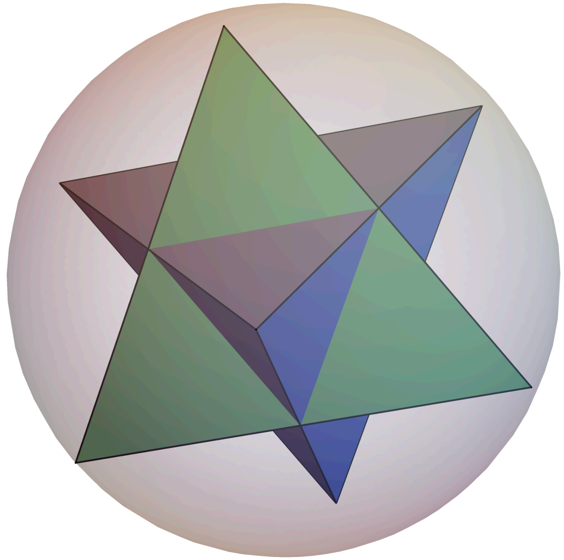

Bridging Applied and Quantitative Topology

Henry Adams

COLORADO STATE UNIVERSITY

AATRN
Applied Algebraic Topology
Research Network


INTERVIEW
SERIES
2022 - 2023
Frédéric Chazal
interviewed by
Steve Oudot
SEP 14TH
Konstantin Mischaikow
interviewed by
Tomas Gedeon
OCT 26TH
Yusu Wang
interviewed by
Tamal Dey
DEC 7TH
Claudia Landi
interviewed by
Barbara Giunti
FEB 1ST
For Zoom coordinates, become a member at AATRN.NET
Leonidas Guibas
interviewed by
Primoz Skraba
JUN 21ST
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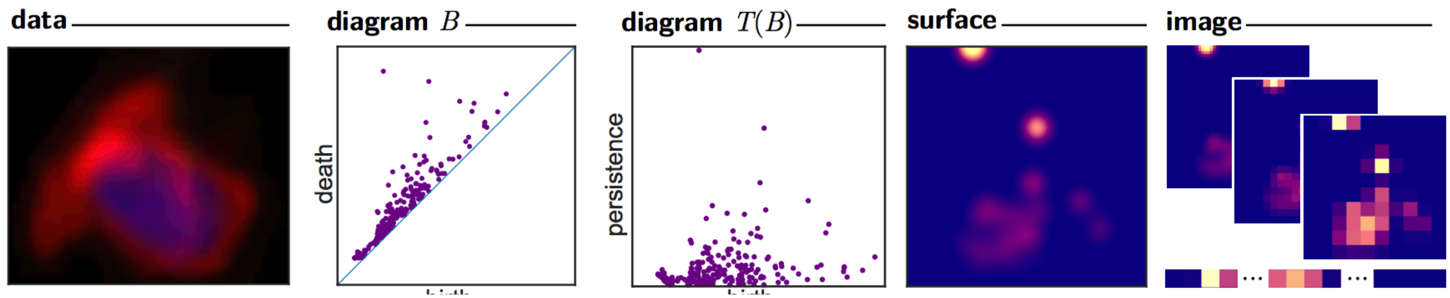
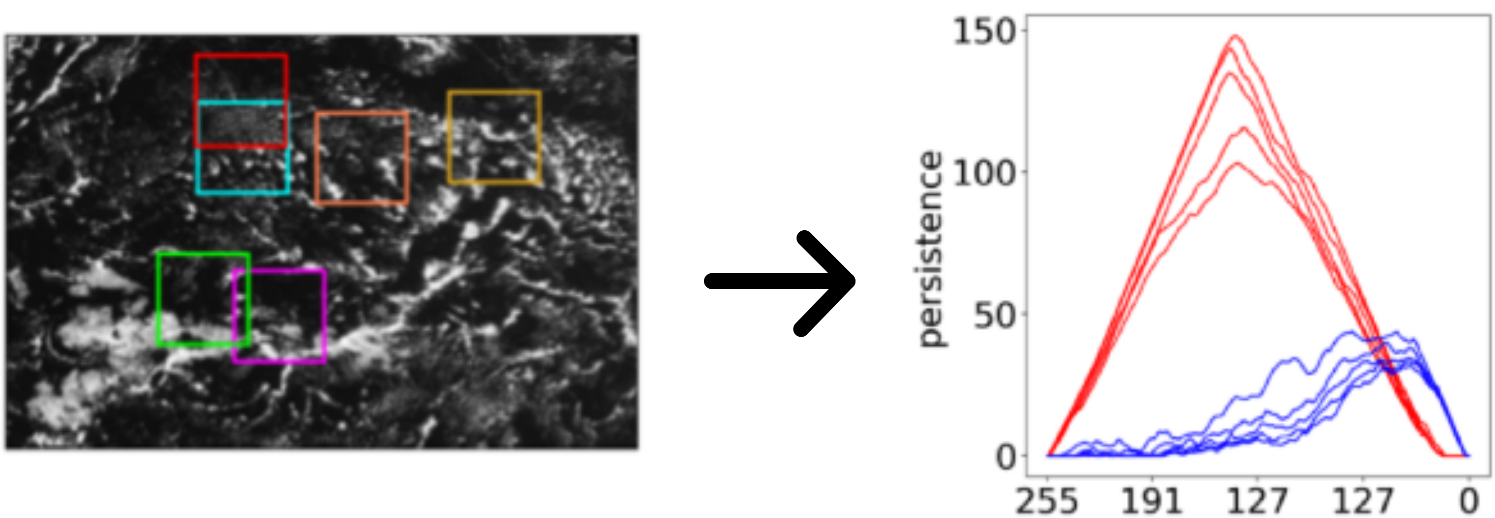
Meet Adetayo (Tayo for short), born January 20!

