Fair Division

Henry Adams

CSU Data Science Seminar

Sources

"Rental Harmony: Sperner's Lemma in Fair Division"
Francis Su, MAA Monthly, 1999

"To Divide the Rent, Start With a Triangle"
Albert Sun, New York Times, 2014

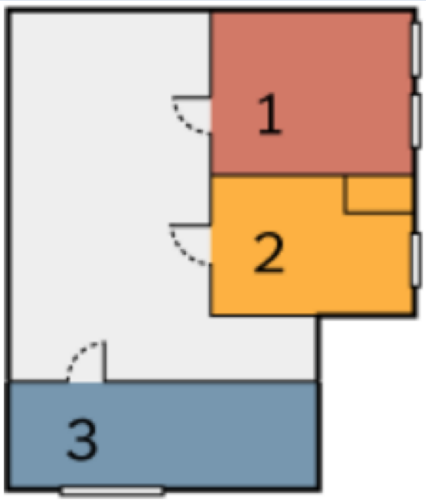
Problem

3 roommates: A, B, C

3 rooms: 1, 2, 3

Rent: $3,000

How do you divide the rent in an envy-free fashion?



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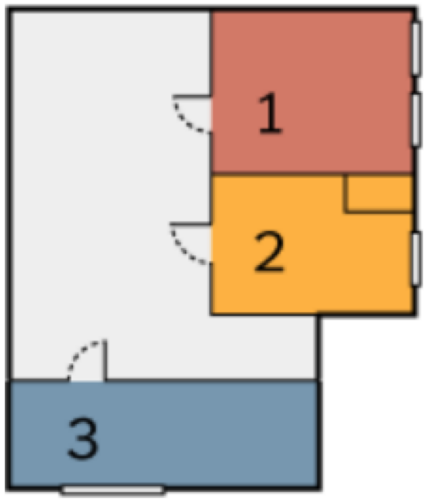


How do you divide the rent in an envy-free fashion?

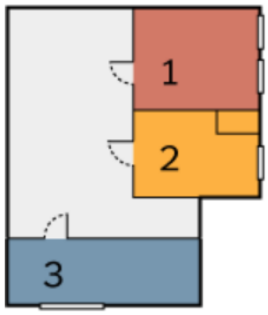
Remark Can be done with any number of rooms and roommates.

Assumption that will be relaxed

Assume that each roommate prefers a free room over a non-free room.

Algorithm

- Draw triangle of rent divisions.



1: 0
2: 0
3: 3000

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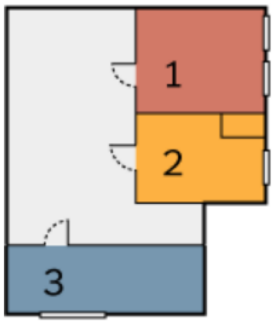
- Draw triangle of rent divisions.
- Subdivide to acceptable level.



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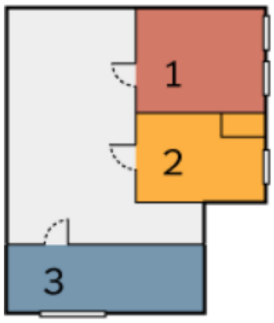

