

Bridging Applied and Quantitative Topology

Henry Adams



**COLORADO STATE
UNIVERSITY**



AATR
Applied Algebraic Topology
Research Network

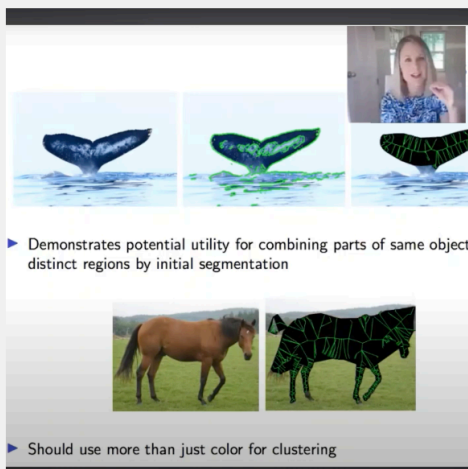


AATR

Applied Algebraic Topology Research Network

Bringing together researchers across the world to develop and use applied and computational topology.

Find out how you can [participate and join AATR](#) (membership is free and so are all events), or check out all our content on our [YouTube channel](#). We invite you to join the following activities, which we are currently organizing:



▶ Demonstrates potential utility for combining parts of same object distinct regions by initial segmentation

▶ Should use more than just color for clustering

Seminar Series

i We have a weekly Seminar Series (or sometimes a tea time) on Wednesdays.



Interview Series
 AUG 11th KATHRYN HESS INTERVIEWED BY PETER BUBENIK
 LISBETH FAJSTRUP INTERVIEWED BY MARTIN RAUSSEN SEP 29th
 OCT 20th ROBERT ADLER INTERVIEWED BY OMER BOBROWSKI
 SHMUEL WEINBERGER INTERVIEWED BY KATHARINE TURNER FEB 9th
 MAR 9th ROBERT GHRIST INTERVIEWED BY RADMILA SAZDANOVIC
FOR MORE COORDINATES, BECOME AN AATR MEMBER AT topology.ima.jhu.edu

Interview Series

See our upcoming interviews.



AATR + WINCOMPTOP PRESENT

Tutorial-a-thon
 MAKE TDA GO VIRAL!

Tutorial-a-thon

Our next tutorial-a-thon is in 2023. Stay tuned!

AATR: www.aatr.net, 1-2 online events per week.
 Research talks, interviews, poster sessions, tutorial-a-thons, tea times.
 YouTube: 4,750 subscribers, 475 videos, 24 hours watched per day.
 We are now accepting contributed videos from you!



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INTERVIEW series

2021-2022

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Applied Algebraic Topology Research Network

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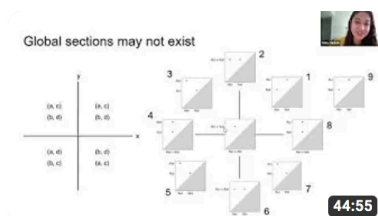
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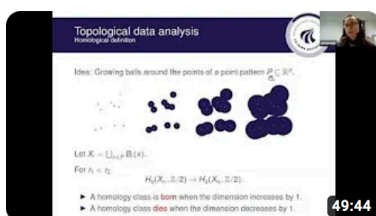
Popular



VIETORIS-RIPS SEMINAR

Abigail Hickok 11/11/22: Persistence Diagram Bundles: A multidimensional...

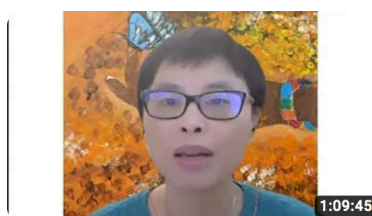
223 views • 3 weeks ago



AATR 2022

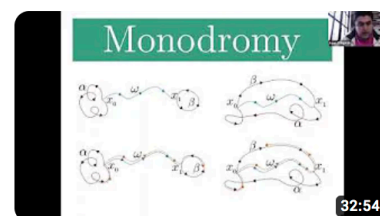
Anne Marie Svane (12/14/2022): Analyzing point processes using topological data...

255 views • 3 weeks ago



Yusu Wang interviewed by Tamal Dey (December 7, 2022)

144 views • 3 weeks ago



TOPOLOGICAL COMPLEXITY SEMINAR

Isaac Ortigoza (12/8/22): Symmetric and symmetrized topological complexity of the...