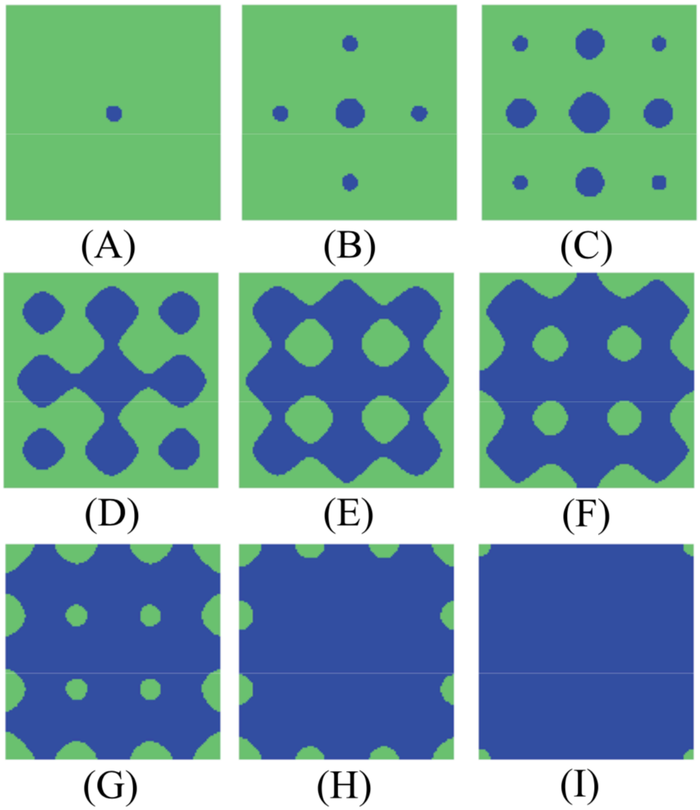
Research Themes

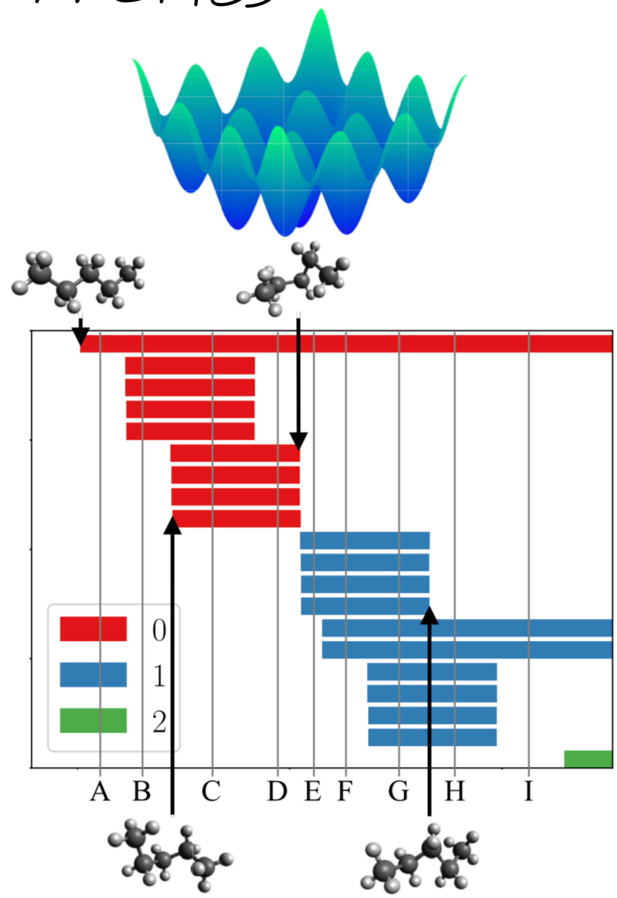


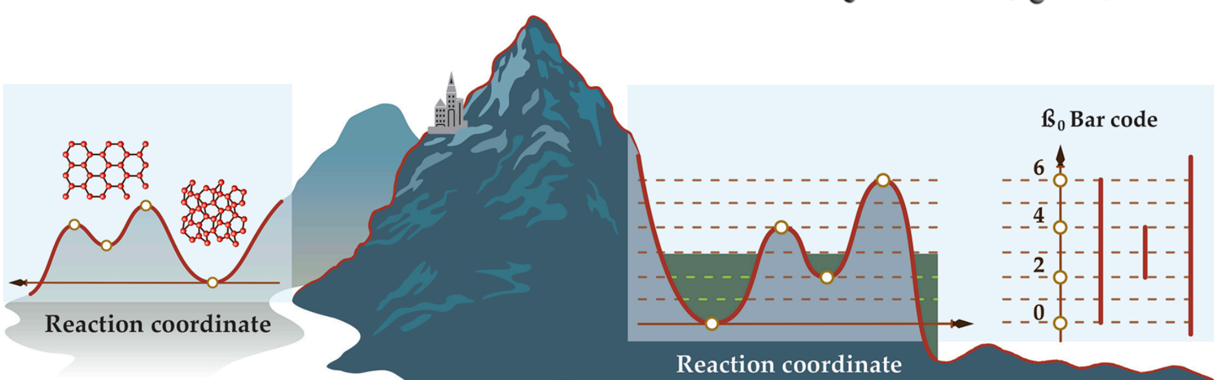


Data Science

