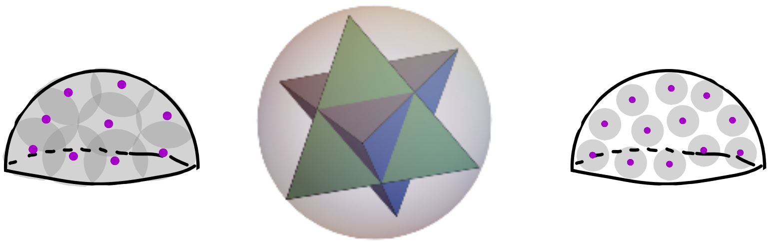
Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes



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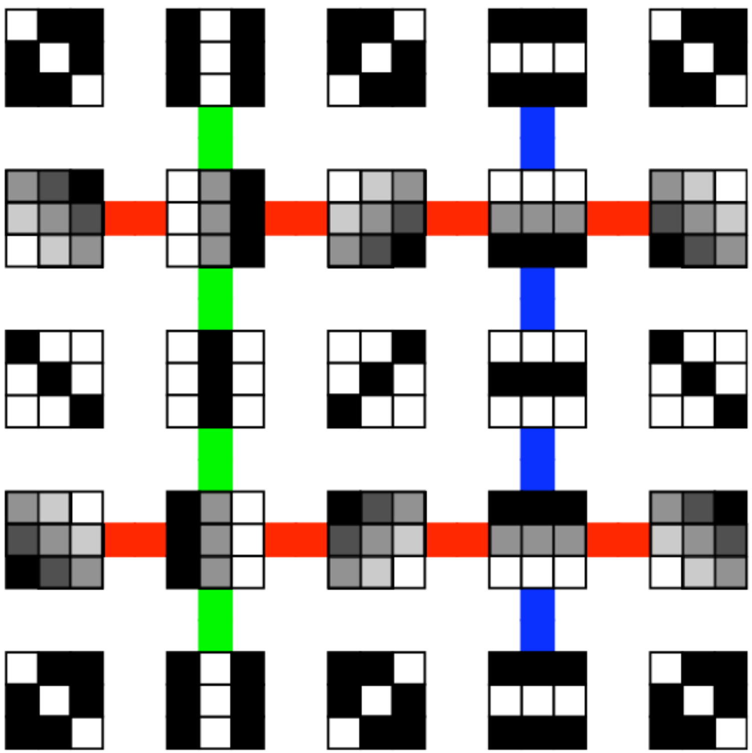
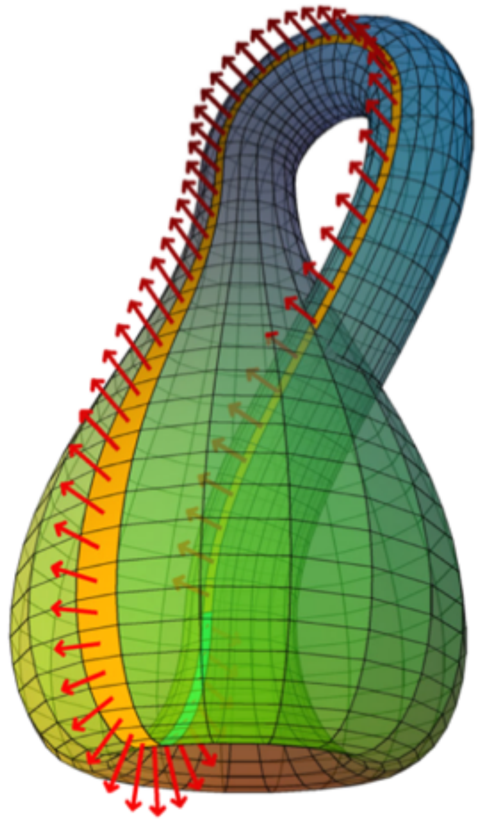
I. Being Gunnar's undergraduate student...





