Cyclic polytopes and nerve complexes

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Abstract

This talk will begin with an introduction to cyclic polytopes, which play an important role in polyhedral combinatorics. A cyclic polytope is a convex hull of points on a moment curve. The upper bound theorem by Peter McMullen and Richard Stanley states that among all simplicial spheres with the same number of vertices, the boundary of a cyclic polytope maximizes the number of faces. I will end by explaining how cyclic polytopes relate to Čech or nerve complexes, a geometric construction used in applied and computational topology.

0.1 Cyclic polytopes

Definition 1. The *moment curve* is $\alpha_d \colon \mathbb{R} \to \mathbb{R}^d$ via $\alpha_d(t) = (t, t^2, t^3, \ldots, t^d)$.

Definition 2. The *cyclic polytope* $C_d(n)$ is the convex hull

$$C_d(n) = \mathrm{conv}\{\alpha_d(t_1), \alpha_d(t_2), \ldots, \alpha_d(t_n)\}$$

of $n > d$ distinct points.

- Combinatorics do not depend on $t_1, \ldots, t_d$.
- $\partial C_d(n)$ is simplicial.
- $\lfloor \frac{d}{2} \rfloor$-neighborly, meaning any subset of size $\leq \frac{d}{2}$ forms a face.
  ($C_3(6)$ has all edges exposed.)
  (No polytope is more than $\lfloor \frac{d}{2} \rfloor$-neighborly.)

Theorem 1 (Upper bound theorem, Peter McMullen 1970)**.** Any d-dimensional polytope with n vertices has no more k-faces than $C_d(n)$ $\forall k$.

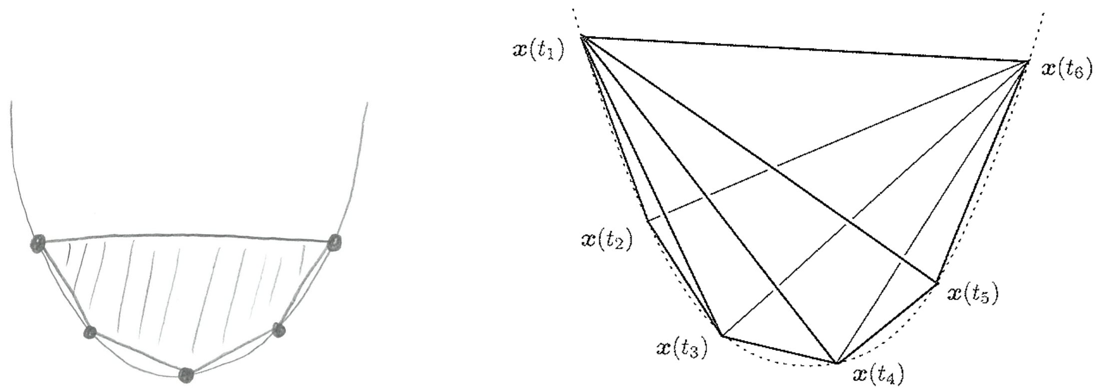


Figure 1: (Left) Drawing of $C_2(5)$. (Right) Figure of $C_3(6)$ on page 11 of [6].

Richard Stanley, 1975, proved the same for simplicial d-spheres (using the Stanley–Reisner ring and homological methods).

Let d be even.

Definition 3. The *trigonometric moment curve* in $\mathbb{R}^d$ is $\gamma_d \colon S^1 \to \mathbb{R}^d$ via

$$\gamma_d(t) = (\cos(2\pi \cdot t), \sin(2\pi \cdot t), \cos(2\pi \cdot 2t), \sin(2\pi \cdot 2t) \ldots, \cos(2\pi \cdot \tfrac{d}{2}t), \sin(2\pi \cdot \tfrac{d}{2}t)).$$

David Gale [4] observed that combinatorially,

$$\begin{aligned} C_d(n) &= \operatorname{conv}\{\gamma_d(t_1), \gamma_d(t_2), \ldots, \gamma_d(t_n)\} \\ &= \operatorname{conv}\{\gamma_d(0), \gamma_d(\tfrac{1}{n}), \ldots, \gamma_d(\tfrac{n-1}{n})\}. \end{aligned}$$

(For the rest of the talk we will think of $C_d(n)$ as $\operatorname{conv}\{\gamma_d(0), \gamma_d(\tfrac{1}{n}), \ldots, \gamma_d(\tfrac{n-1}{n})\}$.)

