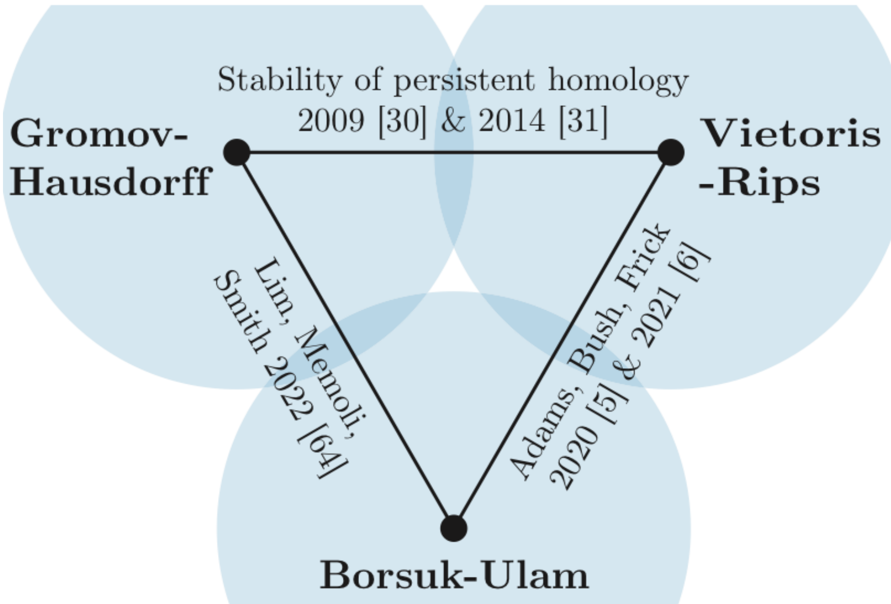
Research Themes

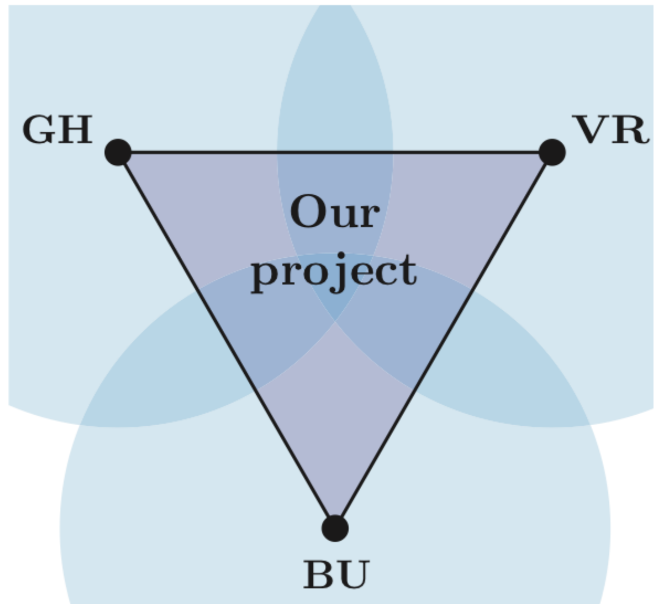


Bridging Applied and Quantitative Topology

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes

arXiv:2301.00246 December 2022



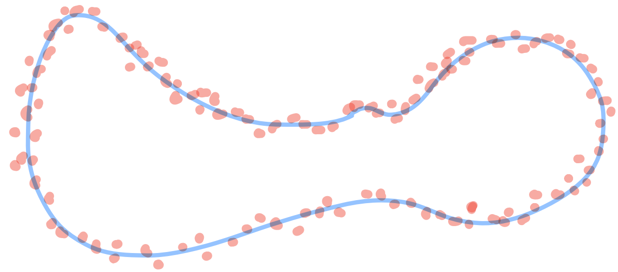


16 authors from 9 institutions:

Henry Adams	Facundo Mémoli	Florian Frick
Johnathan Bush	Nathaniel Clause	Michael Harrison
Michael Moy	Mario Gomez	Amzi Jeffs
Daniel Vargas-Rosario	Sunhyuk Lim	Evgeniya Lagoda
	Qingsong Wang	Nicola Sadovek
	Ling Zhou	Matt Superdock