Research Themes





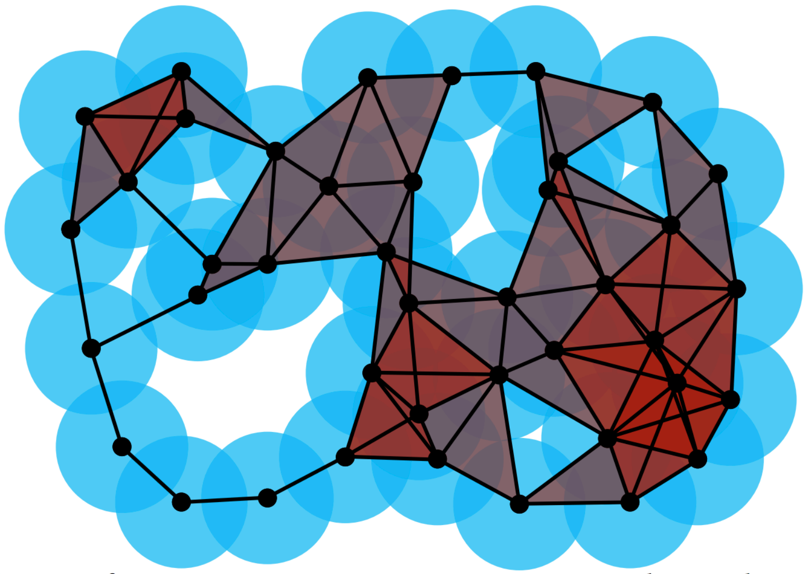


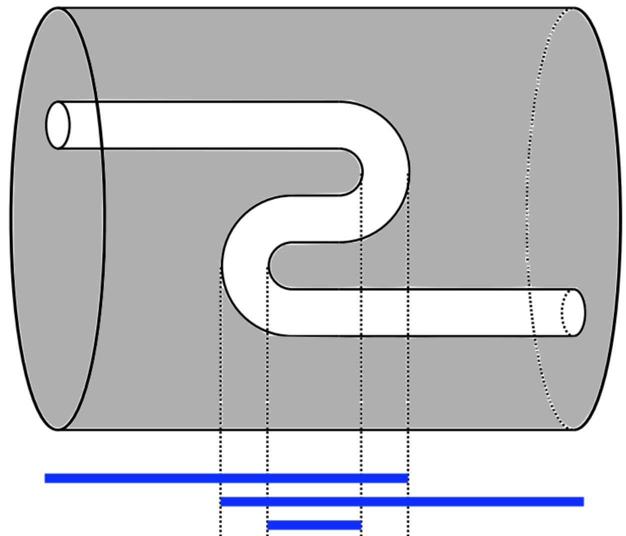
Graph representations of chemical species

