Cyclic polytopes and nerve complexes

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Abstract

This talk will begin with an introduction to cyclic polytopes, which play an important role in polyhedral combinatorics. A cyclic polytope is a convex hull of points on a moment curve. The upper bound theorem by Peter McMullen and Richard Stanley states that among all simplicial spheres with the same number of vertices, the boundary of a cyclic polytope maximizes the number of faces. I will end by explaining how cyclic polytopes relate to Čech or nerve complexes, a geometric construction used in applied and computational topology.

0.1 Cyclic polytopes

Definition 1. The moment curve is $\alpha_d: \mathbb{R} \to \mathbb{R}^d$ via $\alpha_d(t) = (t, t^2, t^3, \ldots, t^d)$.

Definition 2. The cyclic polytope $C_d(n)$ is the convex hull

$$C_d(n) = \text{conv}\{\alpha_d(t_1), \alpha_d(t_2), \ldots, \alpha_d(t_n)\}$$

of $n > d$ distinct points.

- Combinatorics do not depend on $t_1, \ldots, t_d$.
- $\partial C_d(n)$ is simplicial.
- $\lfloor \frac{d}{2} \rfloor$-neighborly, meaning any subset of size $\leq \frac{d}{2}$ forms a face.
  
  ($C_3(6)$ has all edges exposed.)
  
  (No polytope is more than $\lfloor \frac{d}{2} \rfloor$-neighborly.)

Theorem 1 (Upper bound theorem, Peter McMullen 1970). Any $d$-dimensional polytope with $n$ vertices has no more $k$-faces than $C_d(n)$ \forall k.
Richard Stanley, 1975, proved the same for simplicial $d$-spheres (using the Stanley–Reisner ring and homological methods).

Let $d$ be even.

**Definition 3.** The *trigonometric moment curve* in $\mathbb{R}^d$ is $\gamma_d: S^1 \to \mathbb{R}^d$ via

$$\gamma_d(t) = (\cos(2\pi \cdot t), \sin(2\pi \cdot t), \cos(2\pi \cdot 2t), \sin(2\pi \cdot 2t), \ldots, \cos(2\pi \cdot \frac{d}{2}t), \sin(2\pi \cdot \frac{d}{2}t)).$$

David Gale [4] observed that combinatorially,

$$C_d(n) = \text{conv}\{\gamma_d(t_1), \gamma_d(t_2), \ldots, \gamma_d(t_n)\} = \text{conv}\{\gamma_d(0), \gamma_d(\frac{1}{n}), \ldots, \gamma_d(\frac{n-1}{n})\}.$$ (For the rest of the talk we will think of $C_d(n)$ as $\text{conv}\{\gamma_d(0), \gamma_d(\frac{1}{n}), \ldots, \gamma_d(\frac{n-1}{n})\}$.)

**Definition 4.** The *Carathéodory orbitope* is

$$C_d = \text{conv}\{\gamma_d(t) \mid t \in [0, 1]\}.$$
conjugate. This factorization is the classical Fejér–Riesz theorem. Prototypic examples are the group, i.e. a compact subgroup of\(GL_n\), of polynomials that are non-negative on Hermitian Toeplitz matrices or in terms of Hankel matrices. We identify each point \(u \in H\) of the coorbitope, which arise from ternary quartics. These correspond to non-negative linear functions that are non-negative on trigonometric polynomials; that is, \(\sin(\theta)\) and \(\cos(\theta)\).

A characterization of its faces is given in Theorem 5. A real representation of\(G\) is a group homomorphism \(\pi : G \to GL(V)\) for some finite-dimensional real vector space \(V\). We will conclude that this convex body is a projected spectrahedron but not a spectrahedron. Also, exactly one vertex of each triangle at \(O\) of the equilateral triangles for \(O\). The orbitopes of the group \(SO_n\) furnishes a spectrahedral cone dual to \(C\). A real representation of \(G\) is a group homomorphism \(\pi : G \to GL(V)\) for some finite-dimensional real vector space \(V\). We will conclude that this convex body is a projected spectrahedron but not a spectrahedron. Also, exactly one vertex of each triangle at \(O\) of the equilateral triangles for \(O\).

\[ \text{Figure 1: Cross-section of a four-dimensional Carathéodory orbitope.} \]

Given a compact real algebraic group of dimension \(N\), and which is our primary focus in § 5. This convex body is a projected spectrahedron but not a spectrahedron. Also, exactly one vertex of each triangle at \(O\) of the equilateral triangles for \(O\) of the curve \((\cos(\theta), \sin(\theta), \cos(2\theta))\).