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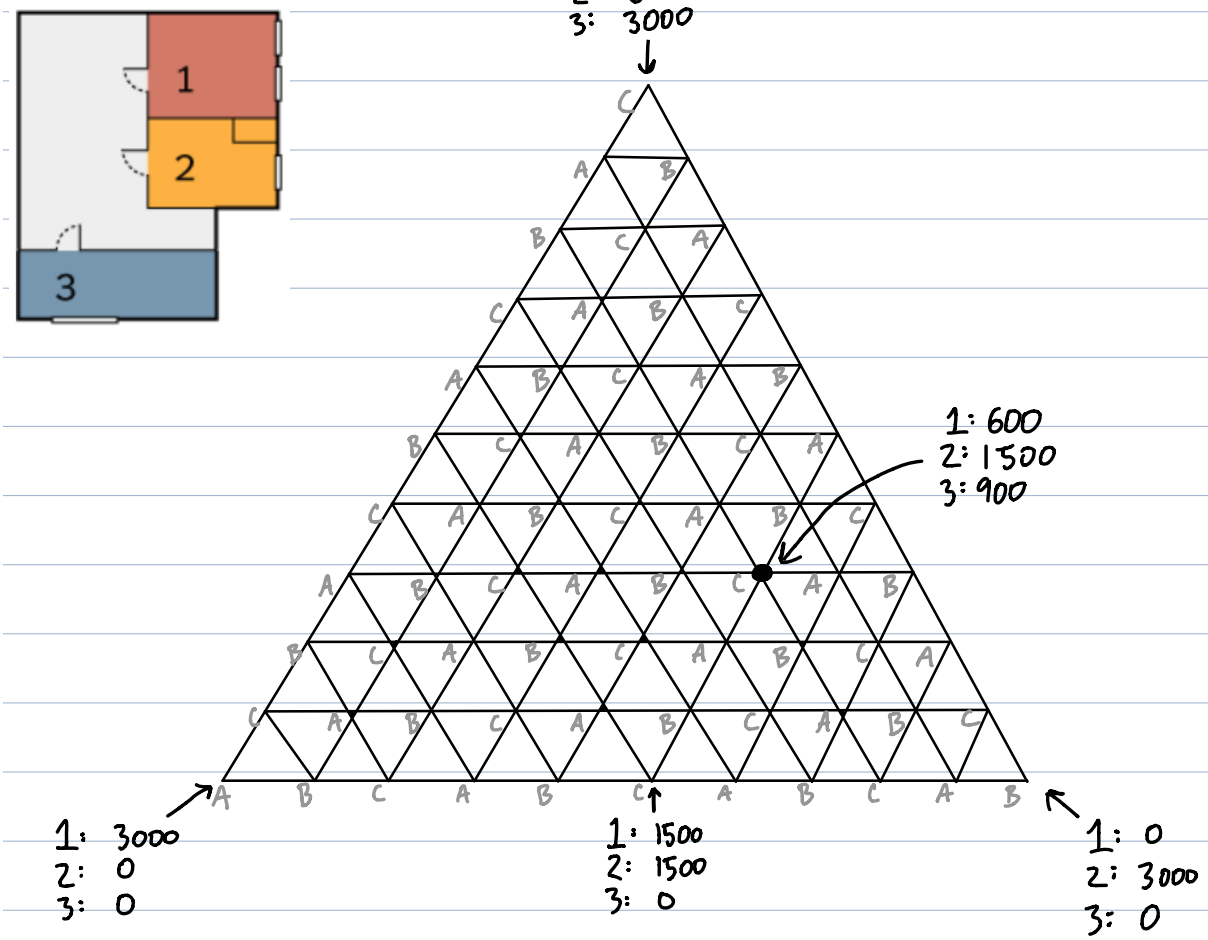
- Subdivide to acceptable level.
- Alternate vertex labels A, B, C.



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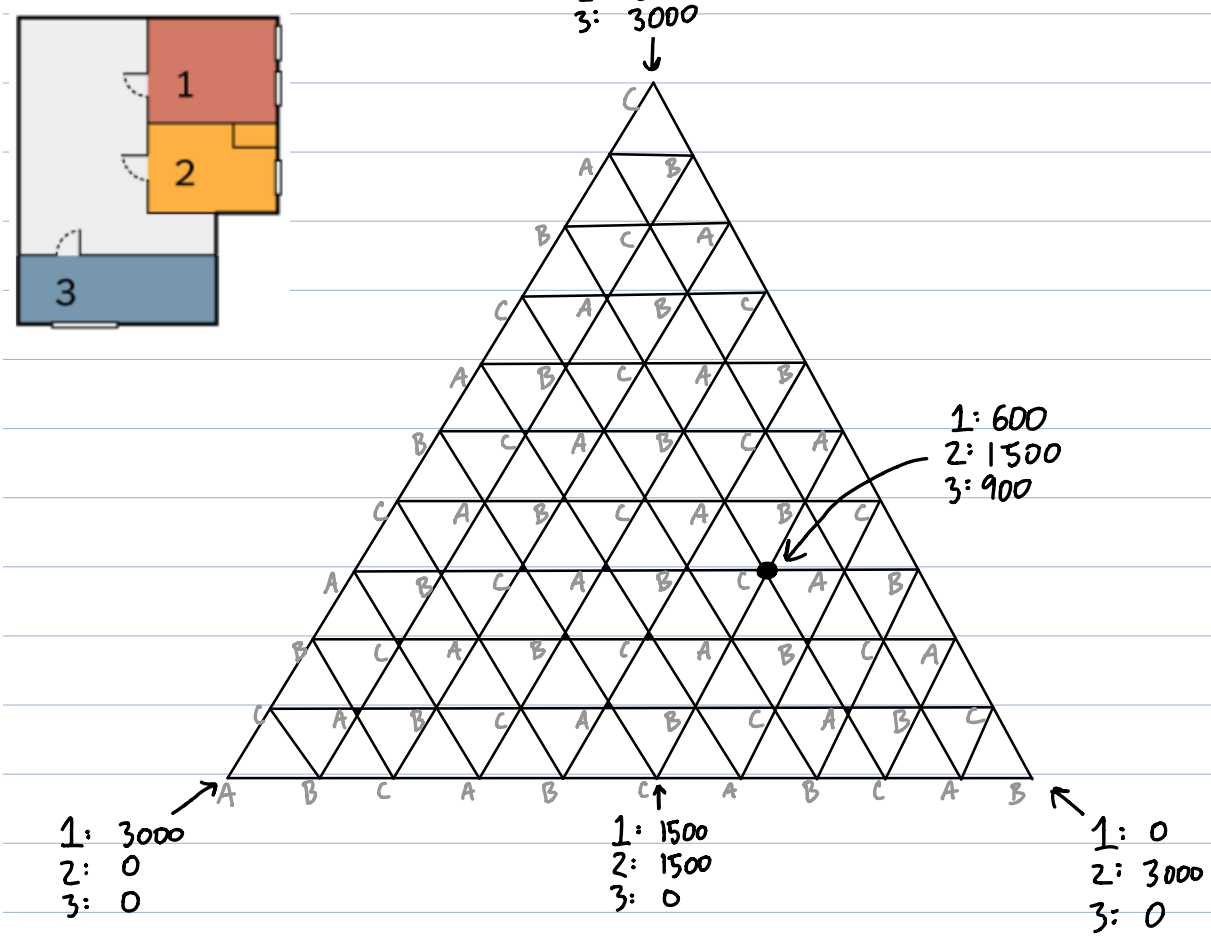
- Alternate vertex labels A, B, C.
- Poll roommates at their vertices.