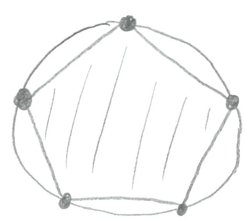


Figure 2: Drawing of $C_2(5)$ on the trigonometric moment curve.

Definition 4. The *Carathéodory orbitope* is

$$C_d = \operatorname{conv}\{\gamma_d(t) \mid t \in [0, 1)\}.$$

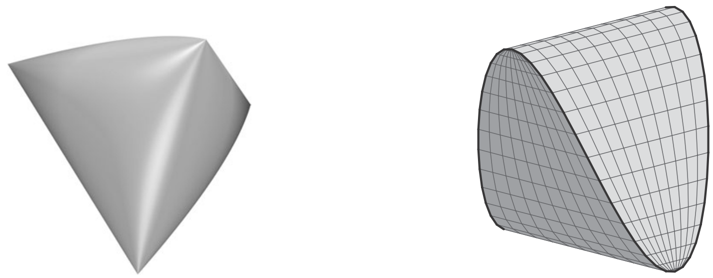


Figure 3: (Left) Figure 1 of [5]: Cross-section of C_4. (Right) Figure 3 of [5]: convex hull of the curve $(\cos(\theta), \sin(\theta), \cos(2\theta))$.

0.2 Nerve simplicial complexes

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of subsets of some topological space.

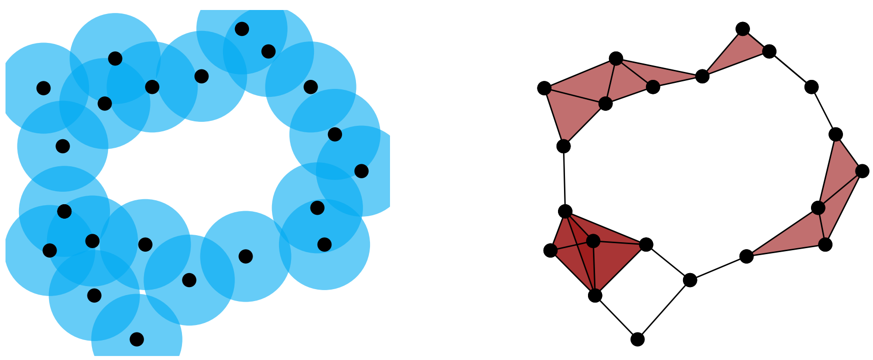


Figure 4: Drawing of $\mathcal{U} = \{\text{balls}\}$ and $\mathcal{N}(\mathcal{U})$.

Definition 5. The *nerve simplicial complex* $\mathcal{N}(\mathcal{U})$ has vertex set I, and $[\alpha_0, \ldots, \alpha_k]$ as a simplex if $\cap_{i=0}^{k} U_{\alpha_i} \neq \emptyset$.

Lemma 1 (Nerve lemma)**.** If each U_α is contractible, and if each intersection $\cap_{i=0}^{k} U_{\alpha_i}$ is empty or contractible, then $\mathcal{N}(\mathcal{U}) \simeq \cup\mathcal{U}$ (for many choices of "reasonable" $\mathcal{U}$).

Definition 6. Let $\mathcal{N}(n, k)$ be the nerve of n equally-spaced closed arcs of length $\frac{k}{n}$ on the circle of unit-circumference.