\[ \text{Figure 2: Cross-section of a four-dimensional Carathéodory orbitope.} \]

\[ \text{Figure 3: (Left) Figure 1 of [5]: Cross-section of } C_4. \text{ (Right) Figure 3 of [5]: convex hull of the curve } (\cos(\theta), \sin(\theta), \cos(2\theta)). \]

\[ \text{Figure 4: Drawing of } U = \{\text{balls}\} \text{ and } N(U). \]

0.2 Nerve simplicial complexes

Let \(U = \{U_\alpha\}_{\alpha \in I}\) be a collection of subsets of some topological space.

\[ \text{Figure 4: Drawing of } U = \{\text{balls}\} \text{ and } N(U). \]

**Definition 5.** The nerve simplicial complex \(N(U)\) has vertex set \(I\), and \([\alpha_0, \ldots, \alpha_k]\) as a simplex if \(\cap_{i=0}^k U_{\alpha_i} \neq \emptyset\).

**Lemma 1** (Nerve lemma). If each \(U_\alpha\) is contractible, and if each intersection \(\cap_{i=0}^k U_{\alpha_i}\) is empty or contractible, then \(N(U) \simeq \cup U\) (for many choices of “reasonable” \(U\)).

**Definition 6.** Let \(N(n, k)\) be the nerve of \(n\) equally-spaced closed arcs of length \(\frac{k}{n}\) on the circle of unit-circumference.
Figure 5: Drawing of $N(6, 1) = S^1$, $N(6, 2) \simeq S^1$, $N(6, 3) \simeq \vee^2 S^2$. I should also draw $N(4, 1) = S^1$, $N(4, 2) = S^2$.

Figure 6: $N(6, 3) \simeq \vee^2 S^2$

- For $1 \leq k < \frac{n}{2}$ we have $N(n, k) \simeq S^1$ by the Nerve lemma.
- $N(n, n - 2) = \partial \Delta_{n-1} = S^{n-2}$.
- $N(n, n - 1) = \Delta_{n-1} \simeq *$.

**Theorem 2** ([2]).

$$N(n, k) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{l+1} < \frac{k}{n} < \frac{l+1}{l+2} \text{ for } l \in \mathbb{N} \\ \vee^{n-k-1} S^{2l} & \text{if } \frac{k}{n} = \frac{l}{l+1} \end{cases}$$
Definition 7. Let \( N(S^1, r) \) be the nerve of all closed arcs of length \( r \) on the circle of unit-circumference.

Theorem 3 \([\Pi]\). \( N(S^1, r) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{l+1} < r < \frac{l+1}{l+2} \\ V^\infty S^{2l} & \text{if } r = \frac{l}{l+1} \end{cases} \) for \( l \in \mathbb{N} \).

0.3 Relationship between \( \partial C_{2l+2}(n) = S^{2l+1} \simeq N(n, k) \)

Let \( \frac{l}{l+1} < \frac{k}{n} < \frac{l+1}{l+2} \).

Theorem 4 \([\Xi]\). Simplicial inclusion \( \partial C_{2l+2}(n) \hookrightarrow N(n, k) \) exists and is a homotopy equivalence.

(Current proof unsatisfyingly requires a priori knowledge of \( N(n, k) \simeq S^{2l+1} \).)

(Analogous inclusion for \( \frac{k}{n} = \frac{l}{l+1} \) allows us to understand \( D_{2n} \) action on \( H_*(N(n, k)) \).)

Conjecture 1 \([\Xi]\). \( N(n, k) \) simplicially collapses to \( \partial C_{2l+2}(n) \).

0.4 Relationship between \( \partial C_{2l+2} = S^{2l+1} \simeq N(S^1, r) \)

Let \( \frac{l}{l+1} < r < \frac{l+1}{l+2} \).

Define \( \gamma_{2l+2}^* \) by \( \gamma_{2l+2} \) on the vertex set, and linear extension to barycentric coordinates.

Define \( p \) by radial projection.

Map \( i \) is defined via \( \sum \lambda_i \gamma_{2l+2}(t_i) \mapsto \sum \lambda_i t_i \). Here the \( \lambda_i \) satisfy \( \sum \lambda_i = 1 \) and \( \lambda_i \geq 0 \), giving a convex combination or barycentric coordinates. For \( i \) to even be continuous, we must change the topology on \( N(S^1, r) \).
Conjecture 2. Map $i \gamma_{2l+2}^*: \mathcal{N}(S^1, r) \to \iota(\partial C_{2l+2}) = S^{2l+1}$ is a deformation retraction.

0.5 Conclusion
All of this has analogues for Vietoris–Rips complexes and Barvinok–Novik orbitopes [3].

References