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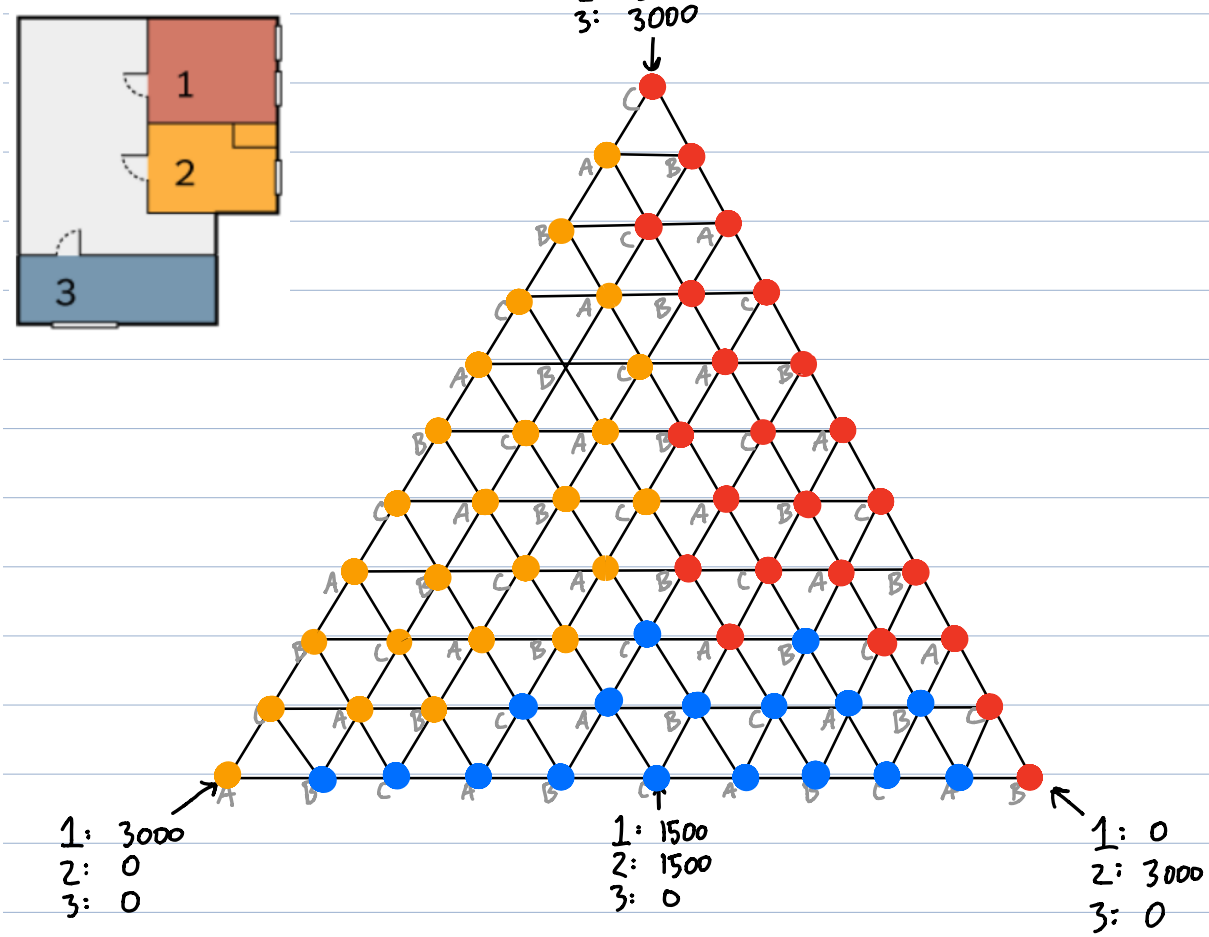

