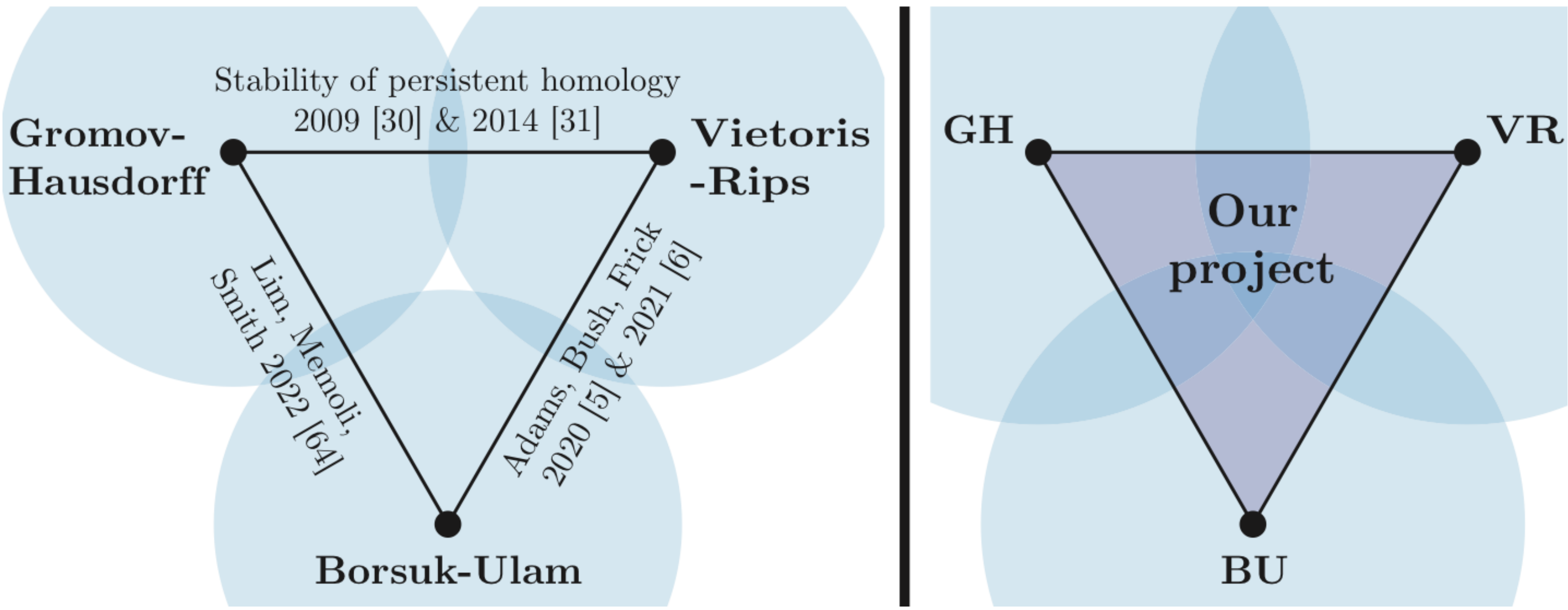
Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes

