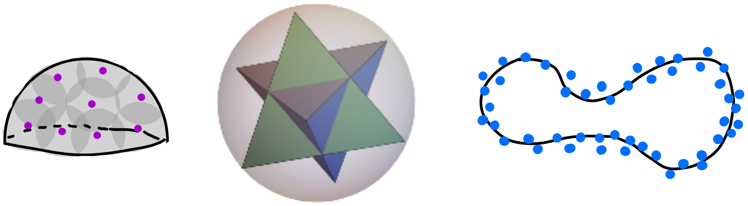
What are Gromov-Hausdorff distances ?



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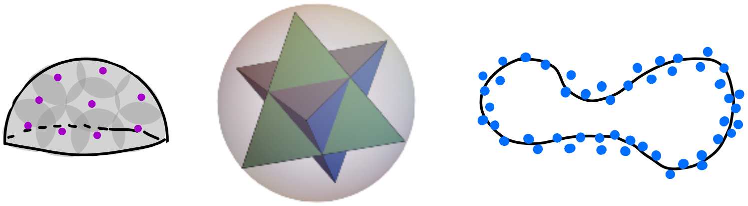

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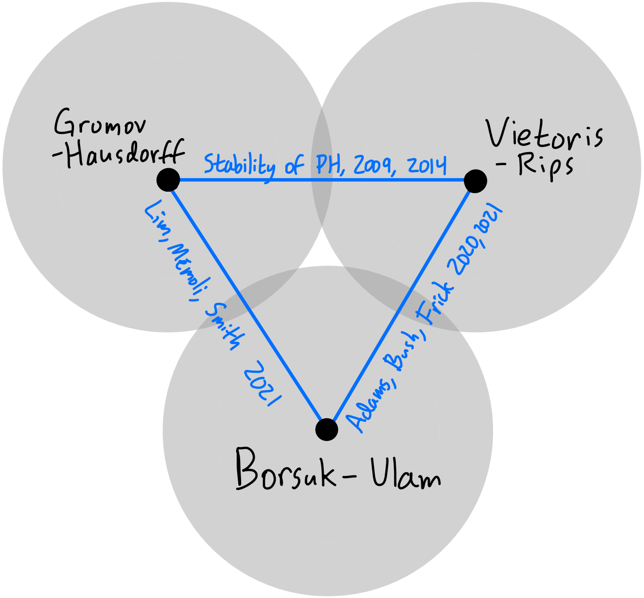
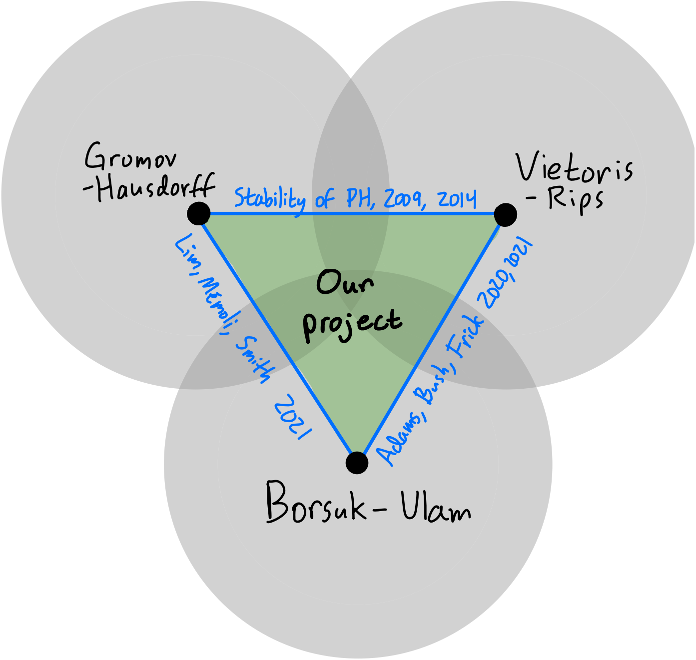
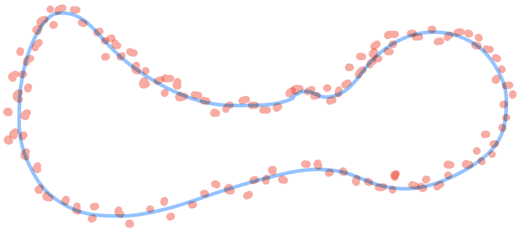
Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes





Gromov–Hausdorff distances

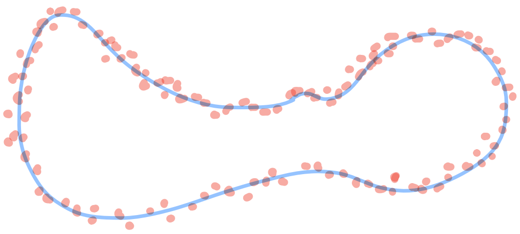
X, Y compact metric spaces



If X and Y are two subsets of the same metric space, then the Hausdorff distance between them is

$$d_H(X, Y) = \inf\{\varepsilon > 0 \mid X \subseteq Y^{\varepsilon} \text{ and } Y \subseteq X^{\varepsilon}\}$$

Gromov–Hausdorff distances



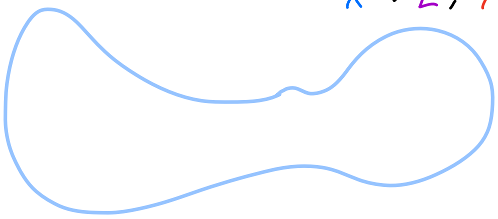
X, Y compact metric spaces

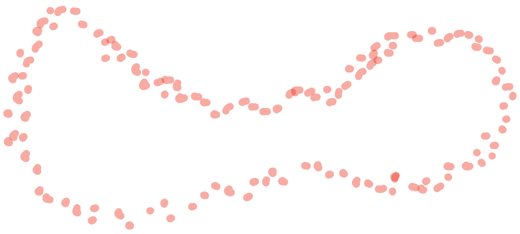
If X and Y are two subsets of the same metric space, then the Hausdorff distance between them is

$$d_H(X, Y) = \inf\{\varepsilon > 0 \mid X \subseteq Y^\varepsilon \text{ and } Y \subseteq X^\varepsilon\}$$

If X and Y are any two metric spaces, then the Gromov-Hausdorff distance between them is

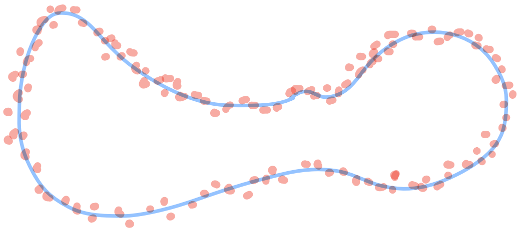
$$d_{GH}(X, Y) = \underset{\substack{\text{isometric embeddings} \\ X \hookrightarrow Z,\ Y \hookrightarrow Z}}{\text{infimum}} \{d_H^Z(X, Y)\}$$





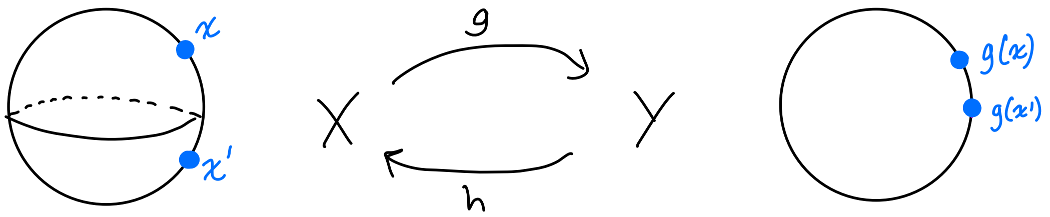
Gromov–Hausdorff distances

X, Y compact metric spaces



Equivalently:

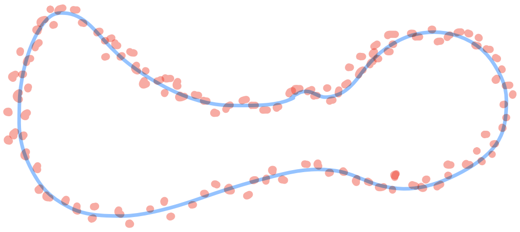
Def $$2 \cdot d_{GH}(X,Y) = \inf_{\substack{g: X \to Y \\ h: Y \to X}} \max\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h)\}.$$



$$\mathrm{dis}(g) = \sup_{x,x' \in X} \left| d(x,x') - d(g(x), g(x')) \right|$$

Gromov–Hausdorff distances

X, Y compact metric spaces



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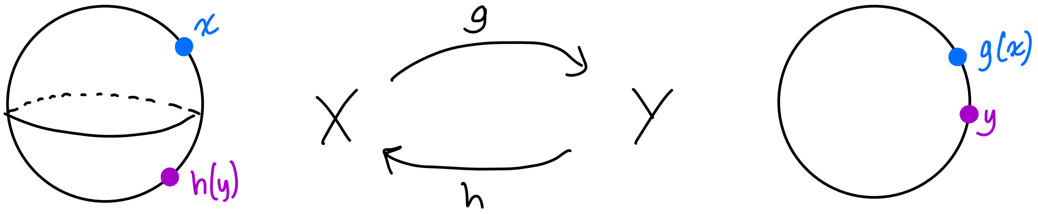


$$\mathrm{dis}(g) = \sup_{x, x' \in X} \left| d(x,x') - d(g(x), g(x')) \right|$$

$$\mathrm{codis}(g,h) = \sup_{\substack{x \in X \\ y \in Y}} \left| d(x, h(y)) - d(g(x), y) \right|$$

Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

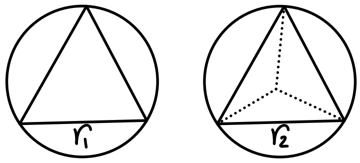
$2\cdot d_{GH}(S^n,S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix

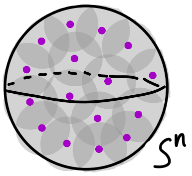
Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$



For $n<k$,

$$\longleftarrow 2\cdot d_{GH}(S^n,S^k) \geq \pi - \mathrm{cov}_{k+1}(S^n)$$



Equality for $1\leq n<k\leq 3$. Proof with discont. Borsuk Ulam generalizes:

Lim, Mémoli, Smith, 2021

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$2\cdot d_{GH}(S^n,S^k)$

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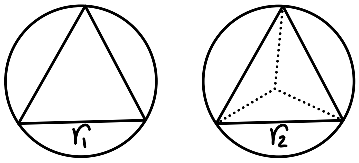
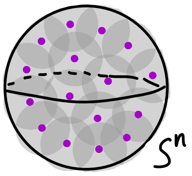
Symmetric matrix
Nonzero entries in $(\pi/2,\pi)$



For $n<k$,
$2\cdot d_{GH}(S^n,S^k)\geq \pi-\mathrm{cov}_{k+1}(S^n)$



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Theorem (Oct, 2021) For $n<k$,

$$2\cdot d_{GH}(S^n,S^k)\geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k\to VR(S^n;r)\}}_{c_{n,k}}.$$

Lim, Mémoli, Smith, 2021

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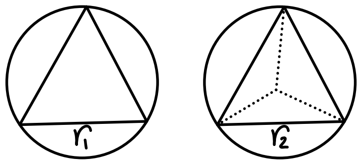
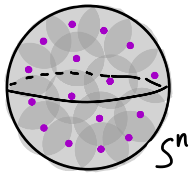


For $n<k$,

← $2\cdot d_{GH}(S^n,S^k)\geq \pi - cov_{k+1}(S^n)$



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Lim, Mémoli, Smith, 2021

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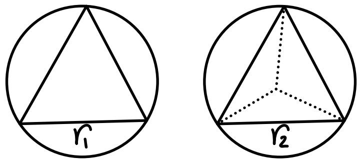

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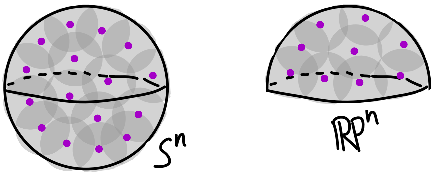

Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

Theorem (Oct, 2021) For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{C_{n,k}} \underset{\uparrow}{\geq} \pi - \mathrm{cov}_k(\mathbb{RP}^n).$$

A., Bush, Frick, 2021

<u>Def</u> $\mathrm{cov}_k(X) :=$ infimum r s.t. k balls of radius $\frac{r}{2}$ cover X.

Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

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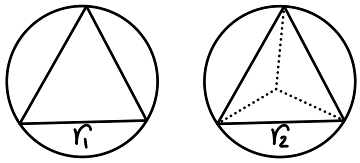

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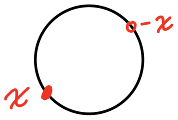
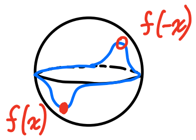
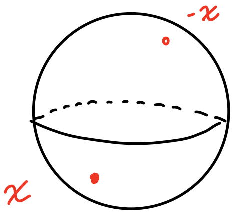
Borsuk-Ulam theorems

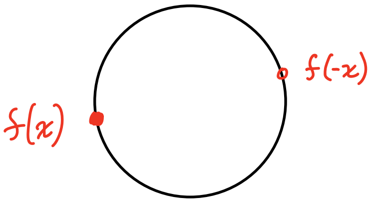




Def A map $f: S^k \to S^n$ is *odd* if $f(-x) = -f(x)$ $\forall x \in X$

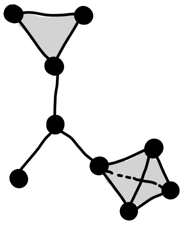
Borsuk-Ulam: There is no cont. odd $S^k \to S^n$ for $k > n$.





Vietoris – Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X;r)$ has vertex set X, all simplices of diameter $\leq r$.



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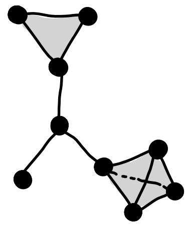
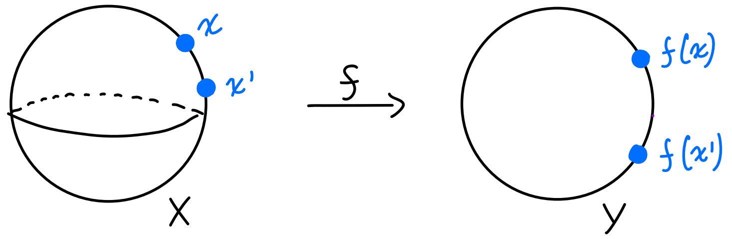


Function $f: X \to Y$ induces a (cont.) simplicial map $f: VR(X,r) \longrightarrow VR(Y; \mathrm{dis}(f)+r)$.

$$[x_0, \ldots, x_m] \longmapsto [f(x_0), \ldots, f(x_m)]$$



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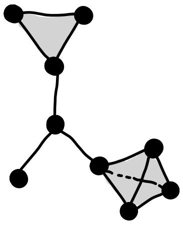

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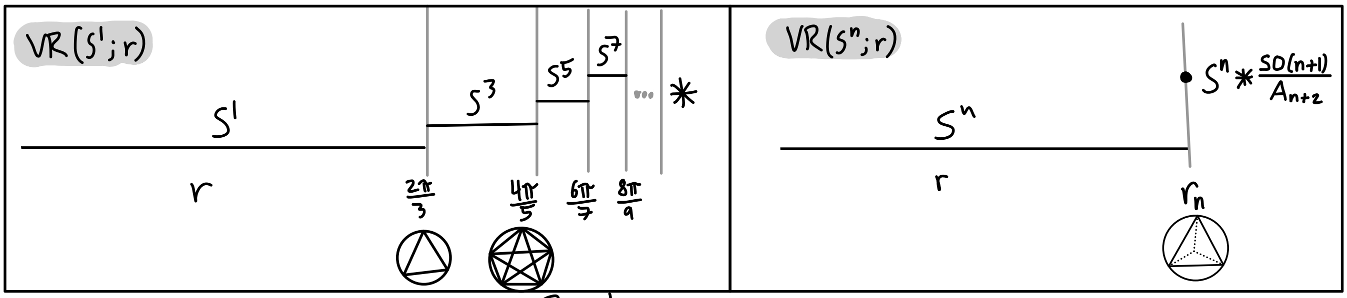