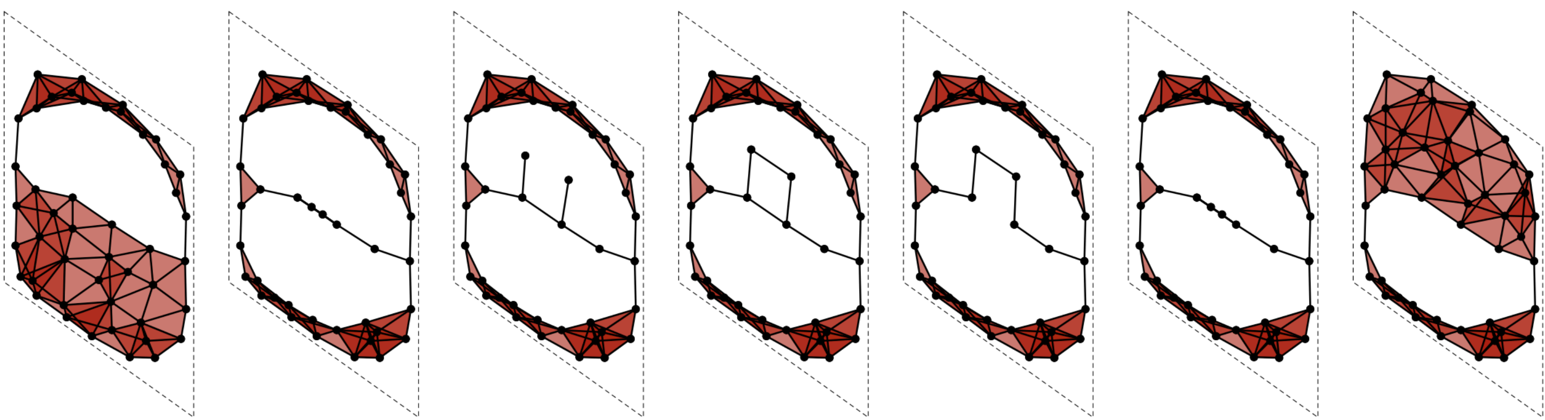
Energy landscape topology

Computational Chemistry

Research Themes

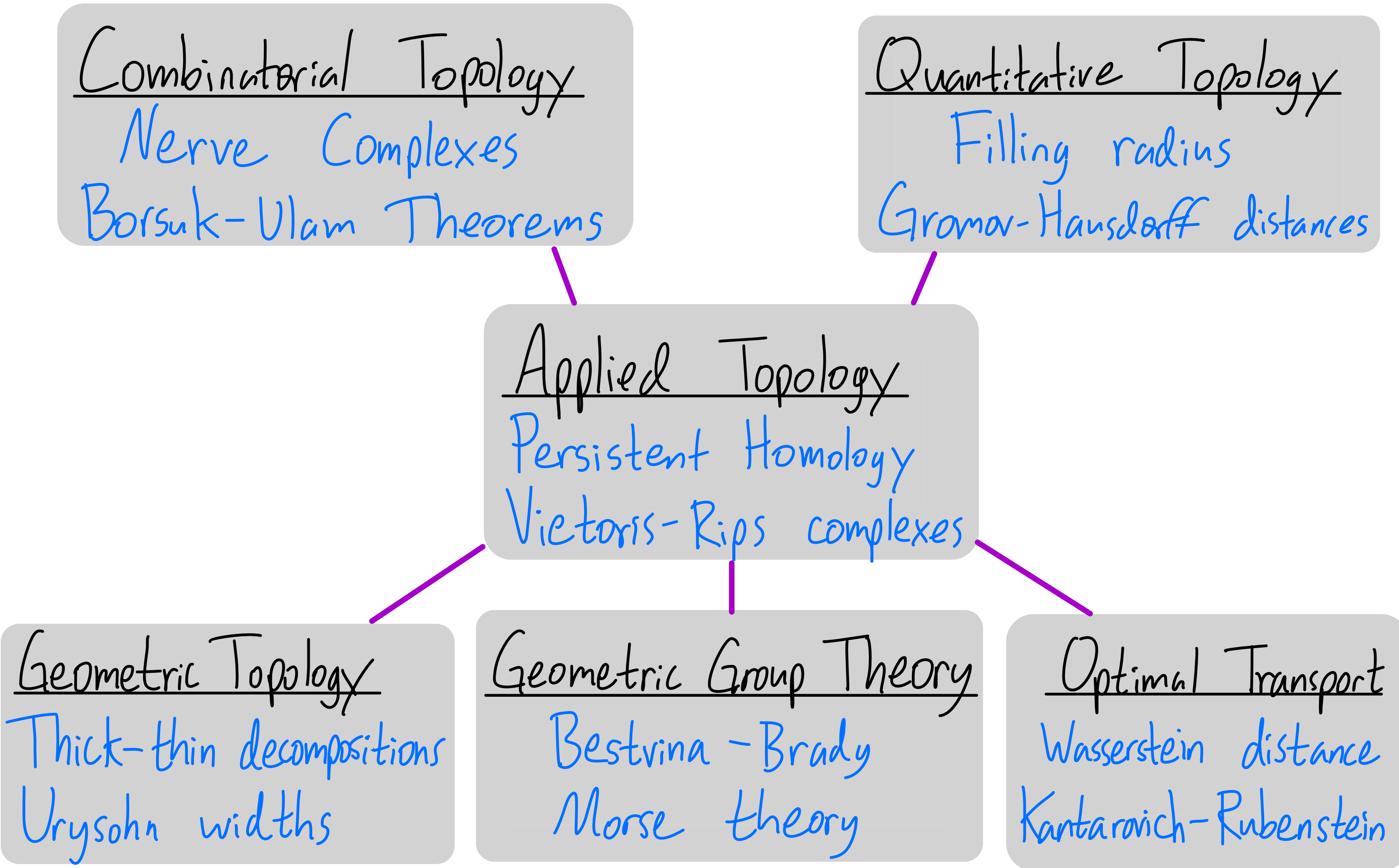






Sensor Networks

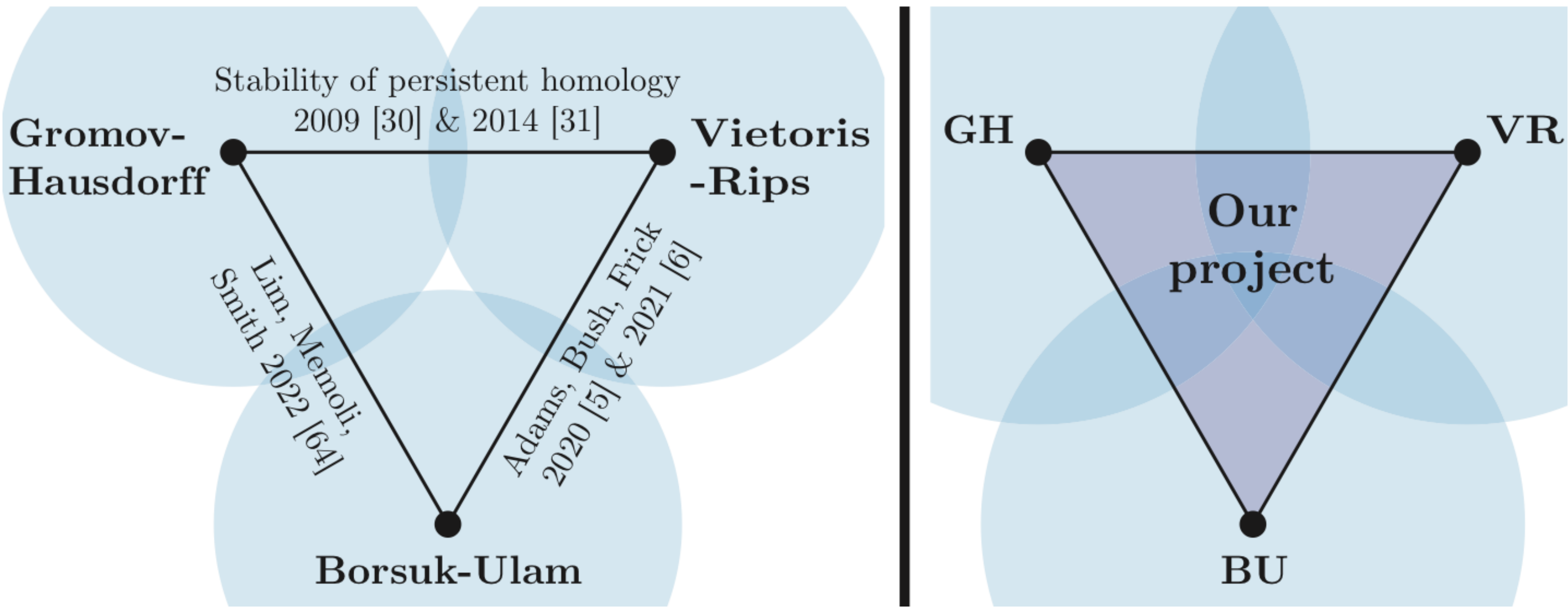
Research Themes



Bridging Applied and Quantitative Topology

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes

arXiv:2301.00246 December 2022



16 authors from 9 institutions:

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Evgeniya Lagoda
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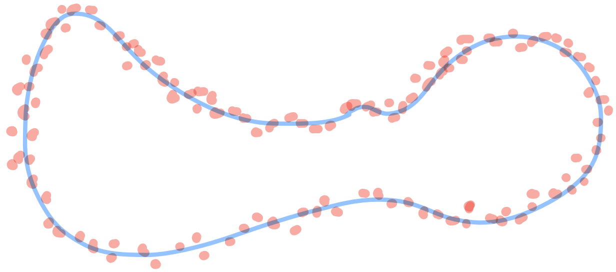
Matt Superdock

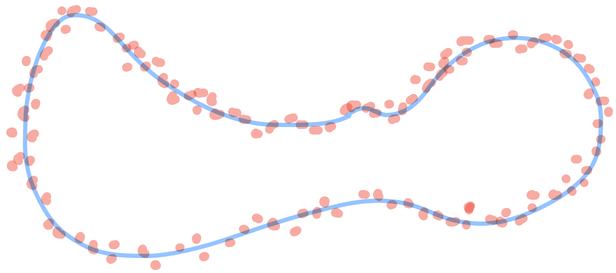
Gromov–Hausdorff distances

X, Y compact metric spaces



If X and Y are two subsets of the same metric space, then the <u>Hausdorff distance</u> between them is