188 views • 1 month ago

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Views 

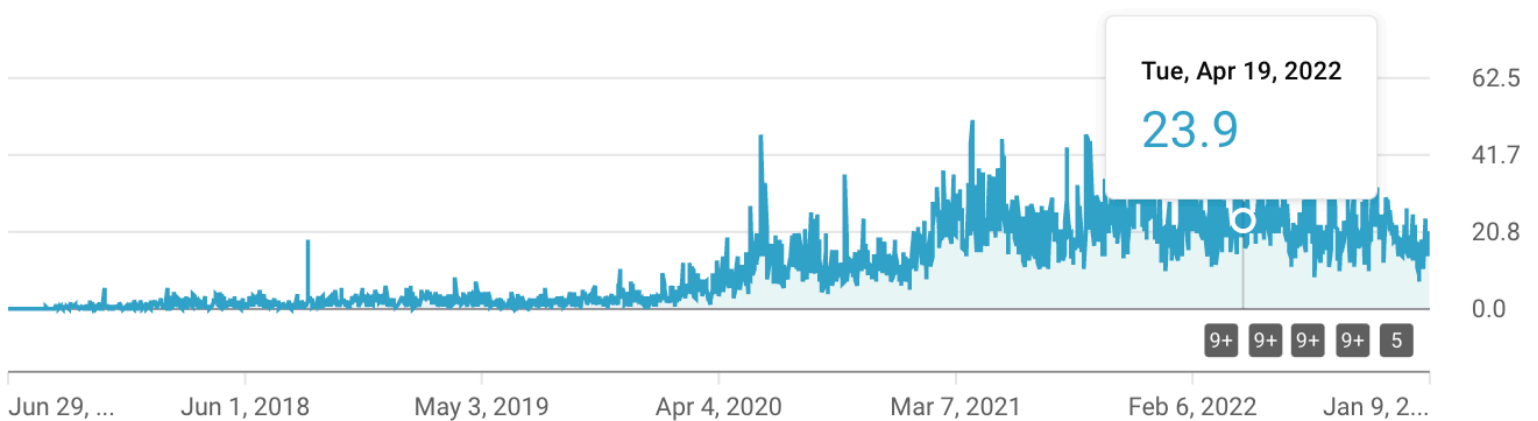
316.7K

Watch time (hours) 

22.4K

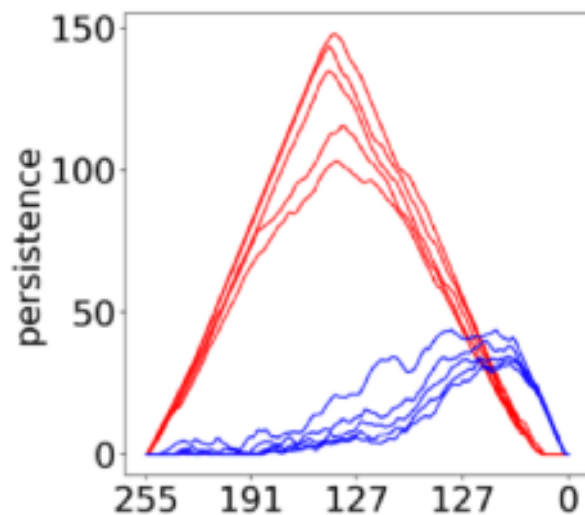
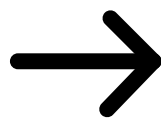
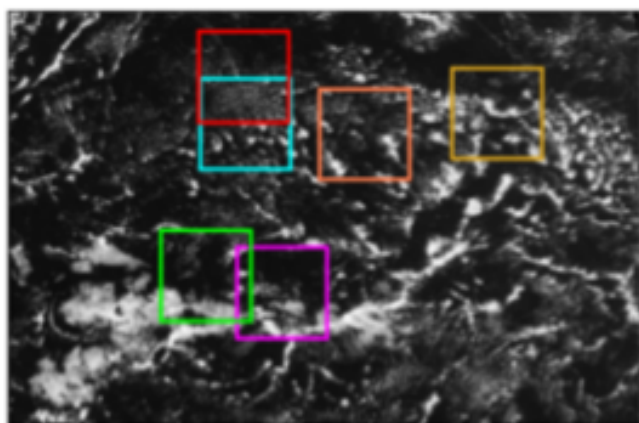
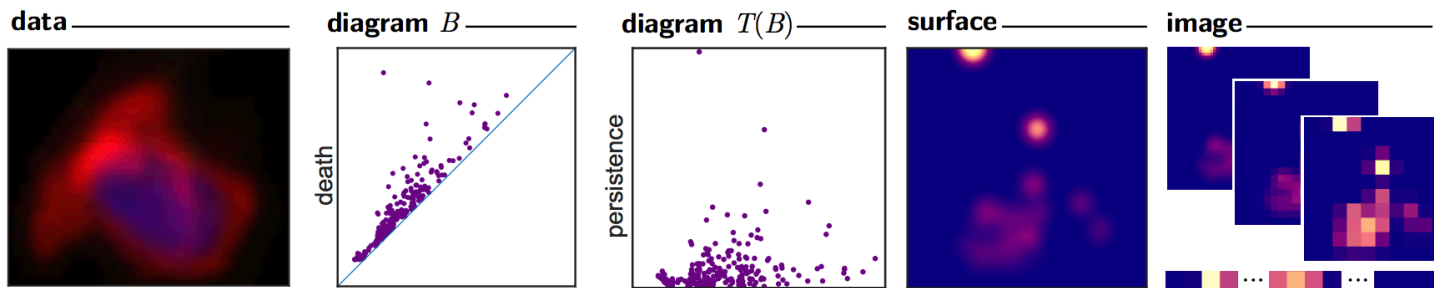
Subscribers 