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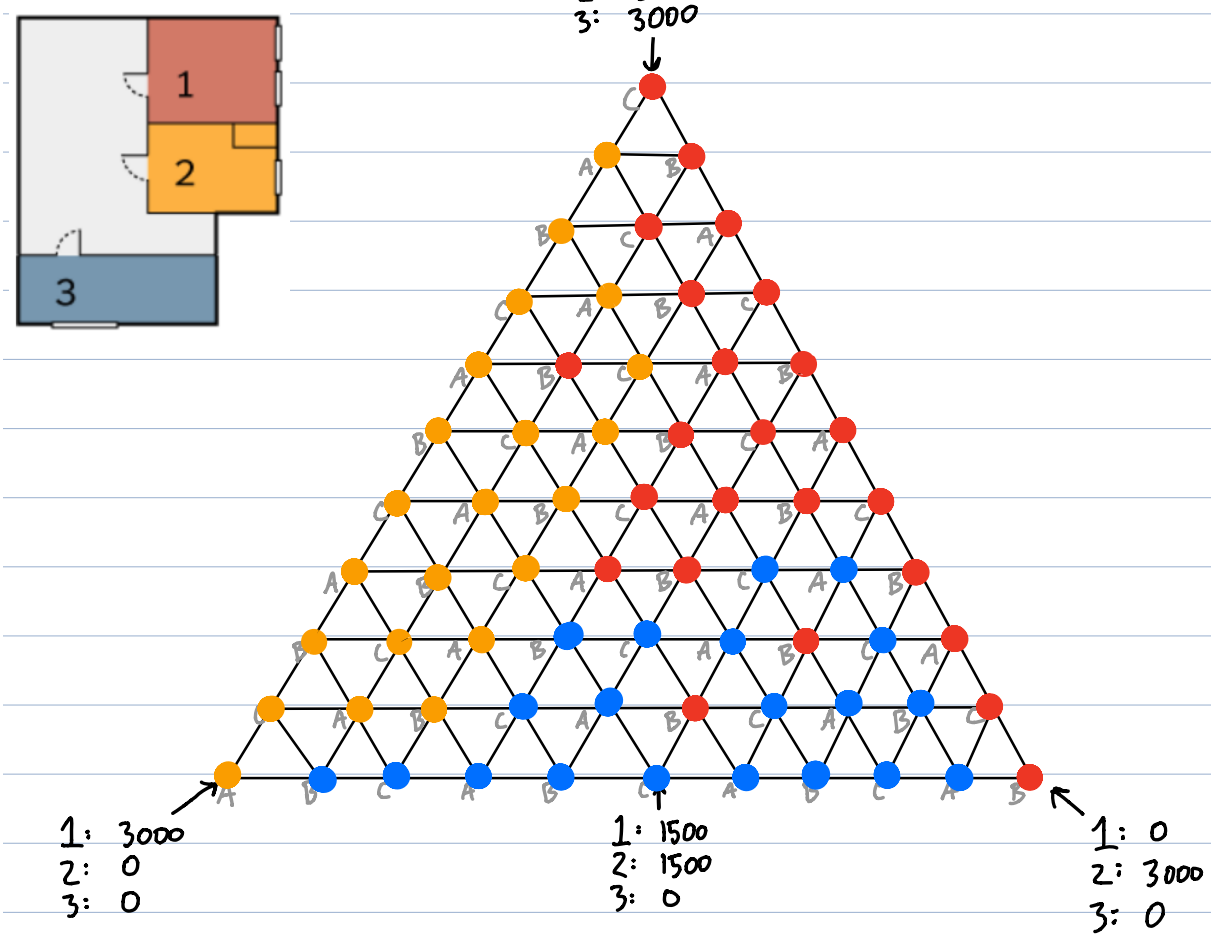