$$d_H(X, Y) = \inf\{\varepsilon > 0 \mid X \subseteq Y^\varepsilon \text{ and } Y \subseteq X^\varepsilon\}$$



Gromov–Hausdorff distances

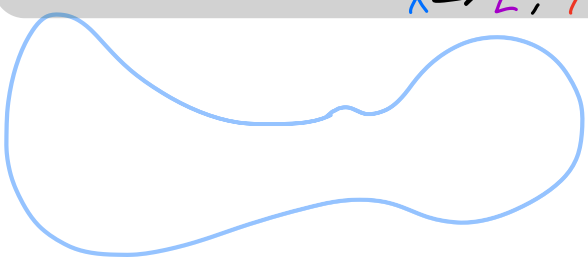
X, Y compact metric spaces

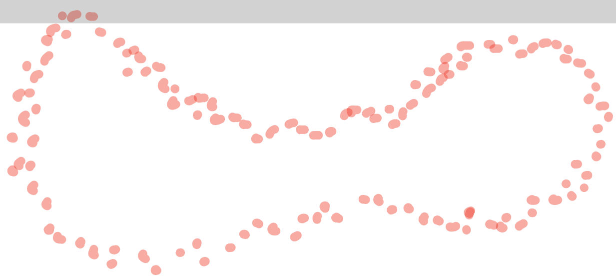
If X and Y are two subsets of the same metric space, then the Hausdorff distance between them is

$$d_H(X,Y) = \inf \{ \varepsilon > 0 \mid X \subseteq Y^{\varepsilon} \text{ and } Y \subseteq X^{\varepsilon} \}$$