arXiv:2301.00246

December 2022



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Johnathan Bush
Michael Moy
Daniel Vargas-Rosario
Facundo Mémoli
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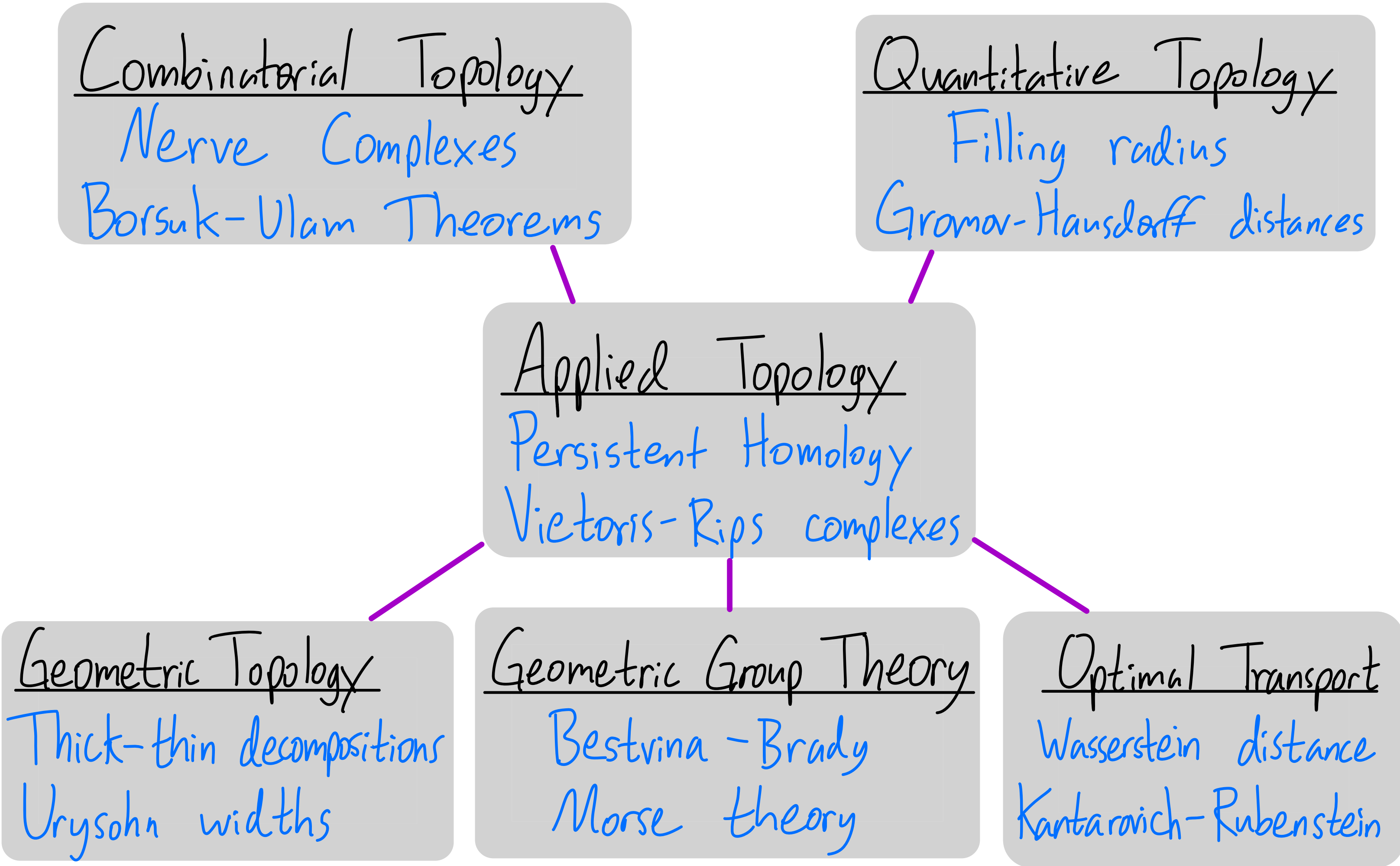

