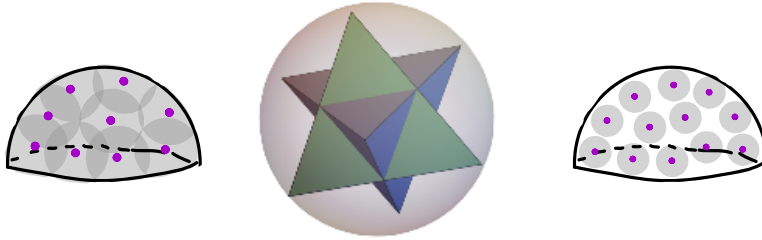


Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes



Henry Adams

Johnathan Bush

Michael Moy

Daniel Vargas-Rosario

Facundo Mémoli

Nathaniel Clause

Mario Gomez Flores

Sunhyuk Lim

Qingsong Wang

Ling Zhou

Florian Frick

Michael Harrison

Amzi Jeffs

Evgeniya Lagoda

Tatiana Levinson

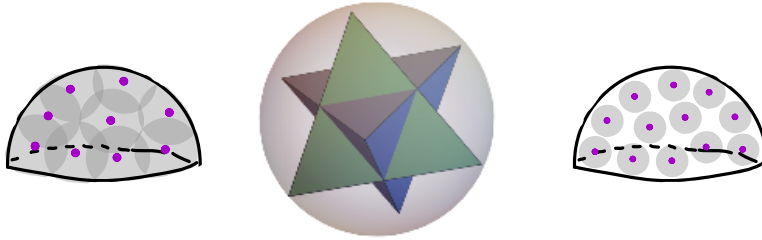
Nicola Sadovsek

Matthew Superdock



AATR.N: www.aatr.n.net , 1-2 live talks per week
YouTube: 4,350 subscribers, 24 hours watched per day
Contributed Videos: "This is my research"

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes



Henry Adams

Johnathan Bush

Michael Moy

Daniel Vargas-Rosario

Facundo Mémoli

Nathaniel Clause

Mario Gomez Flores

Sunhyuk Lim

Qingsong Wang

Ling Zhou

Florian Frick

Michael Harrison

Amzi Jeffs

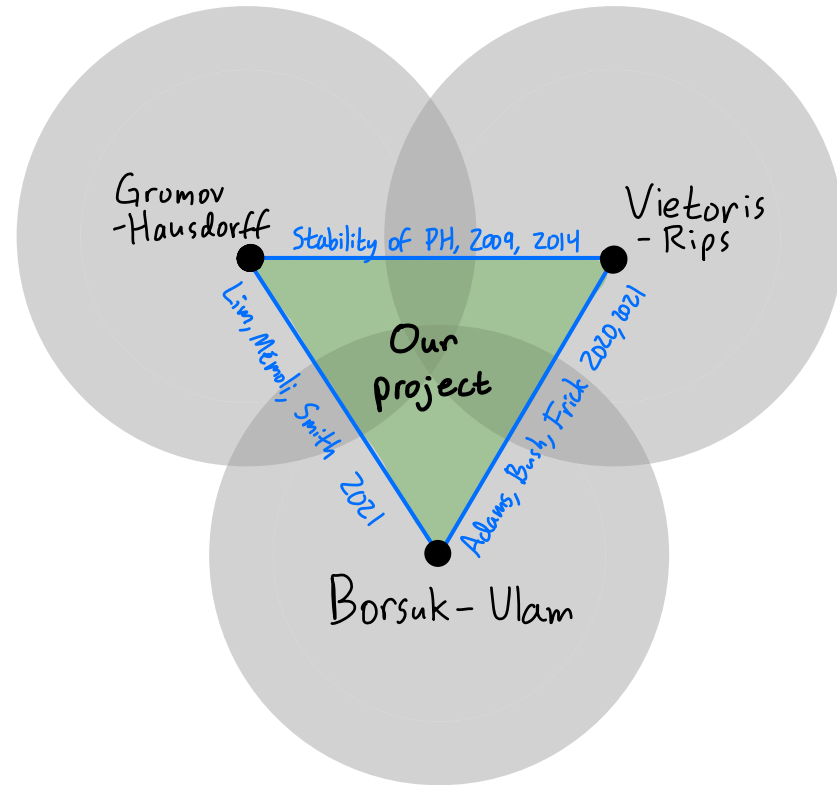
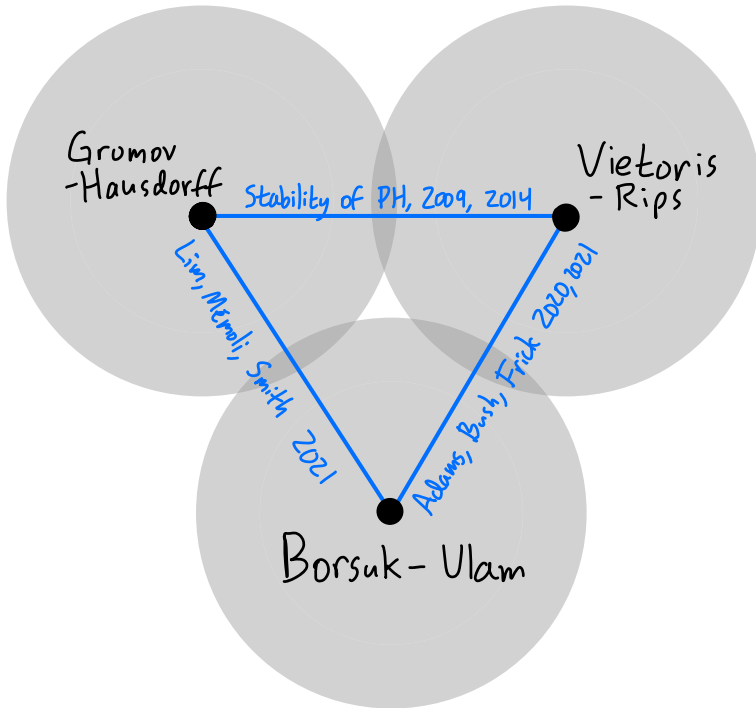
Evgeniya Lagoda