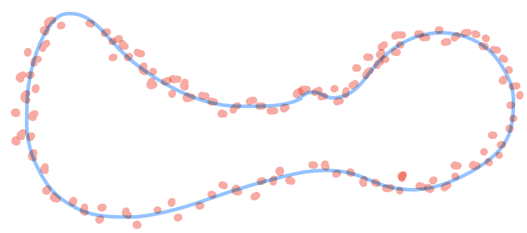
If X and Y are any two metric spaces, then the Gromov-Hausdorff distance between them is

$$d_{GH}(X,Y) = \underset{\substack{\text{isometric embeddings} \\ X \hookrightarrow Z,\ Y \hookrightarrow Z}}{\text{infimum}} \{ d_H^Z(X,Y) \}$$





Bridging Applied and Quantitative Topology

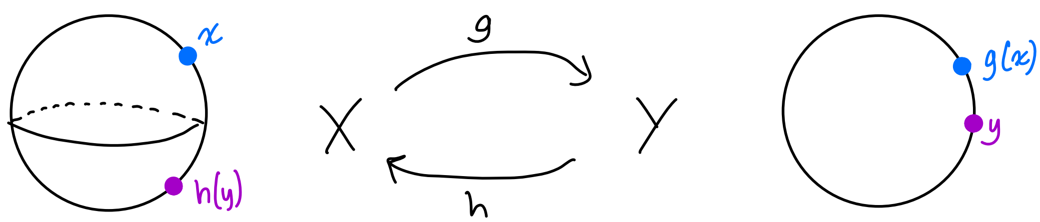


Gromov–Hausdorff distances

X, Y compact metric spaces

Kalton, Ostrovskii 1999

Def $$2 \cdot d_{GH}(X,Y) = \inf_{\substack{g : X \to Y \\ h : Y \to X}} \max\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h)\}.$$



$$\mathrm{dis}(g) = \sup_{x, x' \in X} \left| d(x,x') - d(g(x), g(x')) \right|$$

$$\mathrm{codis}(g,h) = \sup_{\substack{x \in X \\ y \in Y}} \left| d(x, h(y)) - d(g(x), y) \right|$$

Lim, Mémoli, Smith, 2022

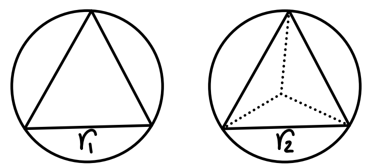
Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

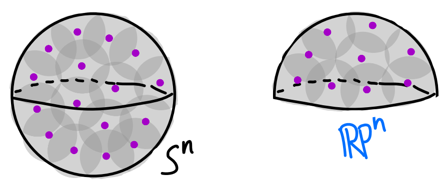
	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$



For $n<k$,
$2 \cdot d_{GH}(S^n, S^k) \geq \pi - \mathrm{cov}_{k+1}(S^n)$



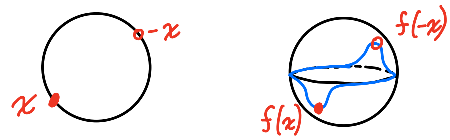
Main Theorem For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}} \geq \pi - \mathrm{cov}_k(\mathbb{RP}^n).$$

A., Bush, Frick, 2021

Def $\mathrm{cov}_k(X) :=$ infimum r s.t. k balls of radius $\frac{r}{2}$ cover X.

Borsuk-Ulam theorems

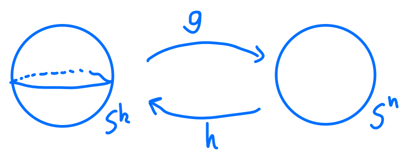


Def $g : S^k \to S^n$ is odd if $g(-x) = -g(x)$ $\forall x \in X$

Borsuk-Ulam: There is no cont. odd $S^k \to S^n$ for $k > n$.

Proof of Main Theorem

$$
\begin{aligned}
2 \cdot d_{GH}(S^n, S^k) &\geq \inf_{g: S^k \to S^n} \operatorname{dis}(g) \\
&= \inf_{\text{odd } g: S^k \to S^n} \operatorname{dis}(g) \\
&\geq C_{n,k}.
\end{aligned}
$$



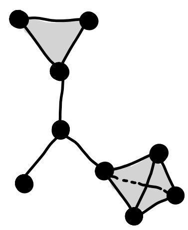
(remaining step)

Theorem (Dubins & Schwarz '81)

Odd $g : S^{n+1} \to S^n$ have $\operatorname{dis}(g) \geq r_n$.