Gromov–Hausdorff distances

X, Y compact metric spaces

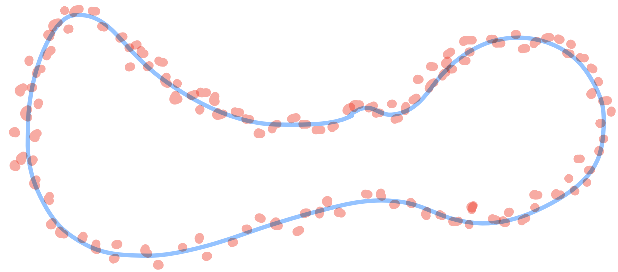


If X and Y are two subsets of the same metric space, then the Hausdorff distance between them is

$$d_H(X,Y) = \inf\{\varepsilon>0 \mid X \subseteq Y^{\varepsilon} \text{ and } Y \subseteq X^{\varepsilon}\}$$

Gromov–Hausdorff distances

X, Y compact metric spaces

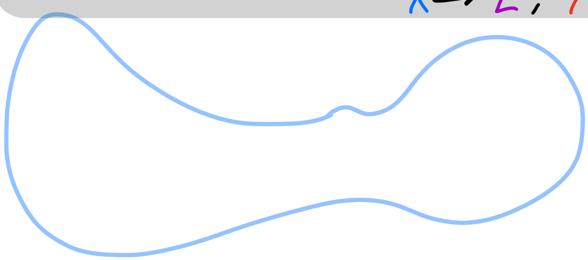


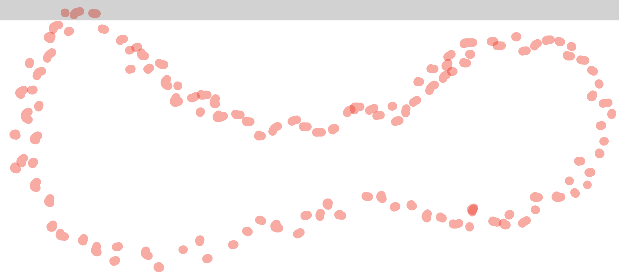
If X and Y are two subsets of the same metric space, then the Hausdorff distance between them is

$$d_H(X, Y) = \inf\{\varepsilon > 0 \mid X \subseteq Y^\varepsilon \text{ and } Y \subseteq X^\varepsilon\}$$

If X and Y are any two metric spaces, then the Gromov-Hausdorff distance between them is

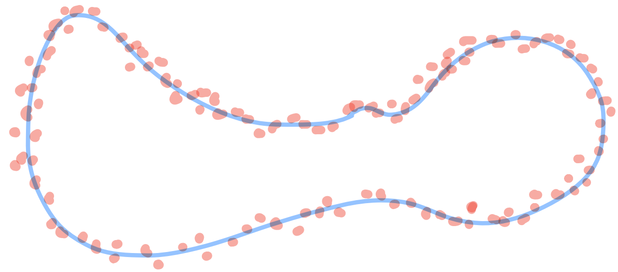
$$d_{GH}(X, Y) = \underset{\substack{\text{isometric embeddings} \\ X \hookrightarrow Z,\ Y \hookrightarrow Z}}{\text{infimum}} \left\{ d_H^Z(X, Y) \right\}$$





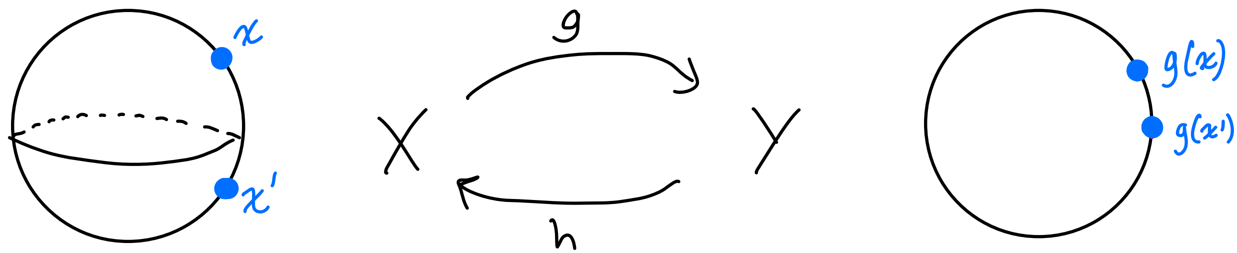
Gromov–Hausdorff distances

X, Y compact metric spaces



Equivalently:

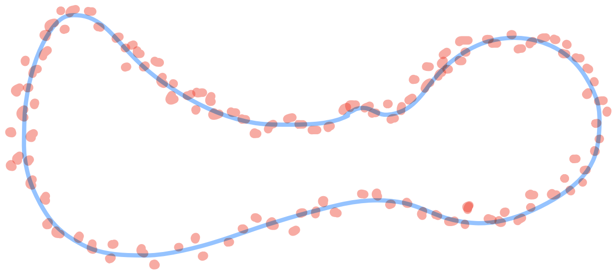
<u>Def</u> $$2 \cdot d_{GH}(X, Y) = \inf_{\substack{g: X \to Y \\ h: Y \to X}} \max\{\operatorname{dis}(g), \operatorname{dis}(h), \operatorname{codis}(g,h)\}.$$



$$\operatorname{dis}(g) = \sup_{x, x' \in X} \left| d(x,x') - d(g(x), g(x')) \right|$$

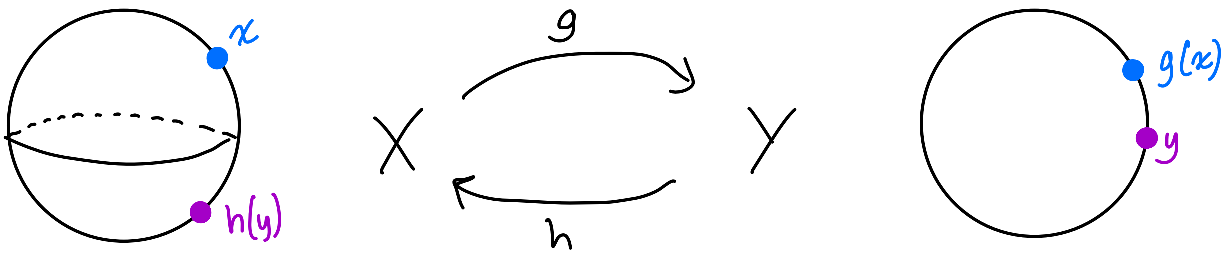
Gromov–Hausdorff distances

X, Y compact metric spaces



Equivalently:

<u>Def</u> $$2 \cdot d_{GH}(X, Y) = \inf_{\substack{g: X \to Y \\ h: Y \to X}} \max \{ \mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h) \}.$$



$$\mathrm{dis}(g) = \sup_{x, x' \in X} \left| d(x, x') - d(g(x), g(x')) \right|$$

$$\mathrm{codis}(g,h) = \sup_{\substack{x \in X \\ y \in Y}} \left| d(x, h(y)) - d(g(x), y) \right|$$

Lim, Mémoli, Smith, 2021

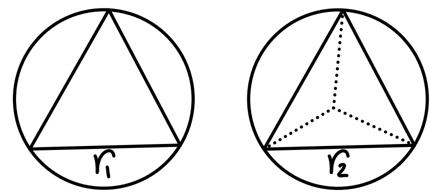
Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$