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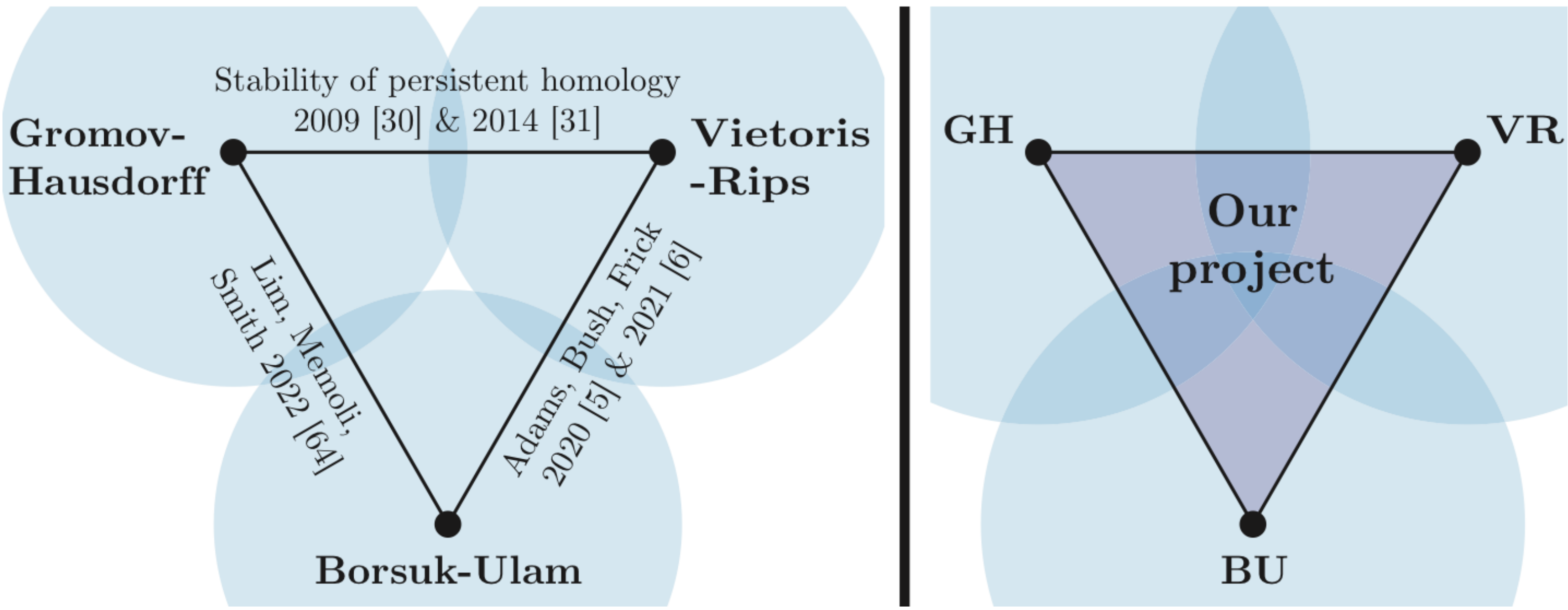

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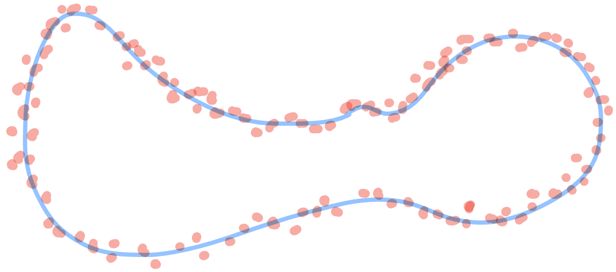
Gromov–Hausdorff distances

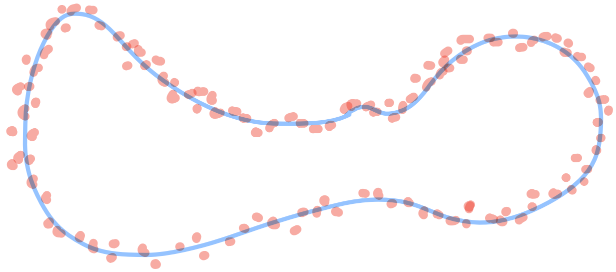
X, Y compact metric spaces



If X and Y are two subsets of the same metric space, then the Hausdorff distance between them is

$$d_H(X, Y) = \inf\{\varepsilon > 0 \mid X \subseteq Y^\varepsilon \text{ and } Y \subseteq X^\varepsilon\}$$