$$C_{1,2k+1} = C_{1,2k} = \frac{2\pi k}{2k+1} \qquad C_{n,n+2} = C_{n,n+1} = r_n$$

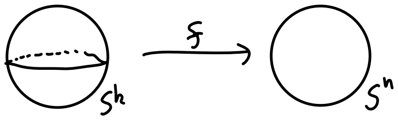
$$\underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n;r)\}}_{C_{n,k}}$$

Dubins & Schwarz '81

Lim, Mémoli, Smith, 2021

(Discont.) odd maps $f: S^{n+1} \to S^n$ have $\mathrm{dis}(f) \geq r_n$.

Generalization (Discont.) odd maps $f: S^k \to S^n$ for $k>n$ have $\mathrm{dis}(f) \geq c_{n,k}$.

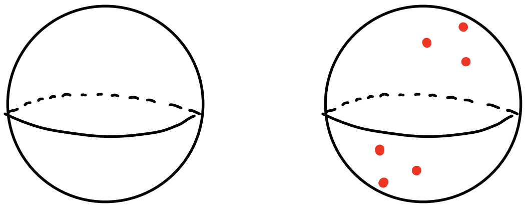


Proof

For $\varepsilon > 0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.

Produce a cont. odd map

$$\begin{tikzcd}
S^k \arrow[r, "\text{partition of unity}"] & VR(X;\varepsilon) \arrow[r, "f"] & VR(S^n; \mathrm{dis}(f)+\varepsilon). \\
 & {[x_0, \ldots, x_m]} \arrow[r, mapsto] & {[f(x_0), \ldots, f(x_m)]}
\end{tikzcd}$$

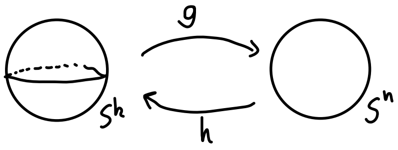


Hence $\mathrm{dis}(f) + \varepsilon \geq c_{n,k}$ $\forall \varepsilon > 0$, so $\mathrm{dis}(f) \geq c_{n,k}$. □

$$\underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}}$$

Theorem (Oct, 2021) For $n<k$,

$$2\cdot d_{GH}(S^n,S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k\to VR(S^n;r)\}}_{C_{n,k}}.$$



Proof of Theorem follows Lim, Mémoli, Smith, 2021

$$\begin{aligned}
2\cdot d_{GH}(S^n,S^k) &= \inf_{\substack{g:S^k\to S^n\\ h:S^n\to S^k}} \max\{\operatorname{dis}(g), \operatorname{dis}(h), \operatorname{codis}(g,h)\}\\
&\geq \inf_{g:S^k\to S^n} \operatorname{dis}(g)\\
&= \inf_{\text{odd } g:S^k\to S^n} \operatorname{dis}(g)\\
&\geq C_{n,k}.
\end{aligned}$$

Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

$2\cdot d_{GH}(S^n, S^k)$

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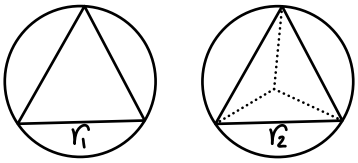

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Question Tight upper bounds on $d_{GH}(S^n, S^k)$ via maps?

Question Bounds on $d_{GH}(X,Y)$ for more general families of G-equivariant metric spaces X, Y ?

Question Relate the p-Gromov-Wasserstein distance $d_{p\text{-}GW}$ to p-Vietoris-Rips thickenings VR_p ?

Question How does the generalization of Dubins & Schwarz relate to Tverberg ?