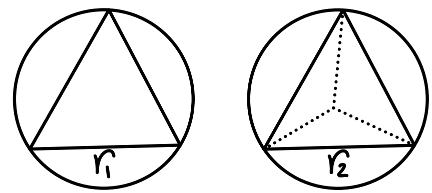


Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$



	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

Proof of some lower bounds:

$$2 \cdot d_{GH}(S^n, S^k) = \inf_{\substack{g: S^k \to S^n \\ h: S^n \to S^k}} \max\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h)\}$$

$$\geq \inf_{g: S^k \to S^n} \mathrm{dis}(g)$$

$$= \inf_{\text{odd } g: S^k \to S^n} \mathrm{dis}(g)$$

Lim, Mémoli, Smith, 2021

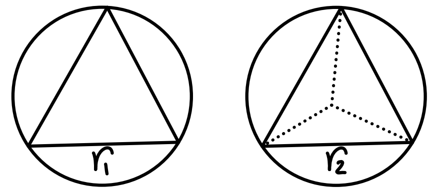
Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

$$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$$



Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

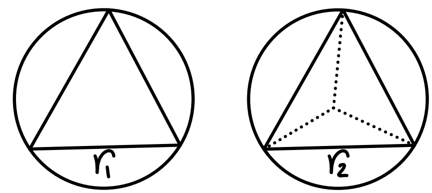
$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix

Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$



Main Theorem

For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}}$$

Lim, Mémoli, Smith, 2021

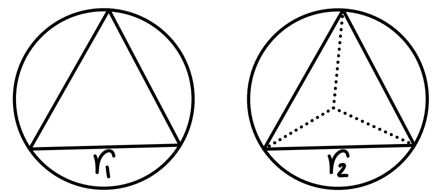
Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

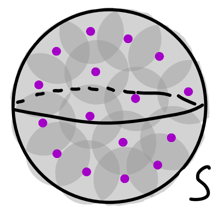
Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$



For $n<k$,

$2 \cdot d_{GH}(S^n, S^k) \geq \pi - \mathrm{cov}_{k+1}(S^n)$



Main Theorem

For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}}$$

Def $\mathrm{cov}_k(X) :=$ infimum r s.t. k balls of radius $\frac{r}{2}$ cover X.

Lim, Mémoli, Smith, 2021

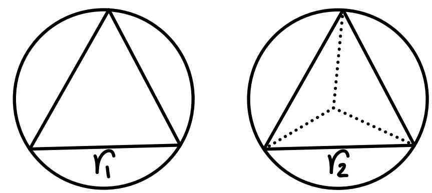
Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

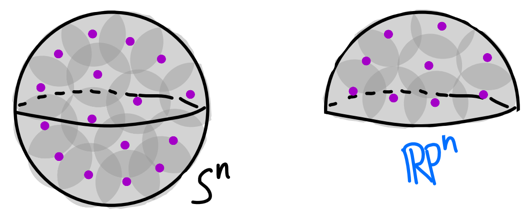
	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$



For $n<k$,
$2 \cdot d_{GH}(S^n, S^k) \geq \pi - \mathrm{cov}_{k+1}(S^n)$



Main Theorem For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}} \geq \pi - \mathrm{cov}_k(\mathbb{RP}^n).$$

A., Bush, Frick, 2021

Def $\mathrm{cov}_k(X) :=$ infimum r s.t. k balls of radius $\frac{r}{2}$ cover X.

Lim, Mémoli, Smith, 2021

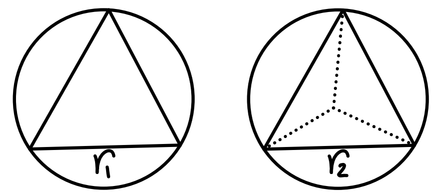
Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$

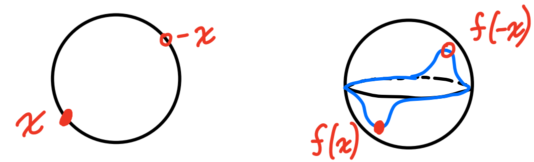


Main Theorem For $n < k$,

$$\underbrace{2 \cdot d_{GH}(S^n, S^k) \geq \inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}$$

$$\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\} = c_{n,k}$$

Borsuk-Ulam theorems

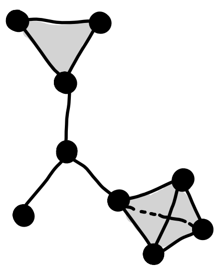


Def A map $f: S^k \to S^n$ is odd if $f(-x) = -f(x)$ $\forall x \in X$

Borsuk-Ulam: There is no cont. odd $S^k \to S^n$ for $k > n$.