Tatiana Levinson

Nicola Sadovsek

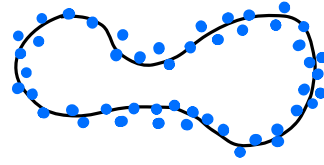
Matthew Superdock

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes

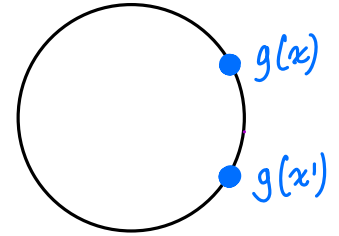
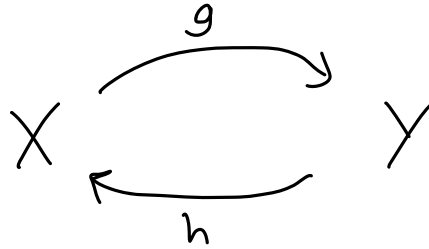
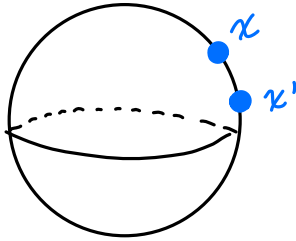


Gromov-Hausdorff distances

X, Y compact metric spaces



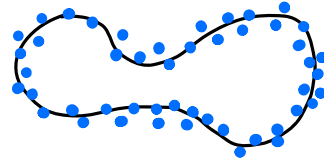
Def 2 $d_{GH}(X, Y) = \inf_{\substack{g: X \rightarrow Y \\ h: Y \rightarrow X}} \max \{ \text{dis}(g), \text{dis}(h), \text{codis}(g, h) \}$



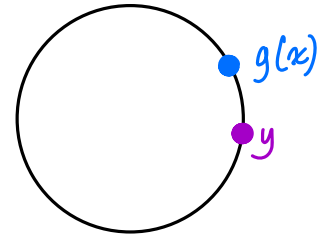
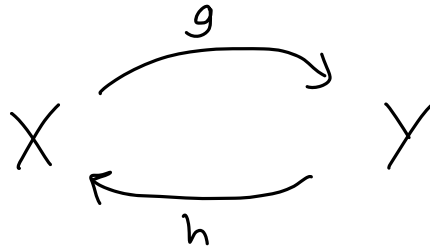
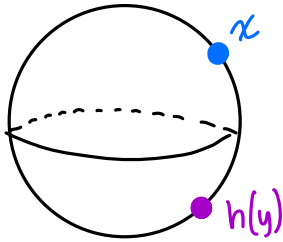
$$\text{dis}(g) = \sup_{x, x' \in X} | d(x, x') - d(g(x), g(x')) |$$

