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Computing Persistent Homology*

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Abstract. We show that the persistent homology of a filtered d-dimensional simplicial complex is simply the standard homology of a particular graded module over a polynomial ring. Our analysis establishes the existence of a simple description of persistent homology groups over arbitrary fields. It also enables us to derive a natural algorithm for computing persistent homology of spaces in arbitrary dimension over any field. This result generalizes and extends the previously known algorithm that was restricted to subcomplexes of $\mathbb{S}^3$ and $\mathbb{Z}_2$ coefficients. Finally, our study implies the lack of a simple classification over non-fields. Instead, we give an algorithm for computing individual persistent homology groups over an arbitrary principal ideal domain in any dimension.

1. Introduction

In this paper we study the homology of a filtered d-dimensional simplicial complex K, allowing an arbitrary principal ideal domain D as the ground ring of coefficients. A filtered complex is an increasing sequence of simplicial complexes, as shown in Fig. 1. It determines an *inductive system* of homology groups, i.e., a family of Abelian groups $\{G_i\}_{i\geq 0}$ together with homomorphisms $G_i \to G_{i+1}$. If the homology is computed with field coefficients, we obtain an inductive system of vector spaces over the field. Each vector space is determined up to isomorphism by its dimension. In this paper we obtain a simple classification of an inductive system of vector spaces. Our classification is in

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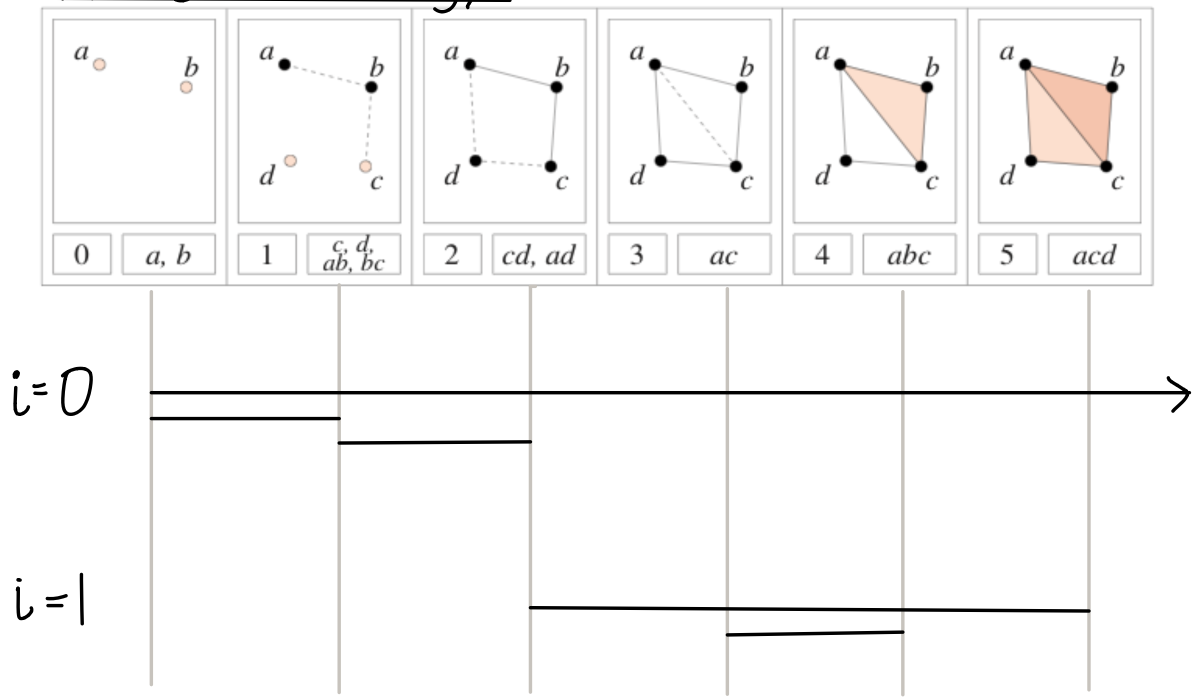
Outline

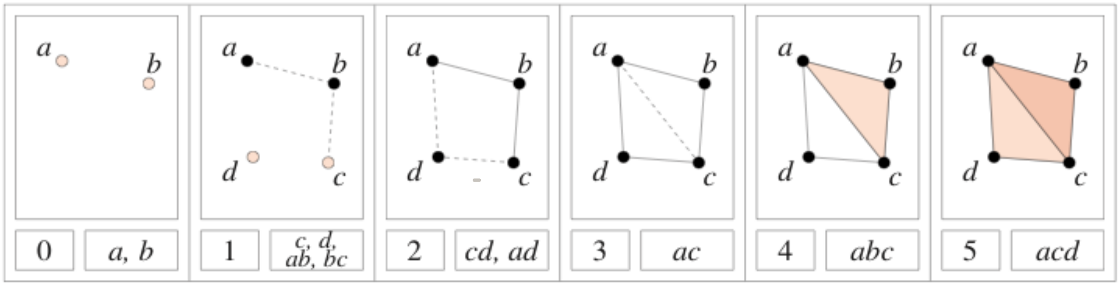
- Persistent homology
- Equivalence with $F[t]$-modules
- Decomposition theorem
- Computation
- Experiments

Persistent homology



Apply i-dimensional homology with ...