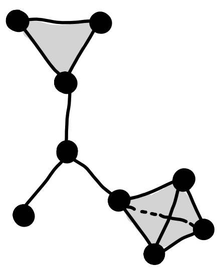
Vietoris – Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X;r)$ has vertex set X, all simplices of diameter $\leq r$.



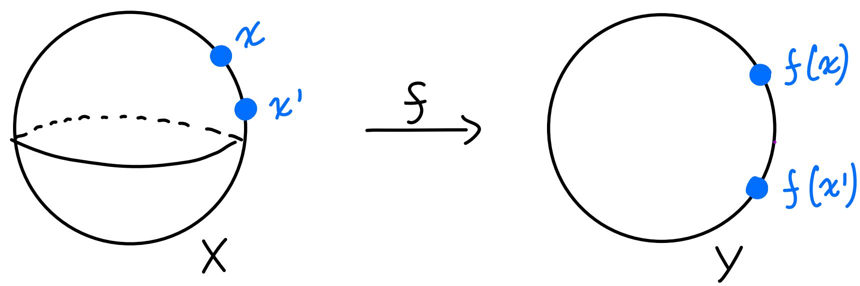
Vietoris – Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X;r)$ has vertex set X, all simplices of diameter $\leq r$.



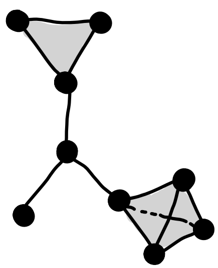
Function $f: X \to Y$ induces a (cont.) simplicial map $f: VR(X, r) \longrightarrow VR(Y; dis(f)+r)$.

$$[x_0, \ldots, x_m] \longmapsto [f(x_0), \ldots, f(x_m)]$$



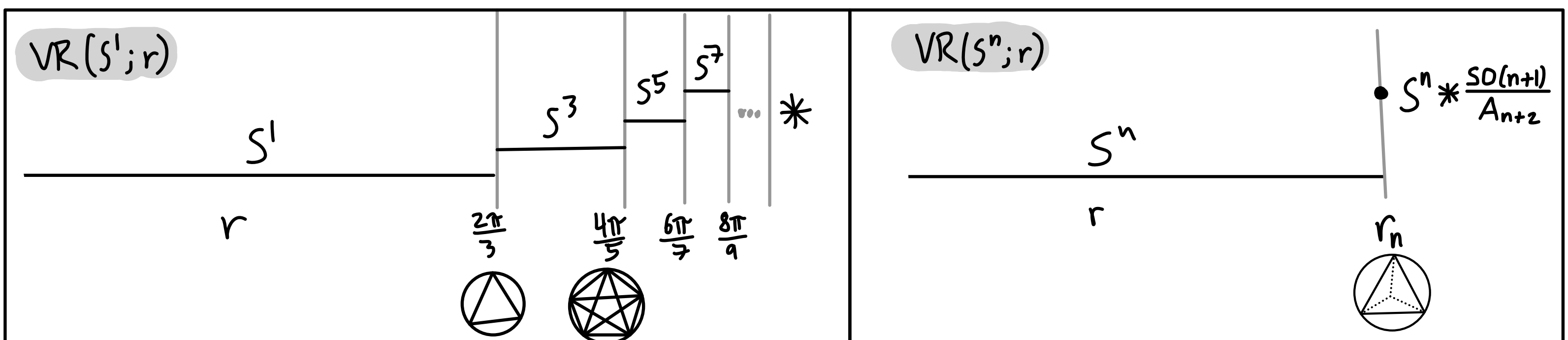
Vietoris – Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X;r)$ has vertex set X, all simplices of diameter $\leq r$.