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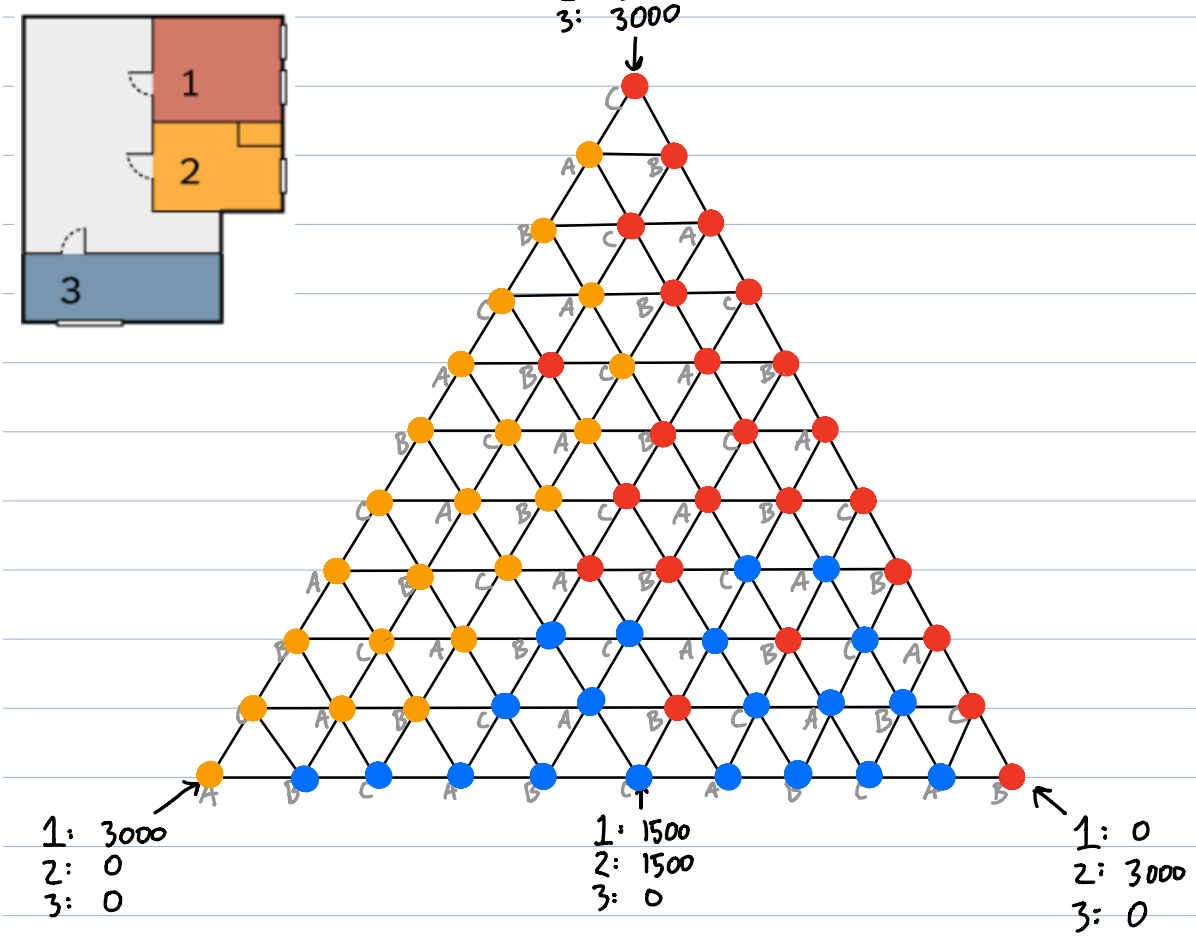

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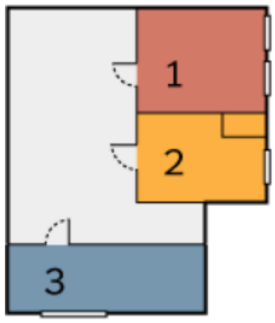
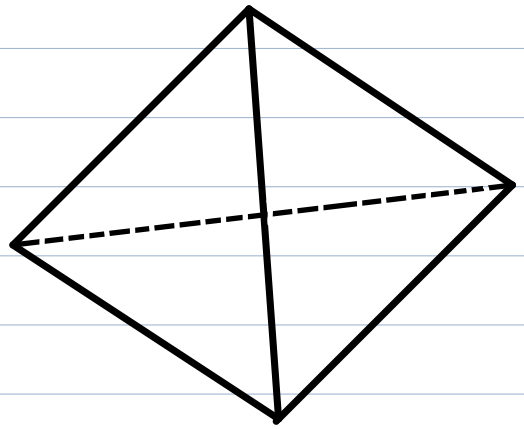


Four Roommates

Tetrahedron instead of triangle



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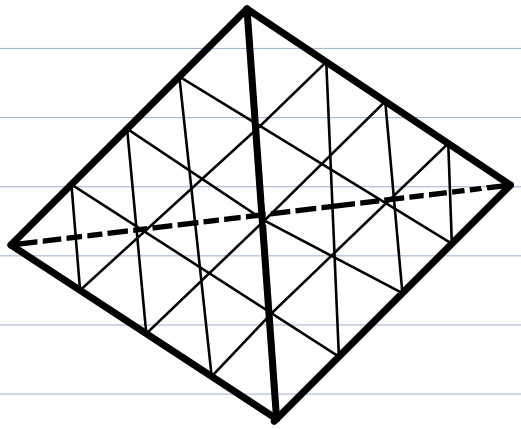


New York Times Applet

What's your total rent? $ 4050

How many of you are there? 2 3 4 5 **6** 7 8

Your names are:

A. Roommate A
B. Roommate B
C. Roommate C
D. Roommate D
E. Roommate E
F. Roommate F

And the rooms in your apartment are:

1. Room 1
2. Room 2
3. Room 3
4. Room 4
5. Room 5
6. Room 6

Start division »

