### Main Theorem

**Theorem.** A Vietoris–Rips complex or a restricted Čech complex on a finite set of points from the circle is homotopy equivalent to either a point, an odd sphere, or a wedge sum of spheres of the same even dimension.

**Remark:** The homotopy types of Vietoris–Rips complexes of evenly-spaced circular points are proven in [1].

**Remark:** The proof for arbitrary circular points relies on the homotopy types of Vietoris–Rips complexes of evenly-spaced circular vertices with connectivity parameter $2\pi k/n$. That is, simplex $\sigma$ is in $\pi_k(n,k)$ when $\text{diam}(\sigma) \leq 2\pi k/n$.

### Notation for evenly-spaced points

**Definition.** Let $\text{VR}(n,k)$ be the Vietoris–Rips complex on $n$ evenly-spaced circular points with connectivity parameter $2\pi k/n$. The proof uses [1, 2, 3].

### Example.

**Example.** $\text{VR}(9,3)$ is the clique complex of the graph below, giving $\text{VR}(9,3) \simeq \bigvee_3 S^2$.

To visualize this, note that $\text{VR}(9,3)$ has three maximal 2-simplices.

**Definition.** Let restricted Čech complex $\mathcal{C}(n,k)$ be the nerve of the covering of $S^1$ by $n$ evenly-spaced closed arcs of arc length $2\pi k/n$.

**Example.** $\mathcal{C}(6,3)$ is the nerve of the 6 circular arcs below, giving $\mathcal{C}(6,3) \simeq \bigvee_2 S^2$.

Alternatively, $\mathcal{C}(6,3)$ has 6 maximal 3-simplices, glued together to form $\mathcal{C}(6,3) \simeq \bigvee_2 S^2$.

### Further properties

**Proposition.** Let $k < n$ and write $n = q(n-k) + r$ with $0 \leq r < n-k$. Then 

$$
\text{VR}(n,k) \simeq \bigvee_{q-1} S^{q-1} \quad \text{if } r = 0
$$

**Case of evenly-spaced points**

**Corollary 6.7 from [1].**

Let $k < n/2$ and write $n - k = q(n - 2k) + r$ with $0 \leq r < n - 2k$. Then 

$$
\text{VR}(n,k) \simeq \bigvee_{q-1} S^{q-1} \quad \text{if } r = 0
$$

**Remark:** Note $\text{VR}(n+k,k) \simeq \mathcal{C}(n,k)$. For example, $\text{VR}(9,3) \simeq \bigvee_3 S^2 \simeq \mathcal{C}(6,3)$.

### Arbitrary circular points

When built on an arbitrary finite set of circular points, a Vietoris–Rips or restricted Čech complex is still homotopy equivalent to either a point, an odd sphere, or a wedge sum of spheres of the same even dimension.

**Proof idea:** If $K$ is a simplicial complex and $u$ and $v$ are two distinct vertices with $s_t(u) \subseteq s_t(v)$ (we say $u$ is dominated by $v$), then $K \simeq K \setminus u$. We show how to remove dominated vertices until we are left with a complex equivalent to some $\text{VR}(n,k)$ or $\mathcal{C}(n,k)$.

### References