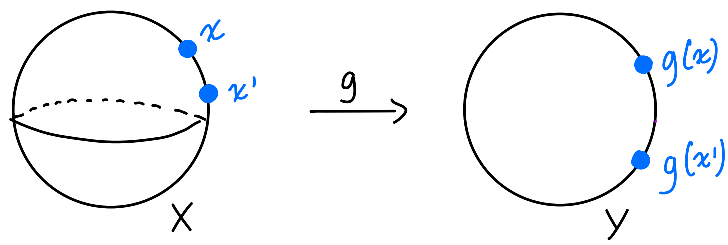
Vietoris – Rips simplicial complexes

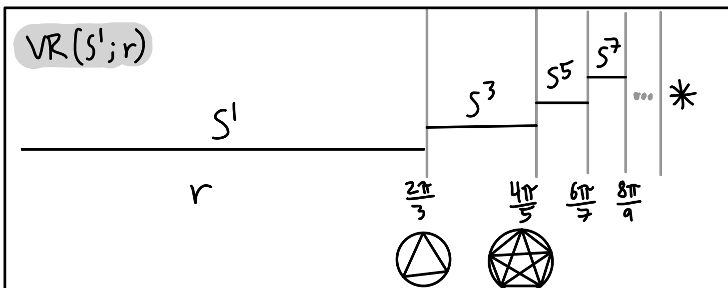
<u>Def</u> X metric space, $r \geq 0$. <u>Vietoris-Rips complex $VR(X;r)$</u> has vertex set X, all simplices of diameter $\leq r$.

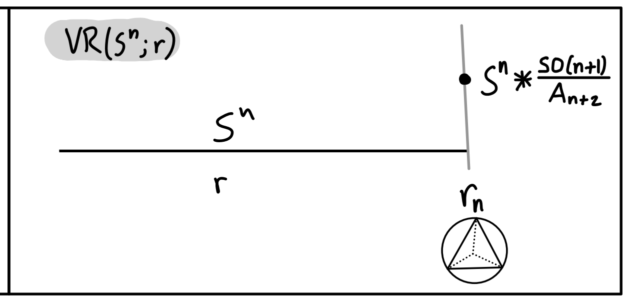


Function $g: X \to Y$ induces a (cont.) simplicial map $g: VR(X,r) \longrightarrow VR(Y; dis(g)+r)$.

$$[x_0, \dots, x_m] \longmapsto [g(x_0), \dots, g(x_m)]$$



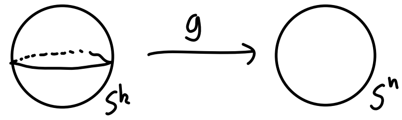




$$c_{1,2k+1} = c_{1,2k} = \frac{2\pi k}{2k+1} \qquad\qquad c_{n,n+2} = c_{n,n+1} = r_n$$

Theorem (generalizing Dubins & Schwarz)

Odd $g: S^k \to S^n$ for $k>n$ have $\mathrm{dis}(g) \geq c_{n,k}$.



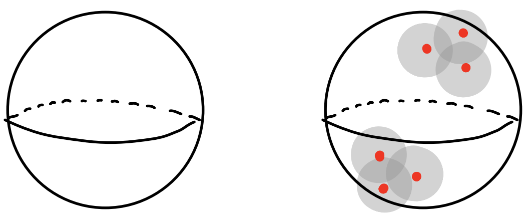
Proof

For $\varepsilon > 0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.

Produce a cont. odd map

$$S^k \xrightarrow{\text{partition of unity}} VR(X;\varepsilon) \xrightarrow{g} VR(S^n;\mathrm{dis}(g)+\varepsilon).$$

$$[x_0, \dots, x_m] \longmapsto [g(x_0), \dots, g(x_m)]$$



Hence $\mathrm{dis}(g) + \varepsilon \geq c_{n,k}$ $\forall \varepsilon > 0$, so $\mathrm{dis}(g) \geq c_{n,k}$. $\square$

Question Tightness of bounds on $d_{GH}(S^n, S^k)$?
on $\mathrm{dis}(g: S^k \to S^n)$?

Question Bounds on $d_{GH}(X,Y)$ for more general families of G-equivariant metric spaces X, Y ?

Question Relate the p-Gromov-Wasserstein distance $d_{p\text{-}GW}$ to p-Vietoris-Rips thickenings VR_p ?

Question How does the generalization of Dubins & Schwarz relate to Tverberg ?

Last section of our paper advertises 12 open questions!
3 follow-up papers in preparation already!