Equivalence



A persistence module

$$\mathbb{V}: \quad V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} V_4 \xrightarrow{f_4} V_5$$

can be thought of as a graded $F[t]$-module with elements $(v_0, v_1, v_2, v_3, v_4, v_5) \in V_0 \oplus \dots \oplus V_5$ and $F[t]$ action given by:

$3 \cdot (v_0, \dots, v_5) =$

$t \cdot (v_0, \dots, v_5) =$

$t^2 \cdot (v_0, \dots, v_5) =$

$(3 + t^2) \cdot (v_0, \dots, v_5) =$

Decomposition

Finitely-generated abelian group

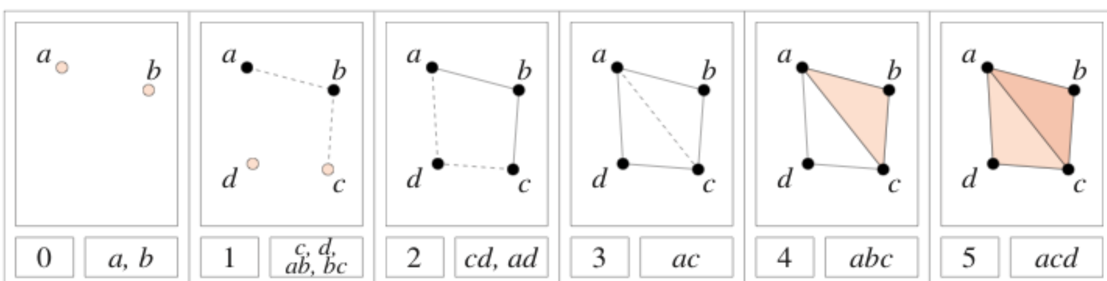
$$G \cong \left(\bigoplus_{i=1}^{n} \mathbb{Z}\right) \oplus \left(\bigoplus_{j=1}^{m} \mathbb{Z}/d_j\mathbb{Z}\right)$$

Finitely-generated F[t]-module

$$\mathbb{V} \cong \left(\bigoplus_{i=1}^{n} F[t]\right) \oplus \left(\bigoplus_{j=1}^{m} F[t]/(t^{n_j})\right)$$

Finitely-generated graded F[t]-module

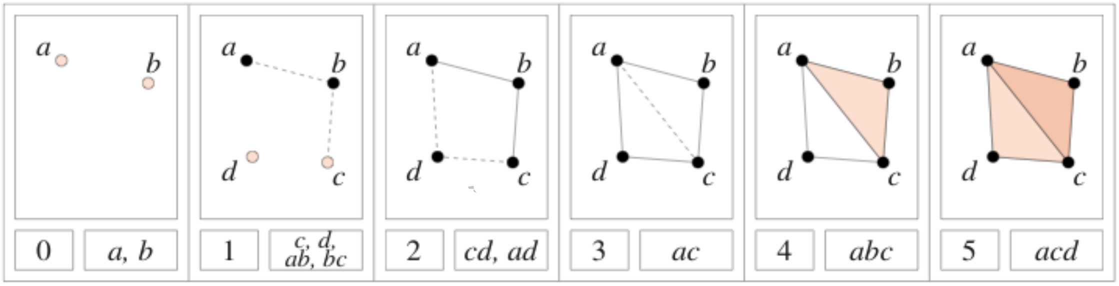
$$\mathbb{V} \cong \left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} F[t]\right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} F[t]/(t^{n_j})\right)$$



$i=0$

$i=1$

Computation



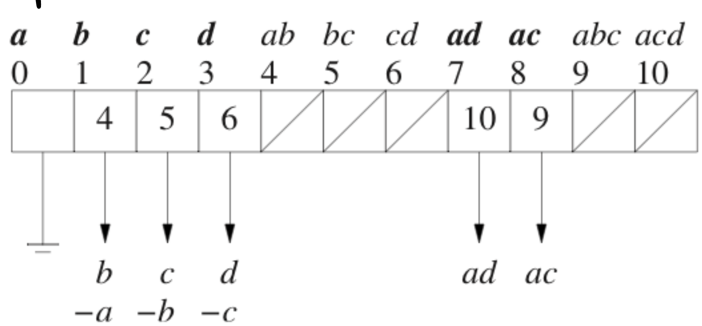
$$
M_1 = \begin{array}{c|ccccc}
 & ab & bc & cd & ad & ac \\
\hline
d & 0 & 0 & t & t & 0 \\
c & 0 & 1 & t & 0 & t^2 \\
b & t & t & 0 & 0 & 0 \\
a & t & 0 & 0 & t^2 & t^3
\end{array}
$$

In our example, we have

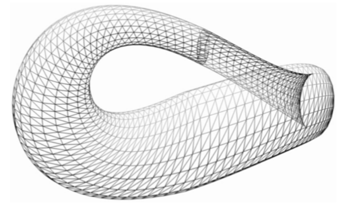
$$
\tilde{M}_1 = \begin{array}{c|ccccc}
 & cd & bc & ab & z_1 & z_2 \\
\hline
d & \boxed{\mathrm{t}} & 0 & 0 & 0 & 0 \\
c & t & \boxed{1} & 0 & 0 & 0 \\
b & 0 & t & \boxed{\mathrm{t}} & 0 & 0 \\
a & 0 & 0 & t & 0 & 0
\end{array}, \tag{8}
$$

where $z_1 = ad - cd - t \cdot bc - t \cdot ab$ and $z_2 = ac - t^2 \cdot bc - t^2 \cdot ab$ form a homogeneous basis for Z_1.

Sparse computation



Experiments



Klein bottle
Electrostatic charge
Jet engine

	$\|K\|$	Length	Filtration (s)	Persistance (s)
K	12,000	1,020	0.03	$<$ 0.01
E	529,225	3,013	3.17	5.00
J	3,029,383	256	24.13	50.23