Gromov-Hausdorff distances



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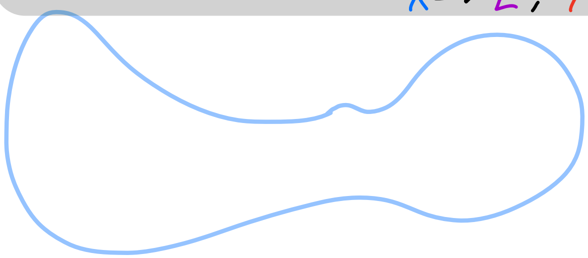
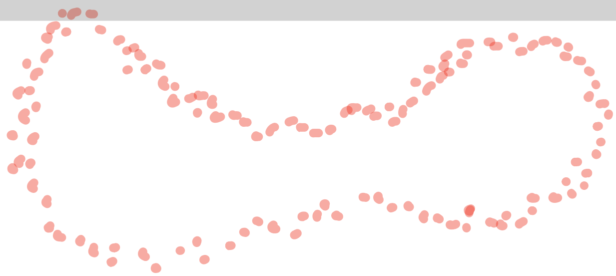
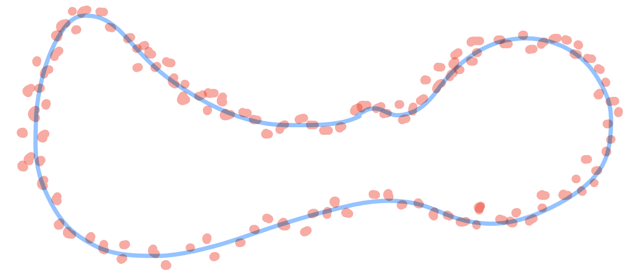
If X and Y are any two metric spaces, then the Gromov-Hausdorff distance between them is

$$d_{GH}(X, Y) = \underset{\substack{\text{isometric embeddings} \\ X \hookrightarrow Z,\ Y \hookrightarrow Z}}{\text{infimum}} \{ d_H^Z(X, Y) \}$$





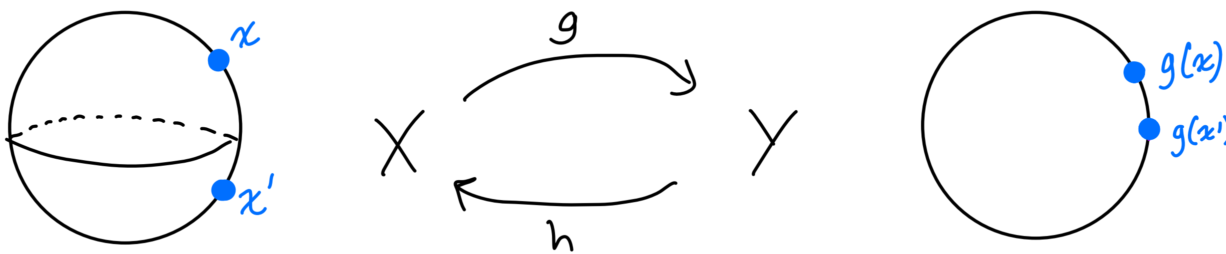
<u>Gromov–Hausdorff distances</u>



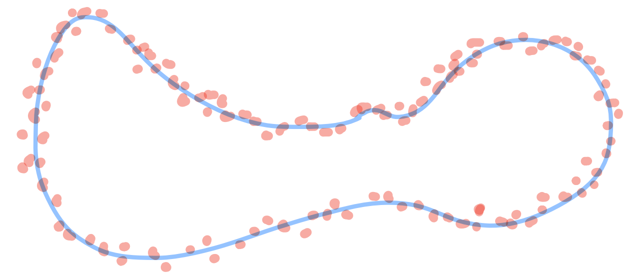
X, Y compact metric spaces

Equivalently:

<u>Def</u> 2. $d_{GH}(X,Y) = \inf\limits_{\substack{g: X \to Y \\ h: Y \to X}} \max\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h)\}.$



$$\mathrm{dis}(g) = \sup_{x, x' \in X} \left| d(x,x') - d(g(x), g(x')) \right|$$



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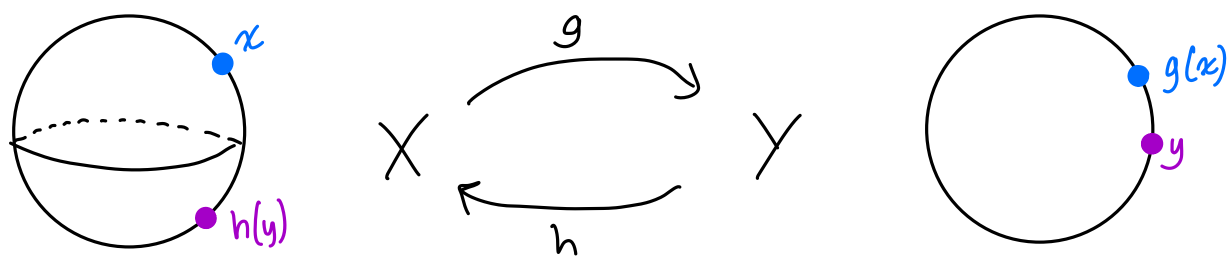