Gromov-Hausdorff distances

X, Y compact metric spaces



Def 2 $d_{GH}(X, Y) = \inf_{\substack{g: X \rightarrow Y \\ h: Y \rightarrow X}} \max \{ \text{dis}(g), \text{dis}(h), \text{codis}(g, h) \}$



$$\text{dis}(g) = \sup_{x, x' \in X} | d(x, x') - d(g(x), g(x')) |$$

$$\text{codis}(g, h) = \sup_{\substack{x \in X \\ y \in Y}} | d(x, h(y)) - d(g(x), y) |$$

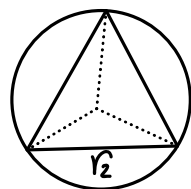
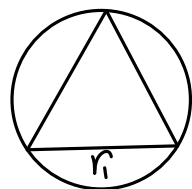
Lim, Memoli, Smith, 2021

Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

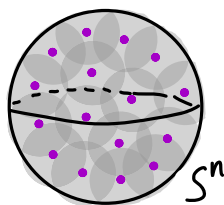
	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	$\geq r_2$				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$



For $n < k$,

$$\leftarrow 2 \cdot d_{GH}(S^n, S^k) \geq \pi - \text{cov}_{k+1}(S^n)$$



Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

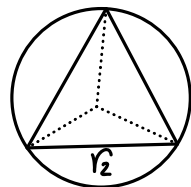
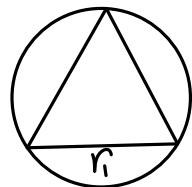
Lim, Memoli, Smith, 2021

Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

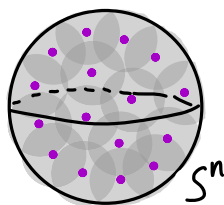
	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0 $\geq r_3$				
S^4				0 $\geq r_4$			
S^5					0 $\geq r_5$		
S^6						0 $\geq r_6$	

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$



For $n < k$,

$$\leftarrow 2 \cdot d_{GH}(S^n, S^k) \geq \pi - \text{cov}_{k+1}(S^n)$$



Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

Theorem (Oct, 2021) For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}.$$

$C_{n,k}$

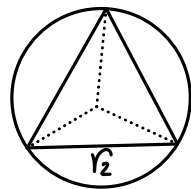
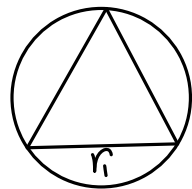
Lim, Memoli, Smith, 2021

Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

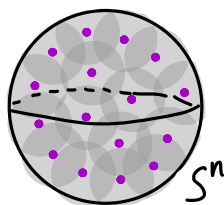
	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3} \geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$	
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$



For $n < k$,

$$\leftarrow 2 \cdot d_{GH}(S^n, S^k) \geq \pi - \text{cov}_{k+1}(S^n)$$



Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

Theorem (Oct, 2021) For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow \text{VR}(S^n; r) \right\}.$$

$C_{n,k}$

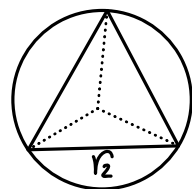
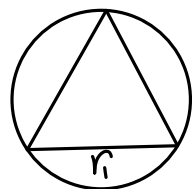
Lim, Memoli, Smith, 2021

Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

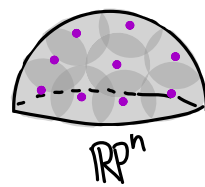
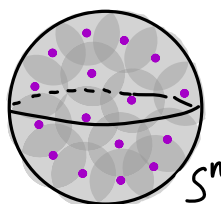
	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3} \geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$	
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$



For $n < k$,

$$\leftarrow 2 \cdot d_{GH}(S^n, S^k) \geq \pi - \text{cov}_{k+1}(S^n)$$



Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

Theorem (Oct, 2021) For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow \text{VR}(S^n; r) \right\}}_{C_{n,k}} \geq \pi - \text{cov}_k(\mathbb{RP}^n).$$

A., Buch, Frick, 2021

Def $\text{cov}_k(X) := \text{infimum } r \text{ s.t. } k \text{ balls of radius } \frac{r}{2} \text{ cover } X.$

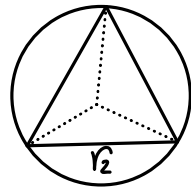
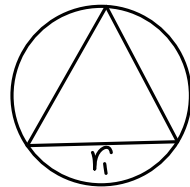
Lim, Memoli, Smith, 2021

Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3} \geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$	
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$



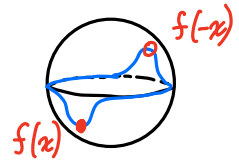
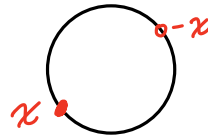
Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