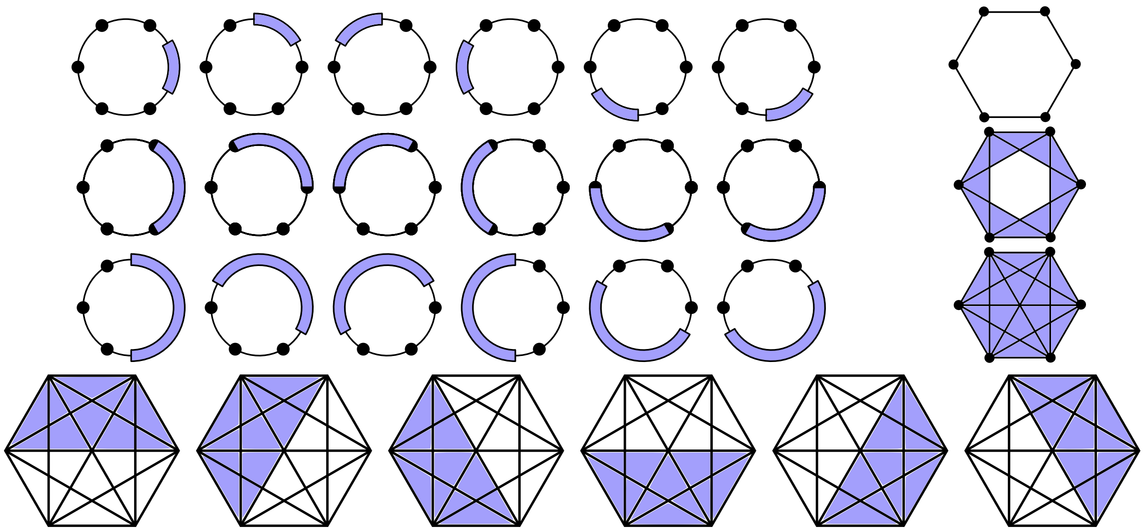


Figure 5: Drawing of $\mathcal{N}(6,1) = S^1$, $\mathcal{N}(6,2) \simeq S^1$, $\mathcal{N}(6,3) \simeq \vee^2 S^2$. I should also draw $\mathcal{N}(4,1) = S^1$, $\mathcal{N}(4,2) = S^2$.

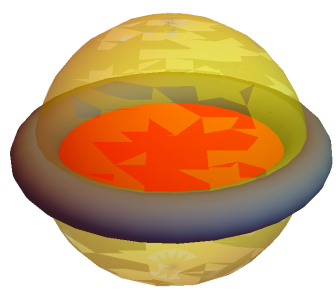
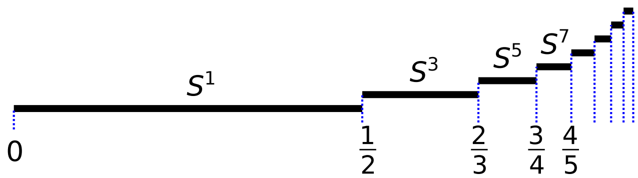


Figure 6: $\mathcal{N}(6,3) \simeq \vee^2 S^2$

- For $1 \le k < \frac{n}{2}$ we have $\mathcal{N}(n,k) \simeq S^1$ by the Nerve lemma.
- $\mathcal{N}(n, n-2) = \partial\Delta_{n-1} = S^{n-2}$.
- $\mathcal{N}(n, n-1) = \Delta_{n-1} \simeq *$.

Theorem 2 ([2]).

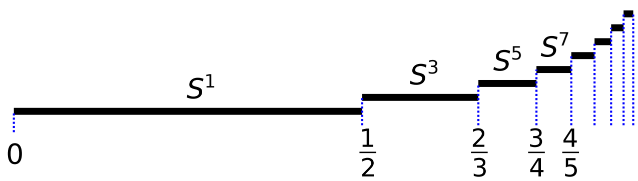
$$\mathcal{N}(n,k) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{l+1} < \frac{k}{n} < \frac{l+1}{l+2} \\ \vee^{n-k-1} S^{2l} & \text{if } \frac{k}{n} = \frac{l}{l+1} \end{cases} \quad \text{for } l \in \mathbb{N}.$$



Definition 7. Let $\mathcal{N}(S^1, r)$ be the nerve of *all* closed arcs of length r on the circle of unit-circumference.