+4.8K



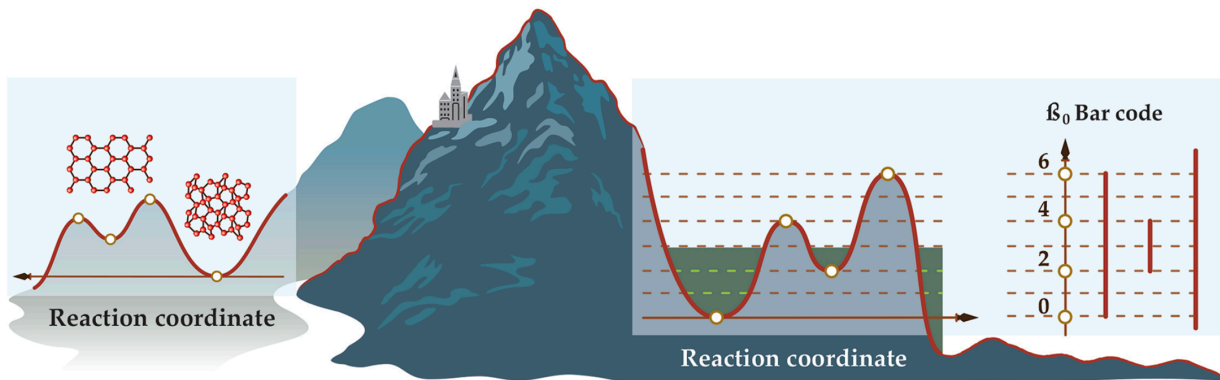
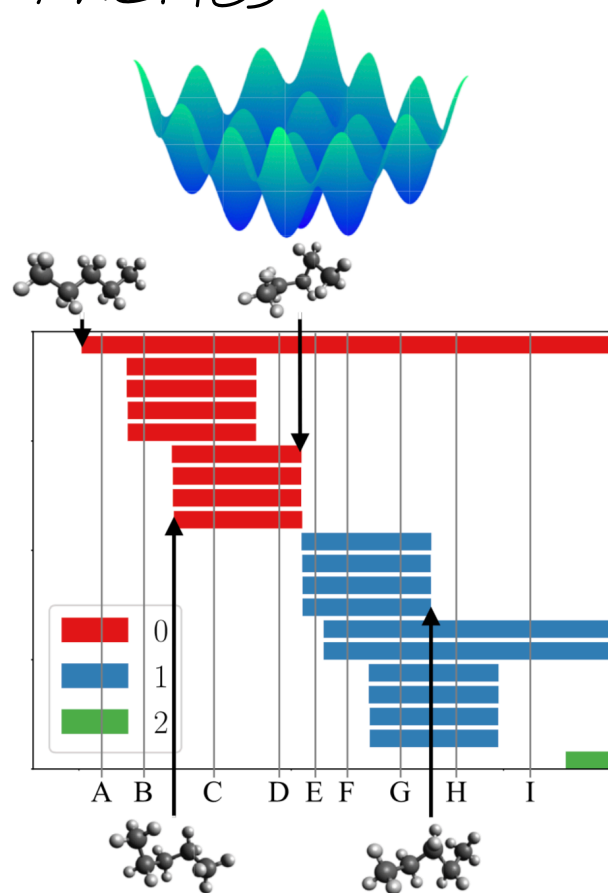
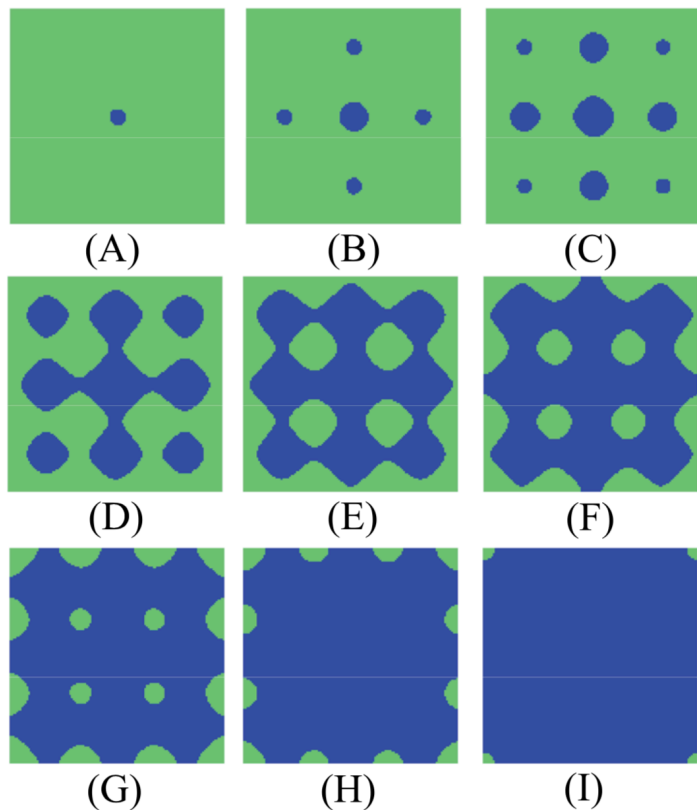
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Research Themes