Function $f: X \to Y$ induces a (cont.) simplicial map $f: VR(X, r) \longrightarrow VR(Y; \operatorname{dis}(f)+r)$.

$$[x_0, \ldots, x_m] \longmapsto [f(x_0), \ldots, f(x_m)]$$



$$C_{1,2k+1} = C_{1,2k} = \frac{2\pi k}{2k+1} \qquad C_{n,n+2} = C_{n,n+1} = r_n$$

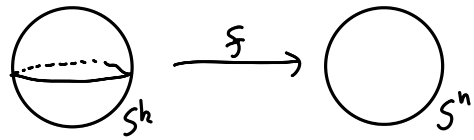
$$\underbrace{\inf \{ r \mid \exists \text{ cont. odd } S^k \to VR(S^n;r) \}}_{C_{n,k}}$$

Dubins & Schwarz '81

Lim, Mémoli, Smith, 2021

(Discont.) odd maps $f: S^{n+1} \to S^n$ have $\mathrm{dis}(f) \geq r_n$.

<u>Generalization</u> (Discont.) odd maps $f: S^k \to S^n$ for $k > n$ have $\mathrm{dis}(f) \geq c_{n,k}$.



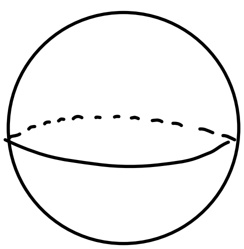
<u>Proof</u>

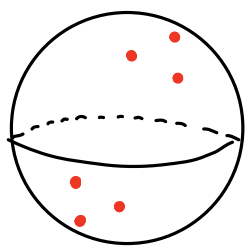
For $\varepsilon > 0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.

Produce a cont. odd map

$$S^k \xrightarrow{\text{partition of unity}} VR(X;\varepsilon) \xrightarrow{f} VR(S^n; \mathrm{dis}(f)+\varepsilon).$$

$$[x_0, \dots, x_m] \longmapsto [f(x_0), \dots, f(x_m)]$$



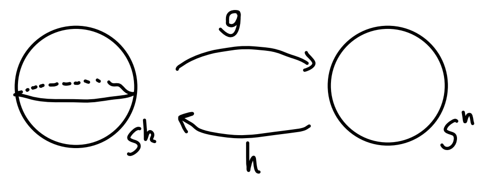


Hence $\mathrm{dis}(f) + \varepsilon \geq c_{n,k}$ $\forall \varepsilon > 0$, so $\mathrm{dis}(f) \geq c_{n,k}$. $\square$

$$\underbrace{\inf \{ r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r) \}}_{c_{n,k}}$$

Main Theorem For $n<k$,

$$2\cdot d_{GH}(S^n,S^k) \ge \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k\to VR(S^n;r)\}}_{C_{n,k}}.$$



Proof of Theorem follows Lim, Mémoli, Smith, 2021

$$2\cdot d_{GH}(S^n,S^k) = \inf_{\substack{g:S^k\to S^n\\ h:S^n\to S^k}} \max\{\mathrm{dis}(g),\ \mathrm{dis}(h),\ \mathrm{codis}(g,h)\}$$

$$\ge \inf_{g:S^k\to S^n} \mathrm{dis}(g)$$

$$= \inf_{\text{odd } g:S^k\to S^n} \mathrm{dis}(g)$$

$$\ge C_{n,k}.$$

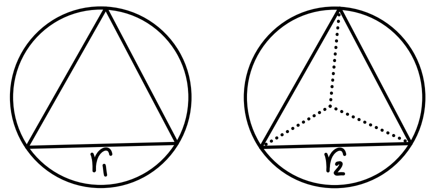
Sphere S^n, geodesic metric, diameter π.

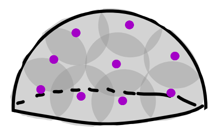
$2\cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$





$\mathbb{RP}^n$

Main Theorem

For $n<k$,

$$2\cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}} \geq \pi - \mathrm{cov}_k(\mathbb{RP}^n).$$

Question Tight upper bounds on $d_{GH}(S^n, S^k)$ via maps?

Question Bounds on $d_{GH}(X, Y)$ for more general families of G-equivariant metric spaces X, Y?

Question Relate the p-Gromov-Wasserstein distance $d_{p\text{-}GW}$ to p-Vietoris-Rips thickenings VR_p?

Question How does the generalization of Dubins & Schwarz relate to Tverberg?

Last section of our paper advertises 12 open questions!
3 follow-up papers in preparation already!