"On the local behavior of natural images" by Carlsson, Ishkhanov, de Silva, Zomorodian, 2008

II. Being Gunnar's graduate student...

Photo from SIAM News, "Engineering mathematics around the world"

III. The time I most annoyed Gunnar...

Stanford University, April 2010

Speaker: David Mumford

Title: Math marching to a different drummer

IV. Trying new things...

IMA New Directions Short Course
Applied Algebraic Topology
June 2009

V. The ability for which I admire Gunnar the most...

Stanford University, July 2012

AATRN: www.aatrn.net , 1-2 live talks per week

YouTube: 4,200 subscribers, 24 hours watched per day

Contributed Videos: "This is my research"

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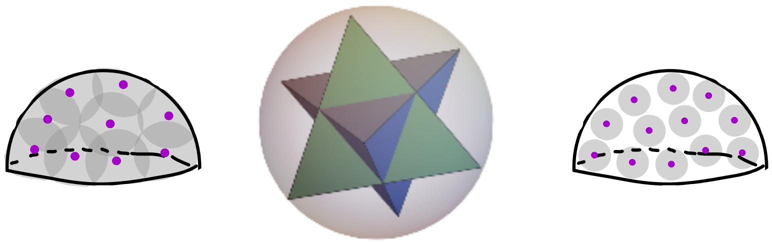

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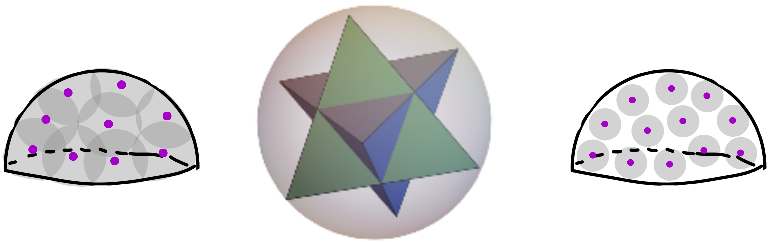

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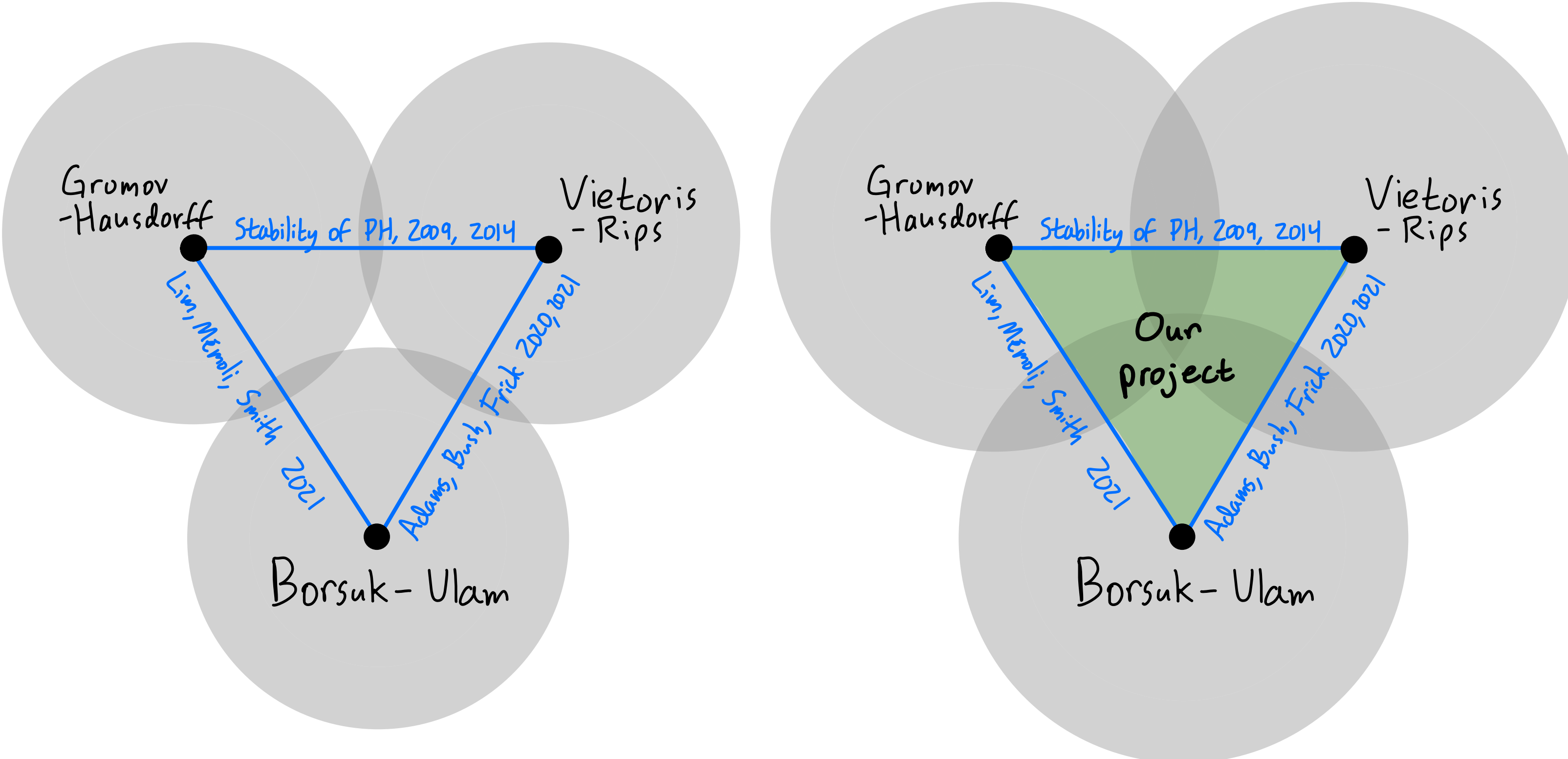
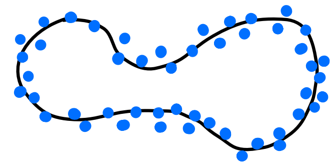
2010 talk by Facundo at Stanford