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$$\mathrm{codis}(g,h) = \sup_{\substack{x\in X \\ y\in Y}} \left| d(x, h(y)) - d(g(x), y) \right|$$

Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

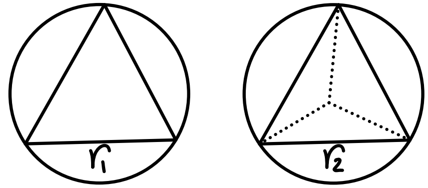
$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix

Nonzero entries in $(\pi/2, \pi)$

$r_n = \cos^{-1}\left(\frac{-1}{n+1}\right)$



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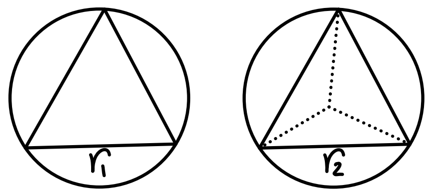

Main Theorem For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}}$$

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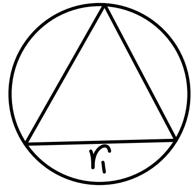
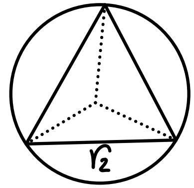
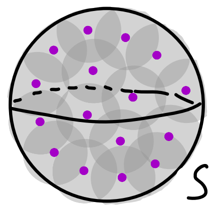




For $n<k$,
$2 \cdot d_{GH}(S^n, S^k) \geq \pi - \mathrm{cov}_{k+1}(S^n)$



S^n

Main Theorem For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}}$$

Def $\mathrm{cov}_k(X) :=$ infimum r s.t. k balls of radius $\frac{r}{2}$ cover X.

Lim, Mémoli, Smith, 2021

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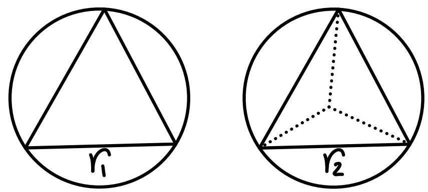

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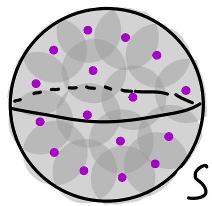

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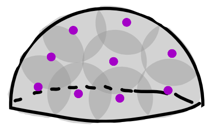

$\mathbb{R}P^n$

Main Theorem For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}} \geq \pi - \mathrm{cov}_k(\mathbb{R}P^n).$$

A., Bush, Frick, 2021

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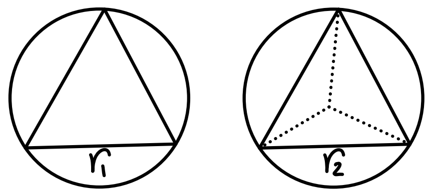

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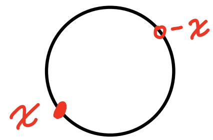
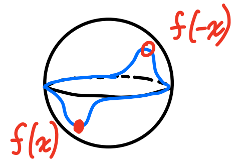
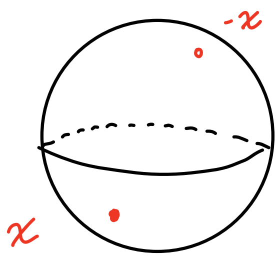
Borsuk-Ulam theorems

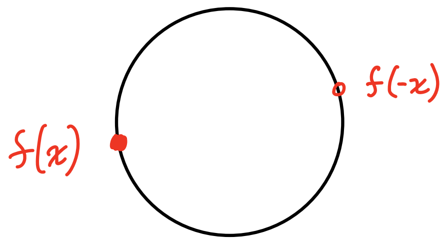




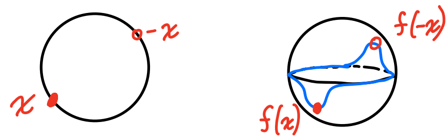
Def A map $f: S^k \to S^n$ is <u>odd</u> if $f(-x) = -f(x)$ $\forall x \in X$

Borsuk-Ulam: There is no cont. odd $S^k \to S^n$ for $k > n$.