Data Science

Research Themes

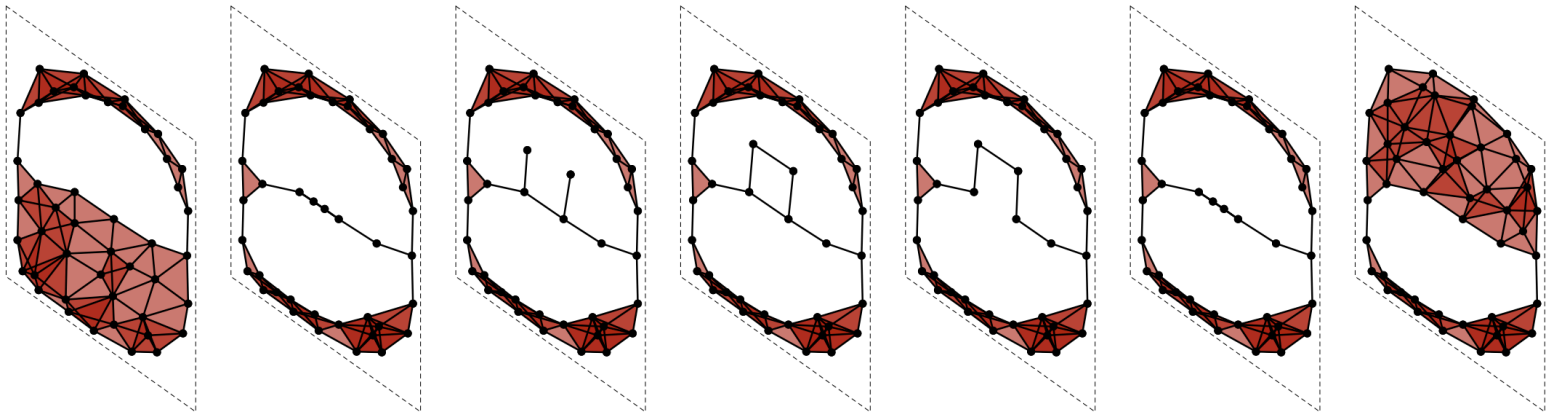
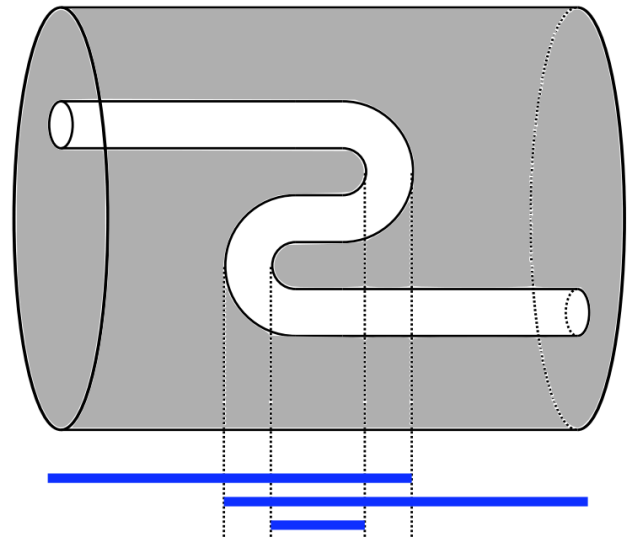
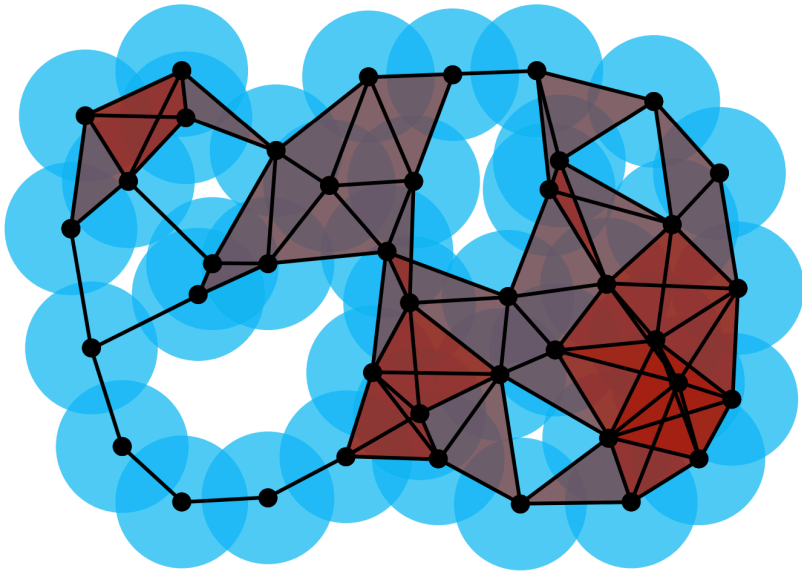