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes

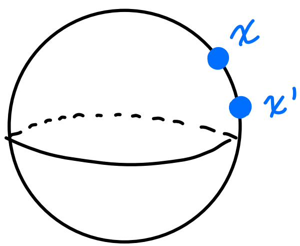


Gromov–Hausdorff distances

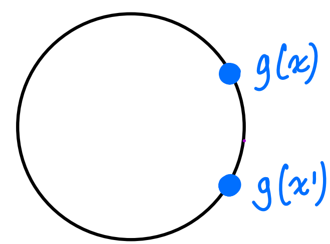


X, Y compact metric spaces

Def $2 \cdot d_{GH}(X, Y) = \inf\limits_{\substack{g: X \to Y \\ h: Y \to X}} \max\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g,h)\}.$



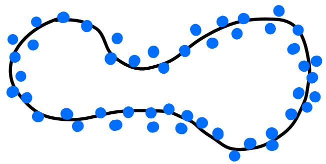
$$\begin{tikzcd} X \arrow[r, bend left, "g"] & Y \arrow[l, bend left, "h"] \end{tikzcd}$$



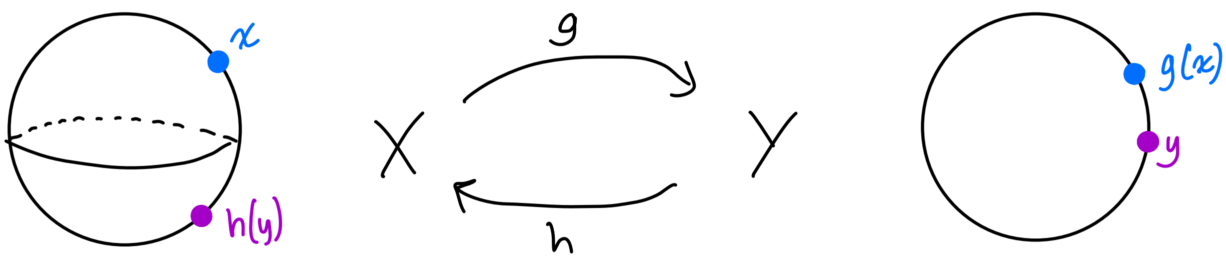
$$\mathrm{dis}(g) = \sup_{x, x' \in X} \left| d(x, x') - d(g(x), g(x')) \right|$$

Gromov–Hausdorff distances

X, Y compact metric spaces



Def 2. $d_{GH}(X,Y) = \inf\limits_{\substack{g: X \to Y \\ h: Y \to X}} \max\{\operatorname{dis}(g), \operatorname{dis}(h), \operatorname{codis}(g,h)\}$.



$$\operatorname{dis}(g) = \sup_{x, x' \in X} \left| d(x,x') - d(g(x), g(x')) \right|$$

$$\operatorname{codis}(g,h) = \sup_{\substack{x \in X \\ y \in Y}} \left| d(x, h(y)) - d(g(x), y) \right|$$

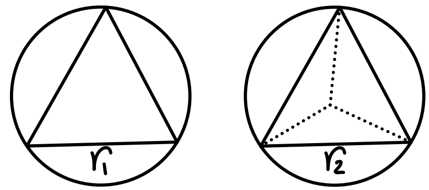
Lim, Mémoli, Smith, 2021

Sphere S^n, geodesic metric, diameter π.