Borsuk-Ulam theorems



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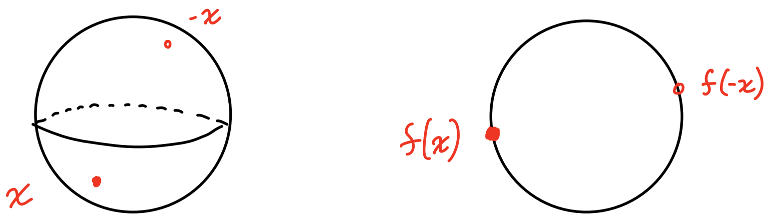
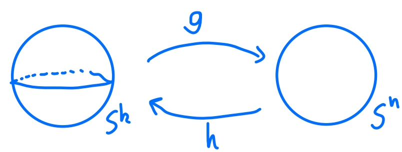


Proof of Main Theorem

$$2 \cdot d_{GH}(S^n, S^k) \geq \inf_{g: S^k \to S^n} \operatorname{dis}(g)$$
$$= \inf_{\text{odd } g: S^k \to S^n} \operatorname{dis}(g)$$
$$\geq C_{n,k}.$$



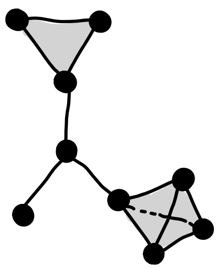
(remaining step)

Theorem (Dubins & Schwarz '81)

Odd $g: S^{n+1} \to S^n$ have $\operatorname{dis}(g) \geq r_n$.

Vietoris – Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X;r)$ has vertex set X, all simplices of diameter $\leq r$.



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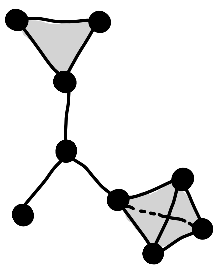
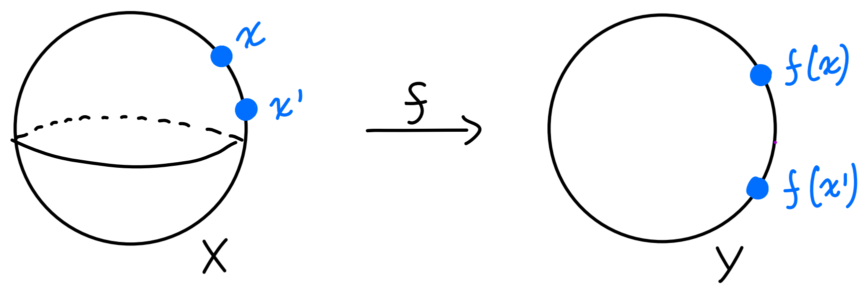


Function $f: X \to Y$ induces a (cont.) simplicial map $f: VR(X, r) \longrightarrow VR(Y; dis(f)+r)$.

$$[x_0, \ldots, x_m] \longmapsto [f(x_0), \ldots, f(x_m)]$$



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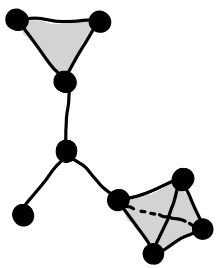