New York Times Applet

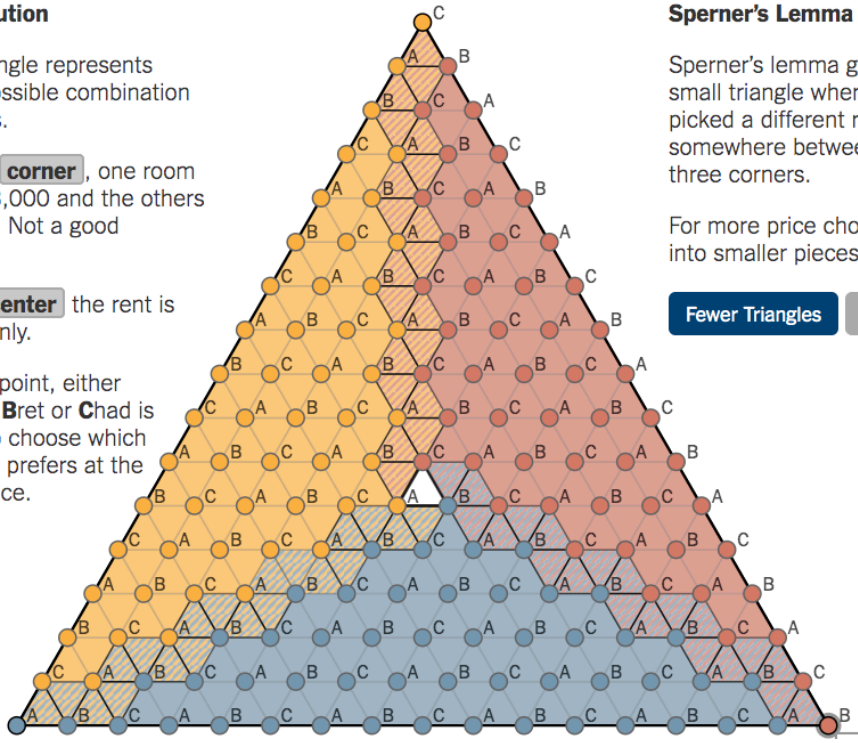
The Solution

This triangle represents every possible combination of prices.

At each **corner**, one room costs $3,000 and the others are free. Not a good solution.

In the **center** the rent is split evenly.

At each point, either **A**shwin, **B**ret or **C**had is asked to choose which room he prefers at the given price.



Sperner's Lemma

Sperner's lemma guarantees that there is a small triangle where every roommate has picked a different room. The "fair" price lies somewhere between the prices at those three corners.

For more price choices, divide the triangle into smaller pieces:

Fewer Triangles | More Triangles

Polling efficiencies, multiple envy-free solutions

New York Times Applet

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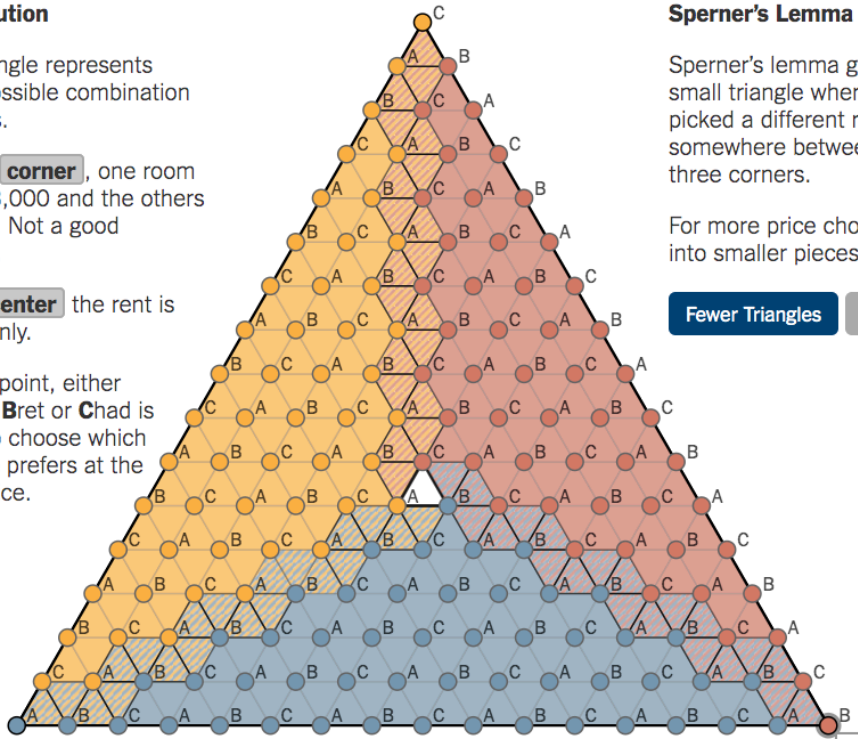

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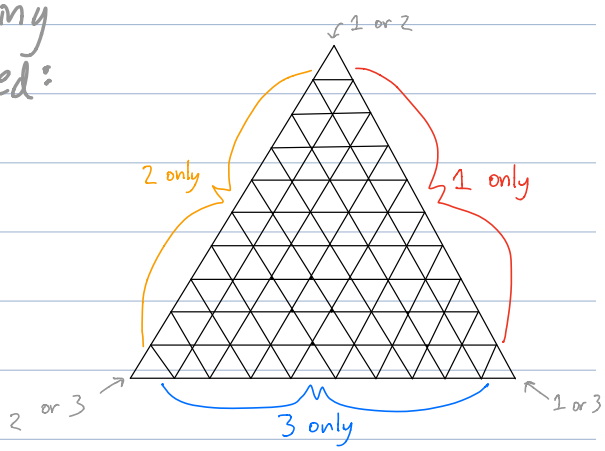
Spliddit Split fare, divide goods, assign (academic) credit, distribute chores, ...