Theorem (Oct, 2021) For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}.$$

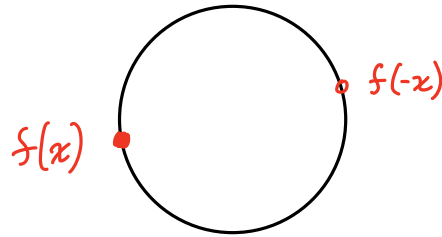
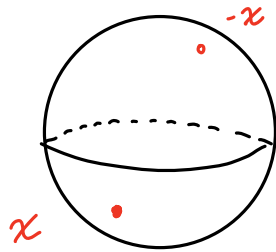
$C_{n,k}$

Borsuk-Ulam theorems



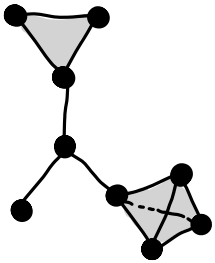
Def A map $f: S^k \rightarrow S^n$ is odd if $f(-x) = -f(x) \quad \forall x \in X$

Borsuk-Ulam: There is no cont. odd $S^k \rightarrow S^n$ for $k > n$.



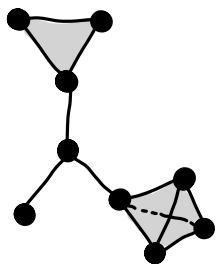
Vietoris-Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X; r)$
has vertex set X , all simplices of diameter $\leq r$.

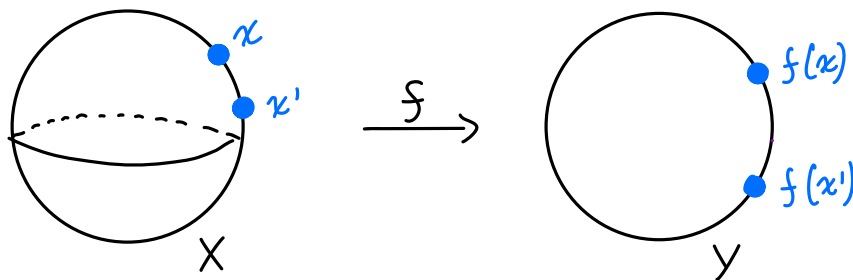


Vietoris-Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X; r)$
has vertex set X , all simplices of diameter $\leq r$.

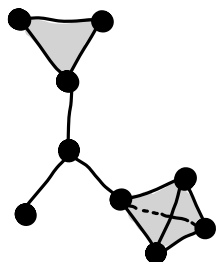


Function $f: X \rightarrow Y$ induces a (cont.) simplicial
map $f: VR(X, r) \rightarrow VR(Y; \text{dis}(f) + r)$.
 $[x_0, \dots, x_m] \mapsto [f(x_0), \dots, f(x_m)]$



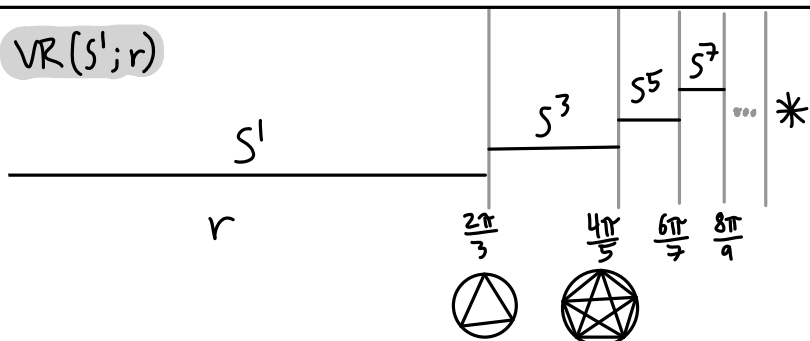
Vietoris-Rips simplicial complexes

Def X metric space, $r \geq 0$. Vietoris-Rips complex $VR(X; r)$
 has vertex set X , all simplices of diameter $\leq r$.