Graph representations
of chemical species

Energy landscape topology

Computational Chemistry

Research Themes



Sensor Networks

Research Themes

Combinatorial Topology

Nerve Complexes
Borsuk-Ulam Theorems

Quantitative Topology

Filling radius
Hypersphericity
Gromov-Hausdorff distances

Applied Topology

Persistent Homology
Vietoris-Rips complexes

Geometric Topology

Thick-thin decompositions
Urysohn Widths

Optimal Transport

Wasserstein Distances
Kantorovich-Rubenstein

Geometric Group Theory

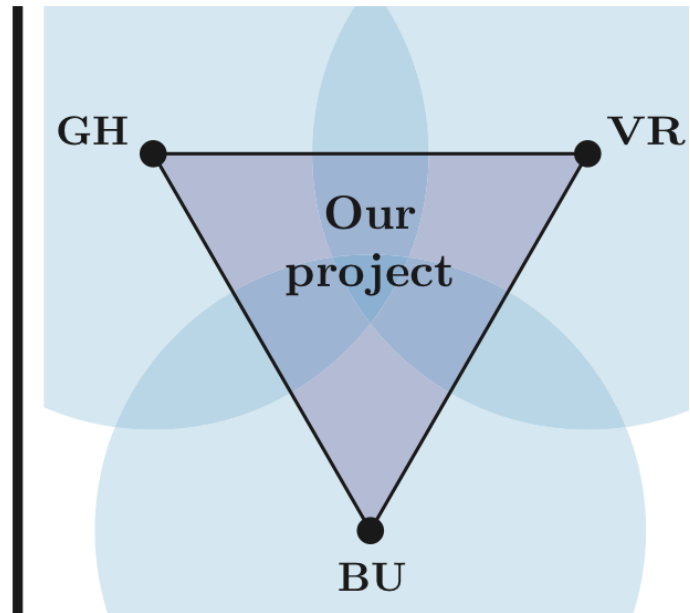
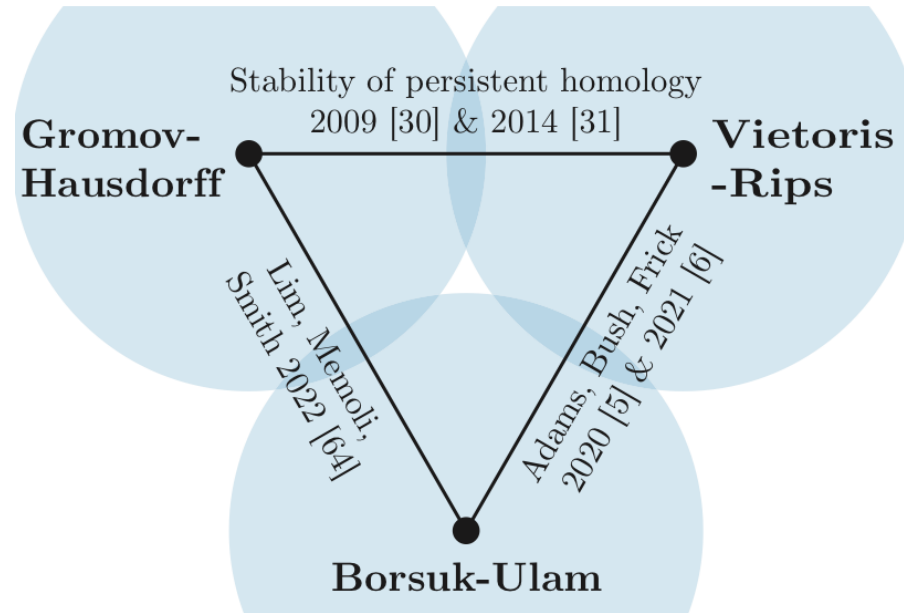
Hyperbolic Groups

Bridging Applied and Quantitative Topology

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes

arXiv:2301.00246

December 2022



16 authors from 9 institutions:

Henry Adams

Johnathan Bush

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Facundo Memoli

Nathaniel Clause

Mario Gomez

Sunhyuk Lim

Qingsong Wang

Ling Zhou

Florian Frick

Michael Harrison

Amzi Jeffs

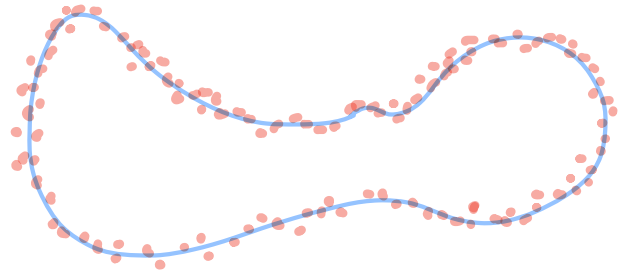
Evgeniya Lagoda

Nicola Sadovek

Matt Superdock

Gromov-Hausdorff distances

X, Y compact metric spaces

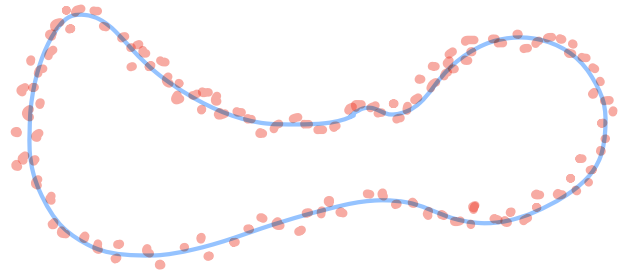


If X and Y are two subsets of the same metric space, then the Hausdorff distance between them is

$$d_H(X, Y) = \inf \left\{ \varepsilon > 0 \mid X \subseteq Y^\varepsilon \text{ and } Y \subseteq X^\varepsilon \right\}$$

Gromov-Hausdorff distances

X, Y compact metric spaces

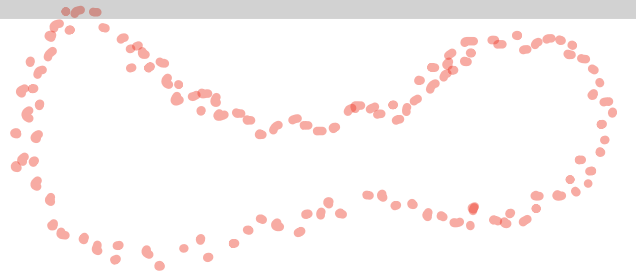
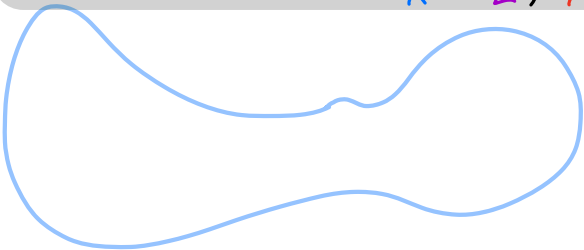


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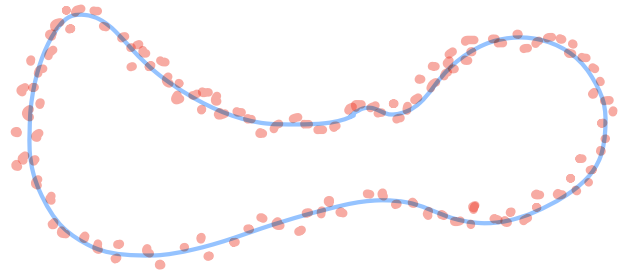
If X and Y are any two metric spaces, then the Gromov-Hausdorff distance between them is

$$d_{GH}(X, Y) = \inf_{\substack{\text{in fimum} \\ \text{isometric embeddings} \\ X \hookrightarrow Z, Y \hookrightarrow Z}} \left\{ d_H(X, Y) \right\}$$