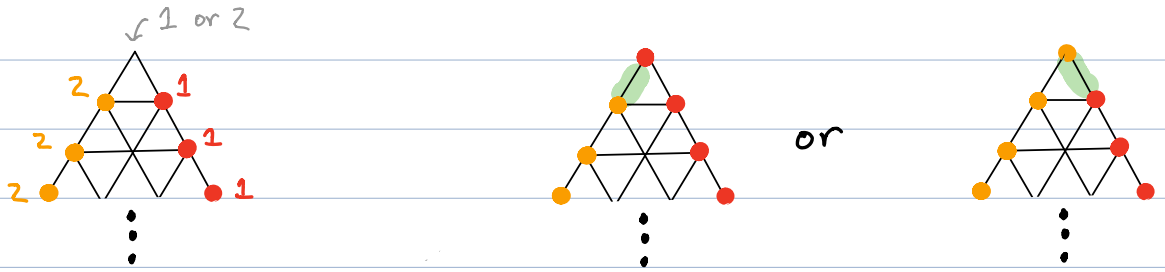
A variant of Sperner's Lemma

Consider a subdivision of a triangle into smaller triangles, such that each vertex is labeled 1, 2, or 3, and such that the following boundary conditions are satisfied:



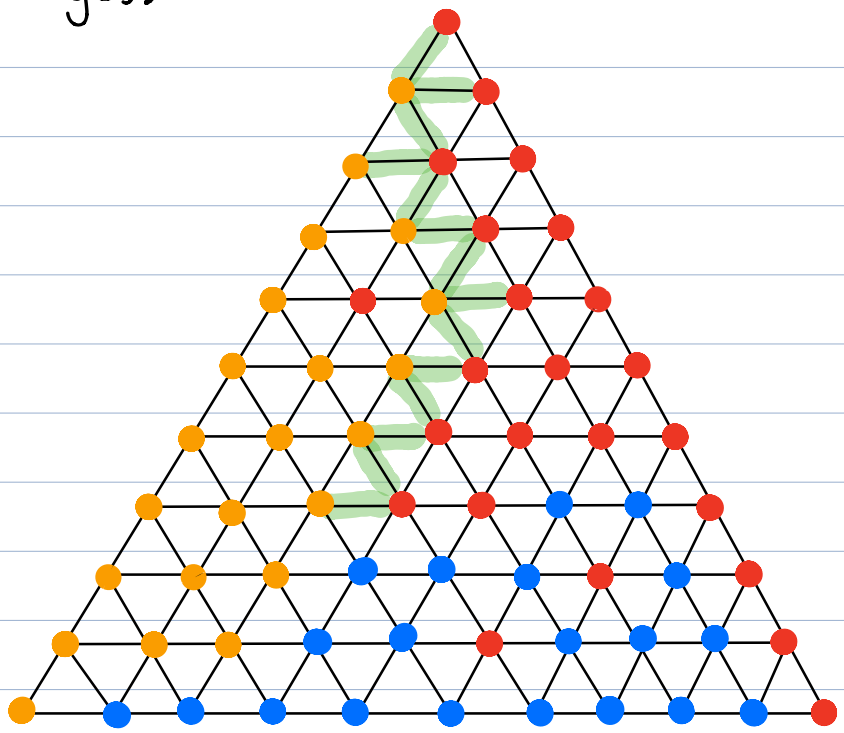
Then there exists a small rainbow triangle with vertex labels 1, 2, 3.

Proof Note there is a unique edge on the boundary with label 1, 2.



The adjacent triangle has either a second 1, 2 edge, or else a vertex with label 3.

We continue to get a path of such 1,2 edges.

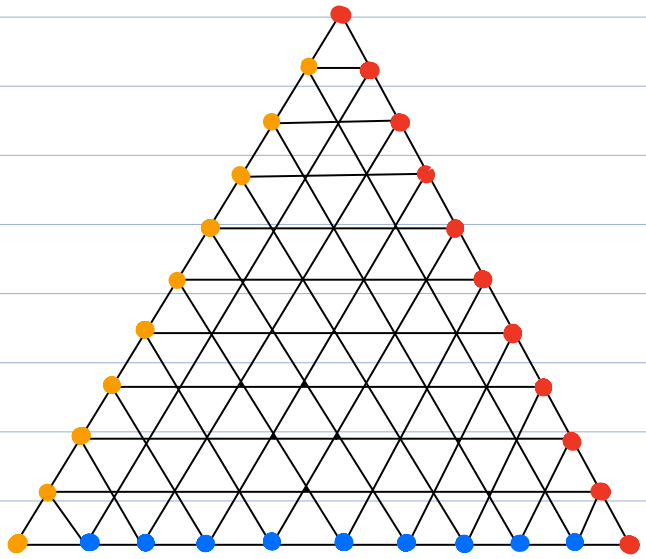


This path must end in a rainbow triangle containing label 3:

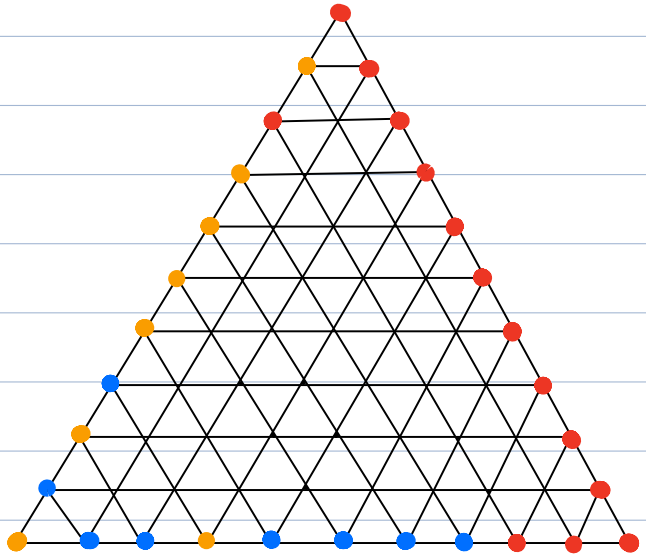
— The path can't end on with a 1,2 edge on the boundary since there is a unique such edge on the boundary.

— The path can't loop back on itself since no triangle has three 1,2 edges.

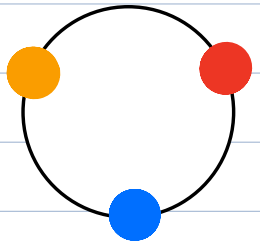
— The path can't go on forever since the triangulation is finite.



Relaxing the boundary assumptions



Only need that the "winding number" around the boundary is nonzero.



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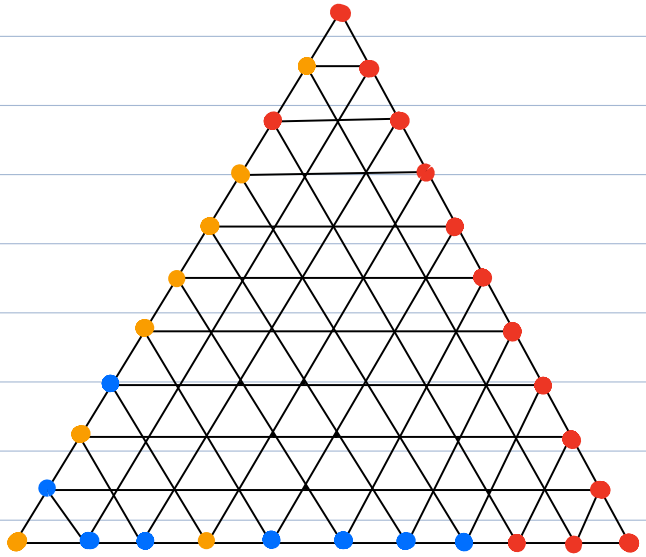

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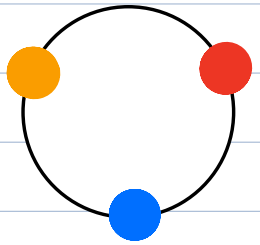
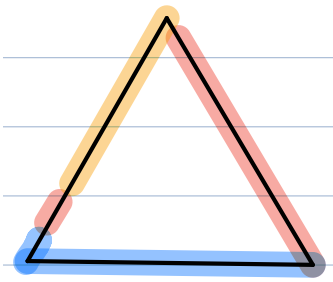
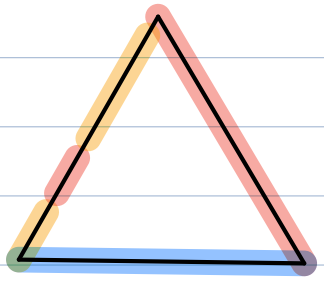
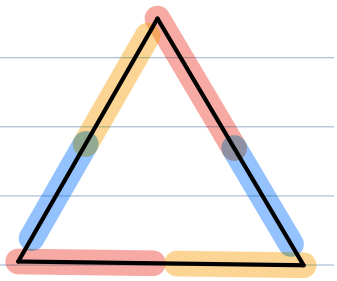
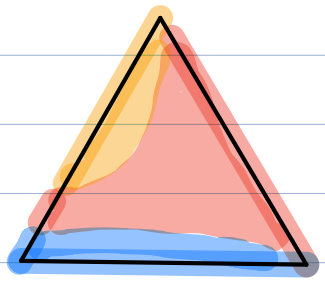


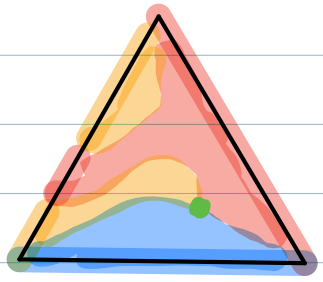








No rainbow triangle



There is a rainbow triangle and an odd # of 1,2 boundary edges.

Thanks!