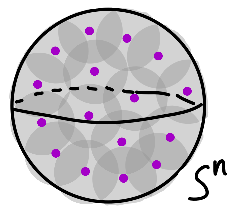
$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$

Symmetric matrix
Nonzero entries in $(\pi/2, \pi)$



For $n<k$,
$$2 \cdot d_{GH}(S^n, S^k) \geq \max\{r_n, \pi - \mathrm{cov}_{k+1}(S^n)\}$$



Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam.

Lim, Mémoli, Smith, 2021

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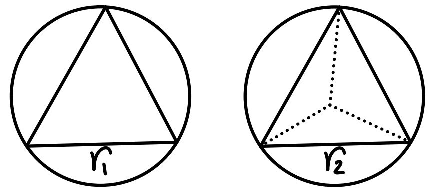

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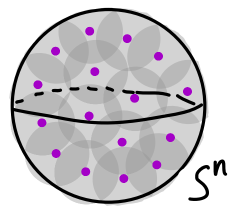

Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

Theorem (Oct, 2021) For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{c_{n,k}}.$$

Lim, Mémoli, Smith, 2021

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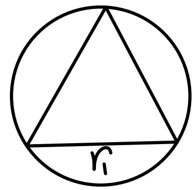

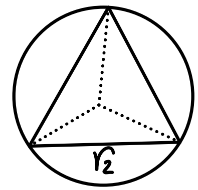

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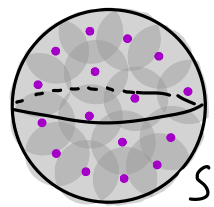

S^n

Equality for $1 \leq n < k \leq 3$. Proof with discont. Borsuk Ulam generalizes:

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Lim, Mémoli, Smith, 2021

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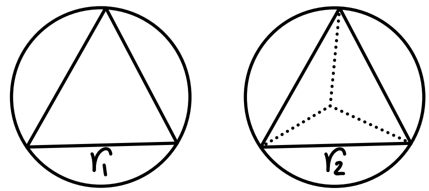
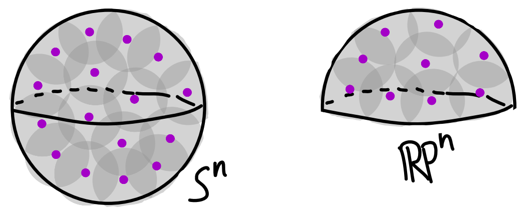


For $n<k$,

$$2 \cdot d_{GH}(S^n, S^k) \geq \max\{r_n, \pi - \mathrm{COV}_{k+1}(S^n)\}$$



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A., Bush, Frick, 2021

Def $\mathrm{COV}_k(X) :=$ infimum r s.t. k balls of radius $\frac{r}{2}$ cover X.

Lim, Mémoli, Smith, 2021

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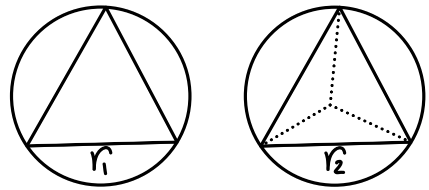

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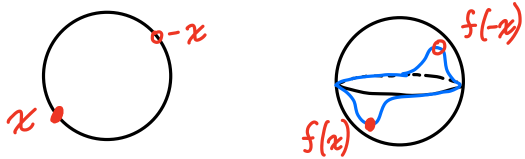
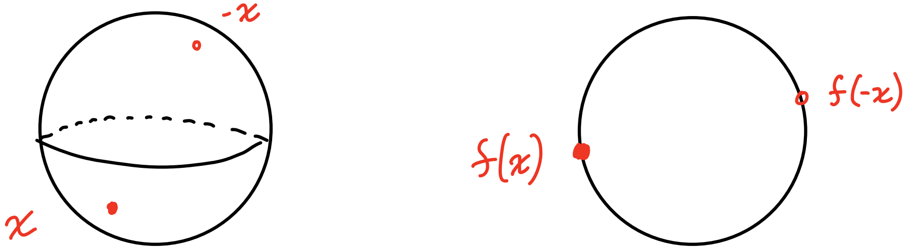
Borsuk-Ulam theorems



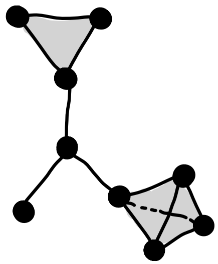
Def A map $f: S^k \to S^n$ is <u>odd</u> if $f(-x) = -f(x)$ $\forall x \in X$

Borsuk-Ulam: There is no cont. odd $S^k \to S^n$ for $k>n$.