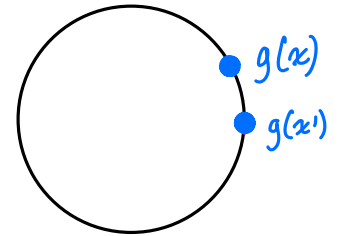
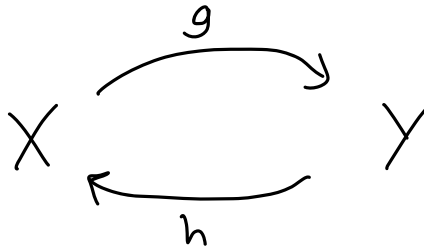
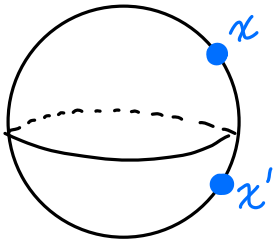
Gromov-Hausdorff distances

X, Y compact metric spaces



Equivalently:

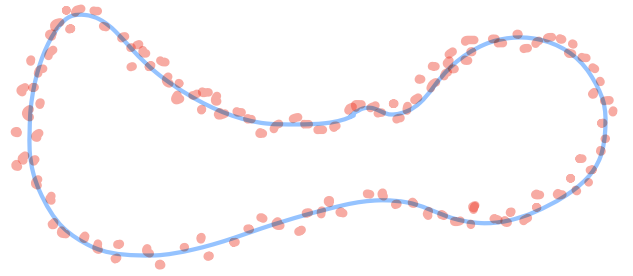
Def 2 $d_{GH}(X, Y) = \inf_{\substack{g: X \rightarrow Y \\ h: Y \rightarrow X}} \max \{ \text{dis}(g), \text{dis}(h), \text{codis}(g, h) \}$.



$$\text{dis}(g) = \sup_{x, x' \in X} | d(x, x') - d(g(x), g(x')) |$$

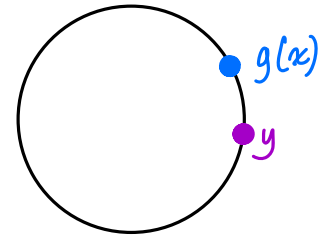
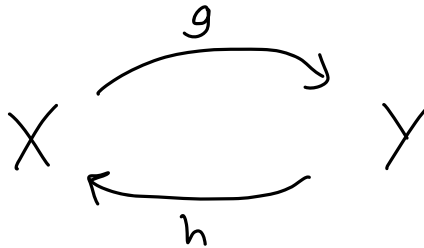
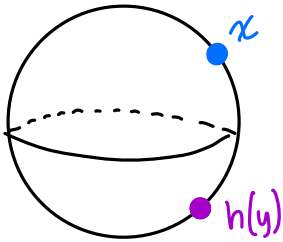
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$$\text{codis}(g, h) = \sup_{\substack{x \in X \\ y \in Y}} | d(x, h(y)) - d(g(x), y) |$$

Lim, Memoli, Smith, 2021

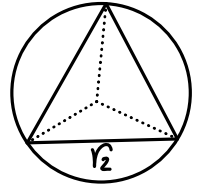
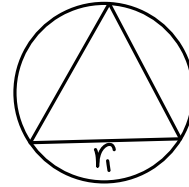
Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0 $\geq r_3$				
S^4				0 $\geq r_4$			
S^5					0 $\geq r_5$		
S^6						0 $\geq r_6$	

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$

$$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$$



Lim, Memoli, Smith, 2021

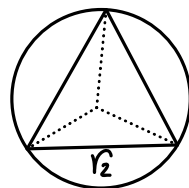
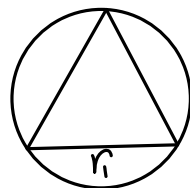
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Proof of some lower bounds:

$$2 \cdot d_{GH}(S^n, S^k) = \inf_{\substack{g: S^k \rightarrow S^n \\ h: S^n \rightarrow S^k}} \max \{ \text{dis}(g), \text{dis}(h), \text{codis}(g, h) \}$$

$$\geq \inf_{g: S^k \rightarrow S^n} \text{dis}(g)$$

$$= \inf_{\text{odd } g: S^k \rightarrow S^n} \text{dis}(g)$$

Lim, Memoli, Smith, 2021

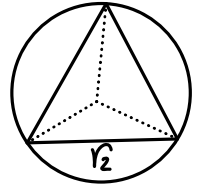
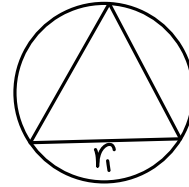
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Lim, Memoli, Smith, 2021

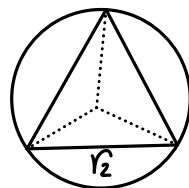
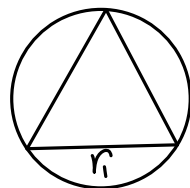
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$$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$$



Main Theorem For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}}_{C_{n,k}}$$