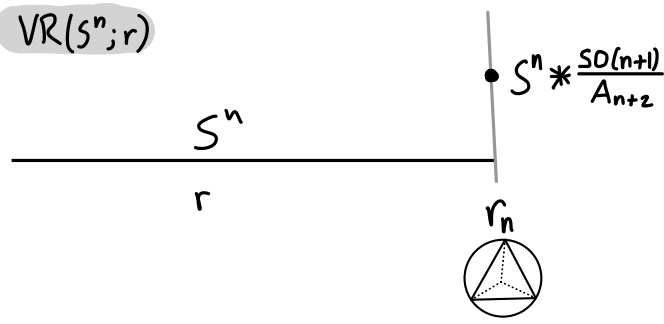
Function $f: X \rightarrow Y$ induces a (cont.) simplicial
 map $f: VR(X, r) \rightarrow VR(Y; \text{dis}(f)+r)$.
 $[x_0, \dots, x_m] \mapsto [f(x_0), \dots, f(x_m)]$

$VR(S^1; r)$



$$C_{1,2k+1} = C_{1,2k} = \frac{2\pi k}{2k+1}$$

$VR(S^n; r)$



$$C_{n,n+2} = C_{n,n+1} = r_n$$

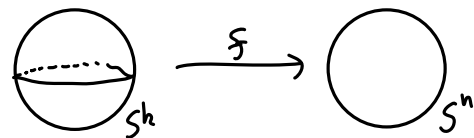
$$\underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}}_{C_{n,k}}$$

Dubins & Schwarz '81

Lim, Memoli, Smith, 2021

(Discont.) odd maps $f: S^{n+1} \rightarrow S^n$ have $\text{dis}(f) \geq r_n$.

Generalization (Discont.) odd maps $f: S^k \rightarrow S^n$ for $k > n$ have $\text{dis}(f) \geq C_{n,k}$.



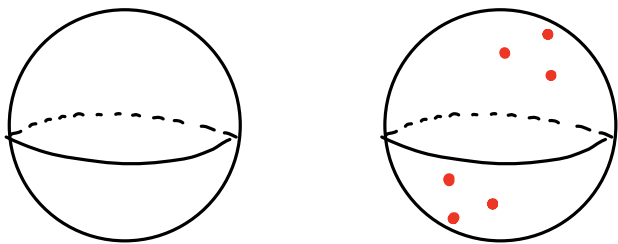
Proof

For $\varepsilon > 0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.

Produce a cont. odd map

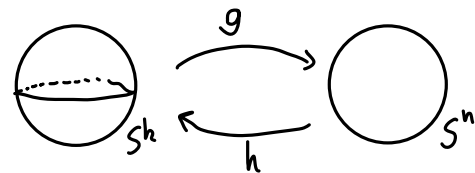
$$S^k \xrightarrow{\text{partition of unity}} \text{VR}(X; \varepsilon) \xrightarrow{f} \text{VR}(S^n; \text{dis}(f) + \varepsilon).$$

$[x_0, \dots, x_m] \longmapsto [f(x_0), \dots, f(x_m)]$



Hence $\text{dis}(f) + \varepsilon \geq C_{n,k} \quad \forall \varepsilon > 0$, so $\text{dis}(f) \geq C_{n,k}$.

Theorem (Oct, 2021) For $n < k$,
 $2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \}}_{C_{n,k}}.$



Proof of Theorem follows Lim, Memoli, Smith, 2021

$$\begin{aligned}
 2 \cdot d_{GH}(S^n, S^k) &= \inf_{\substack{g: S^k \rightarrow S^n \\ h: S^n \rightarrow S^k}} \max \{ \text{dis}(g), \text{dis}(h), \text{codis}(g, h) \} \\
 &\geq \inf_{g: S^k \rightarrow S^n} \text{dis}(g) \\
 &= \inf_{\text{odd } g: S^k \rightarrow S^n} \text{dis}(g) \\
 &\geq C_{n,k}.
 \end{aligned}$$

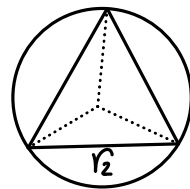
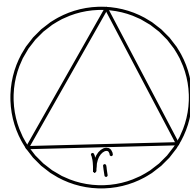
Lim, Memoli, Smith, 2021

Sphere S^n , geodesic metric, diameter π .

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3} \geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$	
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\frac{\pi}{2}, \pi)$



Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

Theorem (Oct, 2021) For $n < k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf \left\{ r \mid \exists \text{ cont. odd } S^k \rightarrow VR(S^n; r) \right\}}_{C_{n,k}}.$$

Question Tight upper bounds on $d_{GH}(S^n, S^k)$ via maps?

Question Bounds on $d_{GH}(X, Y)$ for more general families of G -equivariant metric spaces X, Y ?

Question Relate the p -Gromov-Wasserstein distance d_{p-GW} to p -Vietoris-Rips thickenings VR_p ?

Question How does the generalization of Dubins & Schwarz relate to Tverberg?