Vietoris–Rips simplicial complexes

<u>Def</u> X metric space, $r \geq 0$. <u>Vietoris-Rips complex $VR(X;r)$</u> has vertex set X, all simplices of diameter $\leq r$.



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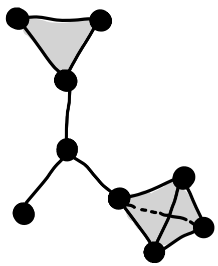
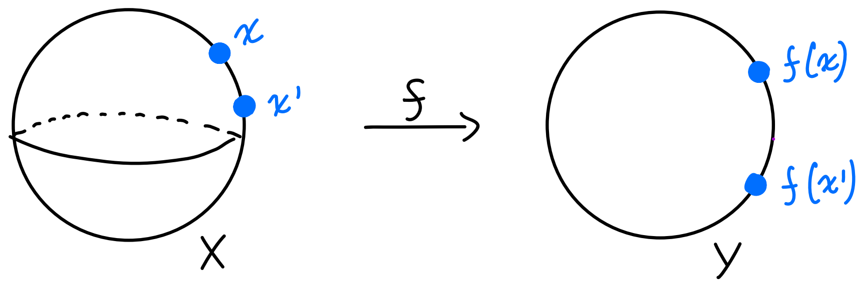


Map $f: X \to Y$ induces a (cont.) simplicial map $f: VR(X, r) \longrightarrow VR(Y; dis(f)+r)$.

$$[x_0, \dots, x_m] \longmapsto [f(x_0), \dots, f(x_m)]$$



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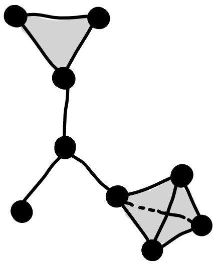

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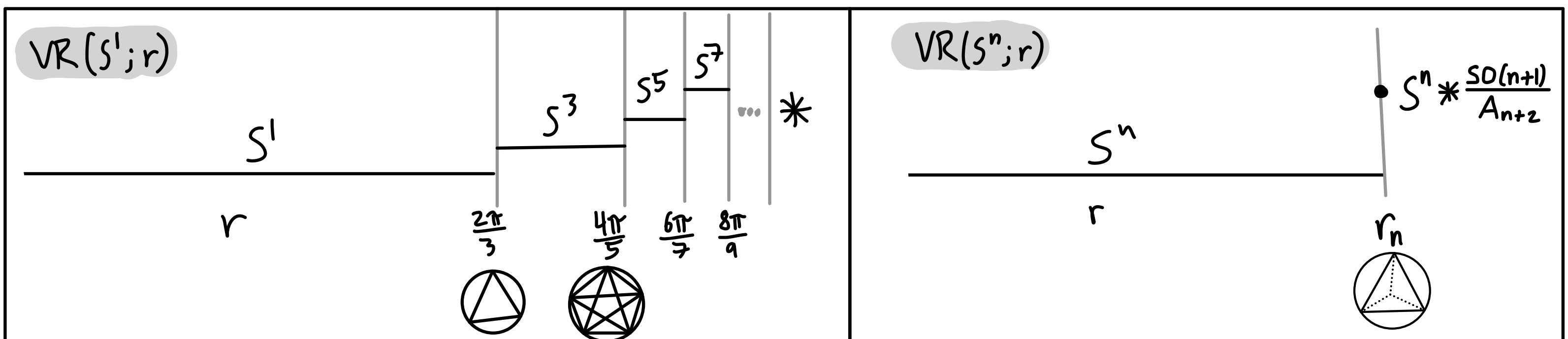


$$C_{1,2k+1} = C_{1,2k} = \frac{2\pi k}{2k+1} \qquad C_{n,n+2} = C_{n,n+1} = r_n$$

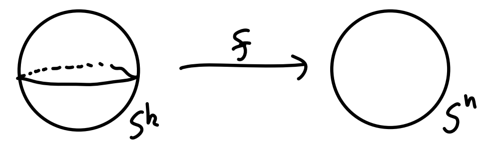
$$\underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n;r)\}}_{C_{n,k}}$$

Dubins & Schwarz '81

Lim, Mémoli, Smith, 2021

(Discont.) odd maps $f: S^{n+1} \to S^n$ have $\mathrm{dis}(f) \geq r_n$.