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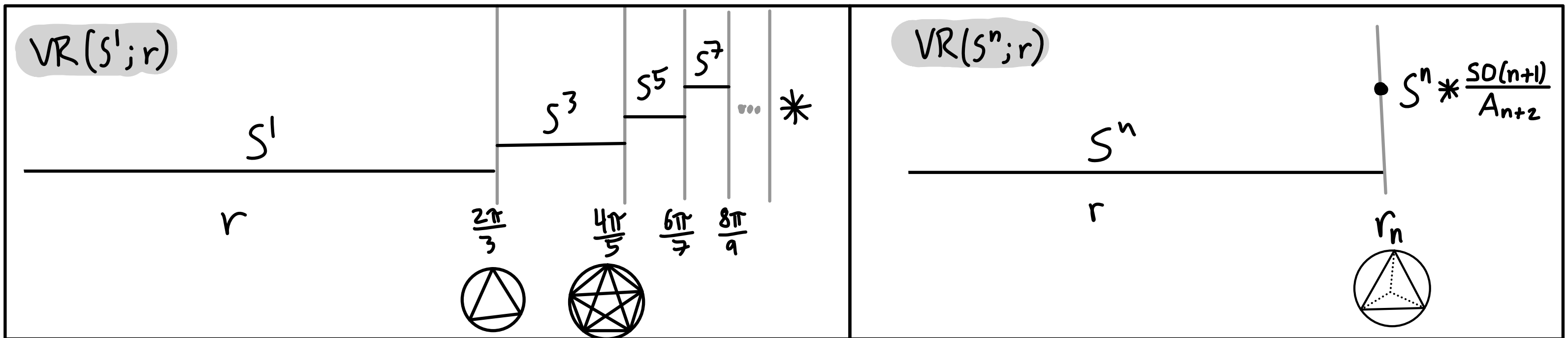
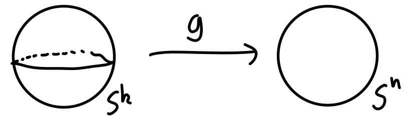


$$C_{1,2k+1} = C_{1,2k} = \frac{2\pi k}{2k+1} \qquad C_{n,n+2} = C_{n,n+1} = r_n$$

$$\underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n;r)\}}_{C_{n,k}}$$

Theorem (generalizing Dubins & Schwarz)

Odd $g: S^k \to S^n$ for $k > n$ have $\mathrm{dis}(g) \geq c_{n,k}$.



Proof

For $\varepsilon > 0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.

Produce a cont. odd map

$$S^k \xrightarrow[\text{of unity}]{\text{partition}} VR(X;\varepsilon) \xrightarrow{g} VR(S^n; \mathrm{dis}(g)+\varepsilon).$$

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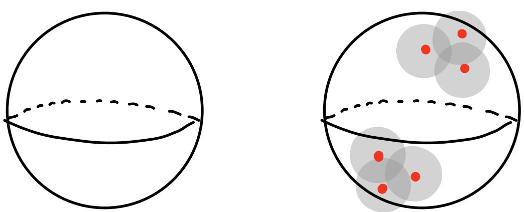


Hence $\mathrm{dis}(g) + \varepsilon \geq c_{n,k}$ $\forall \varepsilon > 0$, so $\mathrm{dis}(g) \geq c_{n,k}$. $\square$

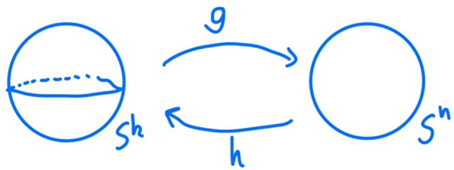
Main Theorem

For $n<k$,

$$2\cdot d_{GH}(S^n,S^k) \ge \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k\to VR(S^n;r)\}}_{C_{n,k}}.$$

Proof of Main Theorem

$$\begin{aligned} 2\cdot d_{GH}(S^n,S^k) &\ge \inf_{g:S^k\to S^n} dis(g) \\ &= \inf_{\text{odd } g:S^k\to S^n} dis(g) \\ &\ge C_{n,k}. \end{aligned}$$



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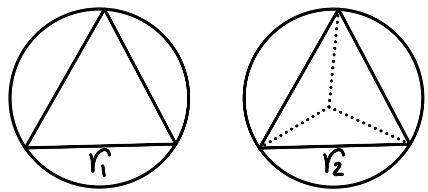

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<u>Question</u> Tight upper bounds on $d_{GH}(S^n, S^k)$ via maps?

<u>Question</u> Bounds on $d_{GH}(X,Y)$ for more general families of G-equivariant metric spaces X, Y?

<u>Question</u> Relate the p-Gromov-Wasserstein distance $d_{p\text{-}GW}$ to p-Vietoris-Rips thickenings VR_p?

<u>Question</u> How does the generalization of Dubins & Schwarz relate to Tverberg?

Last section of our paper advertises 12 open questions!
3 follow-up papers in preparation already!