Lim, Memoli, Smith, 2021

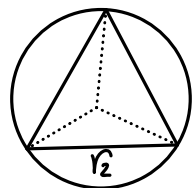
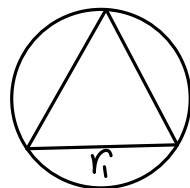
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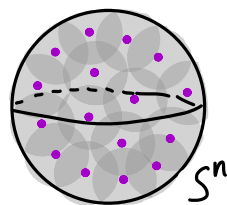
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For $n < k$,

$$\leftarrow 2 \cdot d_{GH}(S^n, S^k) \geq \pi - \text{cov}_{k+1}(S^n)$$



Main Theorem For $n < k$,

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Def $\text{cov}_k(X) := \text{infimum } r \text{ s.t. } k \text{ balls of radius } \frac{r}{2} \text{ cover } X.$

Lim, Memoli, Smith, 2021

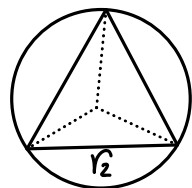
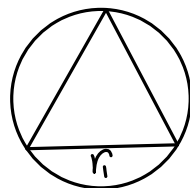
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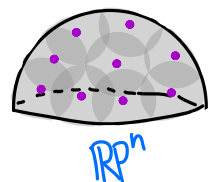
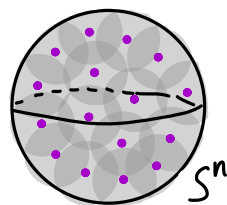
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Main Theorem For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow \text{VR}(S^n; r) \right\}}_{C_{n,k}} \geq \pi - \text{cov}_k(\mathbb{RP}^n).$$

$C_{n,k}$

A., Bush, Frick, 2021

Def $\text{cov}_k(X) := \text{infimum } r \text{ s.t. } k \text{ balls of radius } \frac{r}{2} \text{ cover } X.$

Lim, Memoli, Smith, 2021

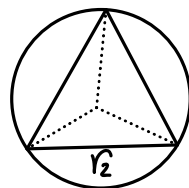
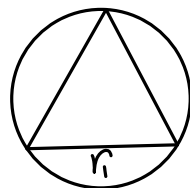
Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3} \geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$	
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$

$$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$$



Main Theorem For $n < k$,

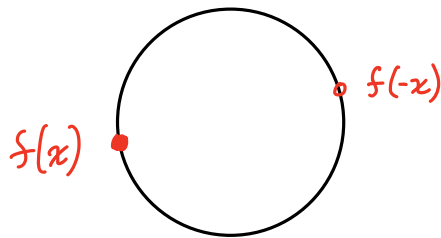
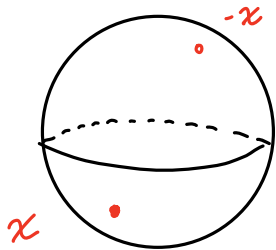
$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}}_{C_{n,k}}$$

Borsuk-Ulam theorems



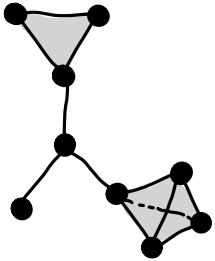
Def A map $f: S^k \rightarrow S^n$ is odd if $f(-x) = -f(x) \quad \forall x \in X$

Borsuk-Ulam: There is no cont. odd $S^k \rightarrow S^n$ for $k > n$.



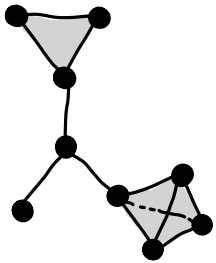
Vietoris-Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X; r)$
has vertex set X , all simplices of diameter $\leq r$.

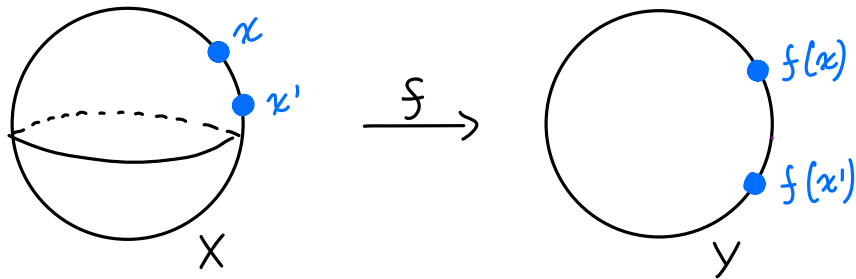


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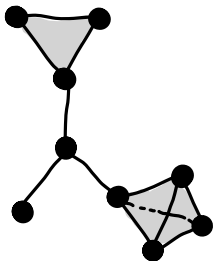


Function $f: X \rightarrow Y$ induces a (cont.) simplicial
map $f: VR(X, r) \rightarrow VR(Y; \text{dis}(f) + r)$.
 $[x_0, \dots, x_m] \mapsto [f(x_0), \dots, f(x_m)]$