<u>Generalization</u> (Discont.) odd maps $f: S^k \to S^n$ for $k>n$ have $\mathrm{dis}(f) \geq c_{n,k}$.



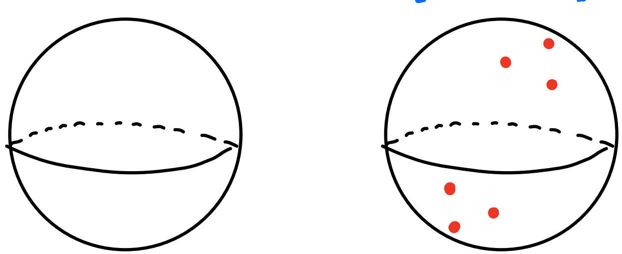
<u>Proof</u>

For $\varepsilon>0$, let $X \subset S^k$ be an $\frac{\varepsilon}{2}$ net with $X = -X$.

Produce a cont. odd map

$$S^k \xrightarrow{\text{partition of unity}} VR(X;\varepsilon) \xrightarrow{f} VR(S^n; \mathrm{dis}(f)+\varepsilon).$$

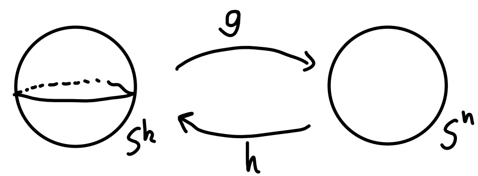
$$[x_0, \ldots, x_m] \longmapsto [f(x_0), \ldots, f(x_m)]$$



Hence $\mathrm{dis}(f)+\varepsilon \geq c_{n,k}$ $\forall \varepsilon>0$, so $\mathrm{dis}(f) \geq c_{n,k}$.

Theorem (Oct, 2021) For $n<k$,

$$2\cdot d_{GH}(S^n, S^k) \geq \underbrace{\inf\{r \mid \exists \text{ cont. odd } S^k \to VR(S^n; r)\}}_{C_{n,k}}.$$



Proof of Theorem follows Lim, Mémoli, Smith, 2021

$$2\cdot d_{GH}(S^n, S^k) = \inf_{\substack{g: S^k\to S^n \\ h: S^n \to S^k}} \max\{dis(g), dis(h), codis(g,h)\}$$

$$\geq \inf_{g: S^k \to S^n} dis(g)$$

$$= \inf_{\text{odd } g: S^k\to S^n} dis(g) \qquad \text{by Lemma 5.5}$$

$$\geq C_{n,k}.$$

<u>Question</u> Tight upper bounds on $d_{GH}(S^n, S^k)$ via maps?

<u>Question</u> Bounds on $d_{GH}(X,Y)$ for more general families of G-equivariant metric spaces X, Y?

<u>Question</u> Relate the p-Gromov-Wasserstein distance $d_{p\text{-}GW}$ to p-Vietoris-Rips thickenings VR_p?

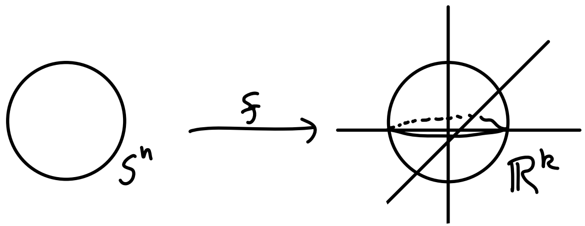
<u>Question</u> How does the generalization of Dubins & Schwarz relate to Tverberg?

Happy birthday, Gunnar!

Aside

A., Bush, Frick 2021 show:

If $f: S^n \to \mathbb{R}^k$ is odd for $k > n$, then there is a set $X \subset S^n$ of diameter $\leq c_{n,k}$ with $\vec{0} \in \operatorname{conv}(f(X))$.



Proof

$$S^n \longrightarrow \mathbb{R}^k \text{ induces}$$

$$VR(S^n; c_{n,k}) \longrightarrow \mathbb{R}^k$$