Theorem 3 ([1])**.**

$$\mathcal{N}(S^1, r) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{l+1} < r < \frac{l+1}{l+2} \\ \bigvee^{\infty} S^{2l} & \text{if } r = \frac{l}{l+1} \end{cases} \quad \text{for } l \in \mathbb{N}.$$



0.3 Relationship between $\partial C_{2l+2}(n) = S^{2l+1} \simeq \mathcal{N}(n, k)$

Let $\frac{l}{l+1} < \frac{k}{n} < \frac{l+1}{l+2}$.

Theorem 4 ([2])**.** Simplicial inclusion $\partial C_{2l+2}(n) \hookrightarrow \mathcal{N}(n, k)$ exists and is a homotopy equivalence.

(Current proof unsatisfyingly requires a priori knowledge of $\mathcal{N}(n, k) \simeq S^{2l+1}$.)
(Analogous inclusion for $\frac{k}{n} = \frac{l}{l+1}$ allows us to understand D_{2n} action on $H_*(\mathcal{N}(n, k))$.)

Conjecture 1 ([2])**.** $\mathcal{N}(n, k)$ simplicially collapses to $\partial C_{2l+2}(n)$.

0.4 Relationship between $\partial C_{2l+2} = S^{2l+1} \simeq \mathcal{N}(S^1, r)$

Let $\frac{l}{l+1} < r < \frac{l+1}{l+2}$.

$$\mathcal{N}(S^1, r) \xrightarrow{\gamma^*_{2l+2}} \mathbb{R}^{2l+2} \setminus \{\vec{0}\} \xrightarrow{p} \partial C_{2l+2} \xrightarrow{i} \mathcal{N}(S^1, r).$$

Define γ^*_{2l+2} by γ_{2l+2} on the vertex set, and linear extension to barycentric coordinates.
Define p by radial projection.

Map i is defined via $\sum \lambda_i \gamma_{2l+2}(t_i) \mapsto \sum \lambda_i t_i$. Here the λ_i satisfy $\sum \lambda_i = 1$ and $\lambda_i \geq 0$, giving a convex combination or barycentric coordinates. For i to even be continuous, we must change the topology on $\mathcal{N}(S^1, r)$.

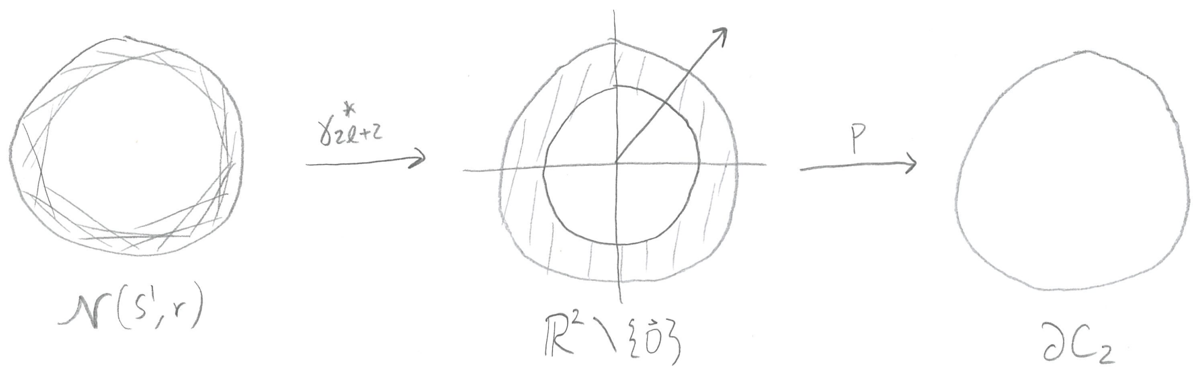


Figure 7: Picture of map $p\gamma^*_{2l+2}$ when $l=0$.

Conjecture 2. Map $ip\gamma^*_{2l+2}\colon \mathcal{N}(S^1,r) \to \iota(\partial C_{2l+2}) = S^{2l+1}$ is a deformation retraction.

0.5 Conclusion

All of this has analogues for Vietoris–Rips complexes and Barvinok–Novik orbitopes [3].

References

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