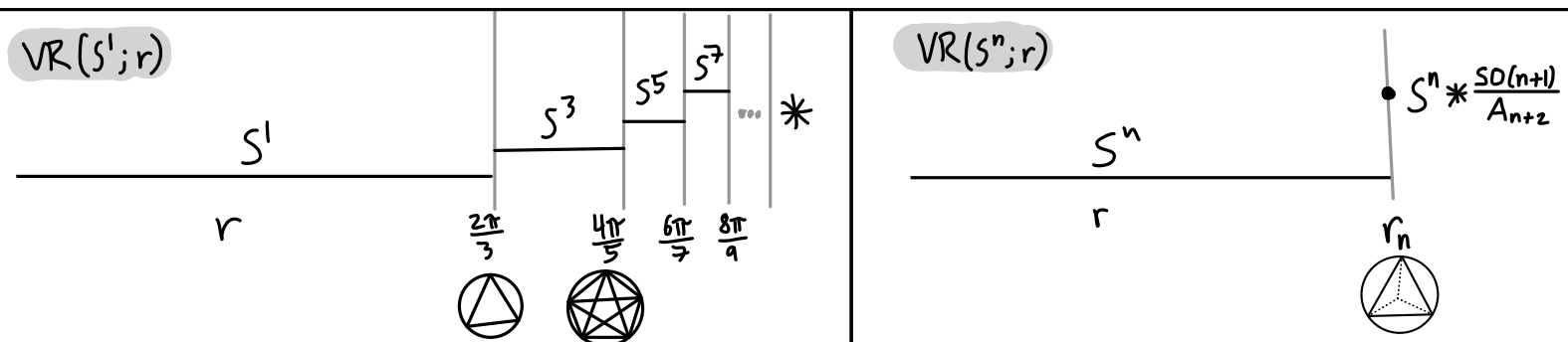
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Function $f: X \rightarrow Y$ induces a (cont.) simplicial map $f: VR(X, r) \rightarrow VR(Y; \text{dis}(f)+r)$.

$[x_0, \dots, x_m] \mapsto [f(x_0), \dots, f(x_m)]$



$$C_{1,2k+1} = C_{1,2k} = \frac{2\pi k}{2k+1}$$

$$C_{n,n+2} = C_{n,n+1} = r_n$$

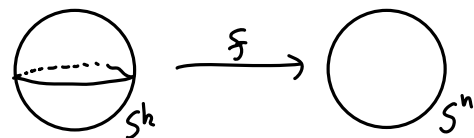
$$C_{n,k} = \inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}$$

Dubins & Schwarz '81

Lim, Memoli, Smith, 2021

(Discont.) odd maps $f: S^{n+1} \rightarrow S^n$ have $\text{dis}(f) \geq r_n$.

Generalization (Discont.) odd maps $f: S^k \rightarrow S^n$ for $k > n$ have $\text{dis}(f) \geq C_{n,k}$.



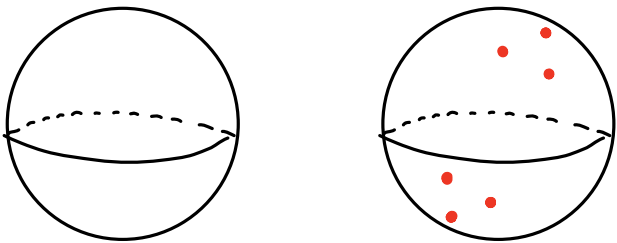
Proof

For $\varepsilon > 0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.

Produce a cont. odd map

$$S^k \xrightarrow{\text{partition of unity}} VR(X; \varepsilon) \xrightarrow{f} VR(S^n; \text{dis}(f) + \varepsilon).$$

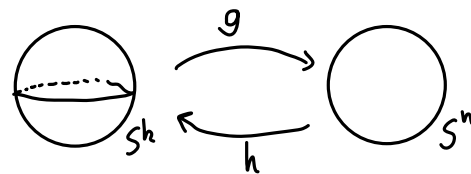
$$[\alpha_0, \dots, \alpha_m] \longmapsto [f(\alpha_0), \dots, f(\alpha_m)]$$



Hence $\text{dis}(f) + \varepsilon \geq C_{n,k} \quad \forall \varepsilon > 0$, so $\text{dis}(f) \geq C_{n,k}$. \square

$$\underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}}_{C_{n,k}}$$

Main Theorem For $n < k$,
 $2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \}}_{C_{n,k}}.$



Proof of Theorem follows Lim, Memoli, Smith, 2021

$$2 \cdot d_{GH}(S^n, S^k) = \inf_{\substack{g: S^k \rightarrow S^n \\ h: S^n \rightarrow S^k}} \max \{ \text{dis}(g), \text{dis}(h), \text{codis}(g, h) \}$$

$$\geq \inf_{g: S^k \rightarrow S^n} \text{dis}(g)$$

$$= \inf_{\text{odd } g: S^k \rightarrow S^n} \text{dis}(g)$$

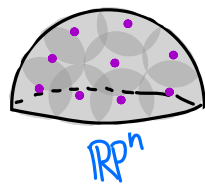
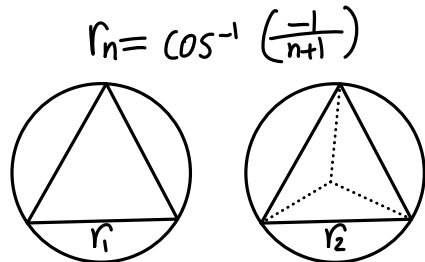
$$\geq C_{n,k}.$$

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$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}}_{C_{n,k}} \geq \pi - \text{cov}_k(RP^n).$$

Question Tight upper bounds on $d_{GH}(S^n, S^k)$ via maps?

Question Bounds on $d_{GH}(X, Y)$ for more general families of G -equivariant metric spaces X, Y ?

Question Relate the p -Gromov-Wasserstein distance d_{p-GW} to p -Vietoris-Rips thickenings VR_p ?

Question How does the generalization of Dubins & Schwarz relate to Tverberg?

Last section of our paper advertises 12 open questions!
3 follow-up papers in preparation already!