Bridging applied and quantitative topology

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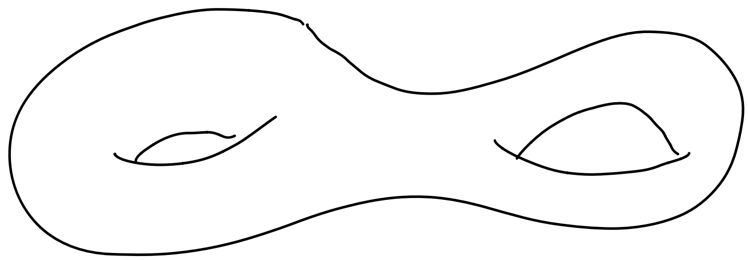
Joint with Michał Adamaszek, Florian Frick, Baris Coskunuzer, Johnathan Bush, Joshua Mirth, Michael Moy, Facundo Mémoli, Qingsong Wang

Upcoming: Hess, Fajstrup, Adler, Weinberger, Ghrist

Bridge #1: Filling Radius

Gromov, 1983, "Filling Riemannian manifolds"

Isosystolic inequality For M an essential n-dimensional Riemannian manifold, $\mathrm{systole}(M) \leq C\,\mathrm{vol}(M)^{1/n}$.



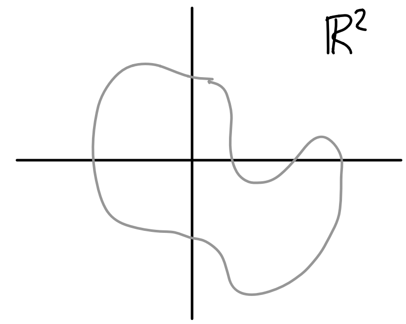
The systole of M is the length of the shortest non-contractible loop.

"Essential" rules out counterexamples like $S^1 \times S^2$.

Proof $\mathrm{sys}(M) \leq 6 \cdot \mathrm{FillingRadius}(M) \leq C\,\mathrm{vol}(M)^{1/n}$

M essential   all M

Filling radius



X a metric space

Kuratowski embedding $X \hookrightarrow L^\infty(X)$

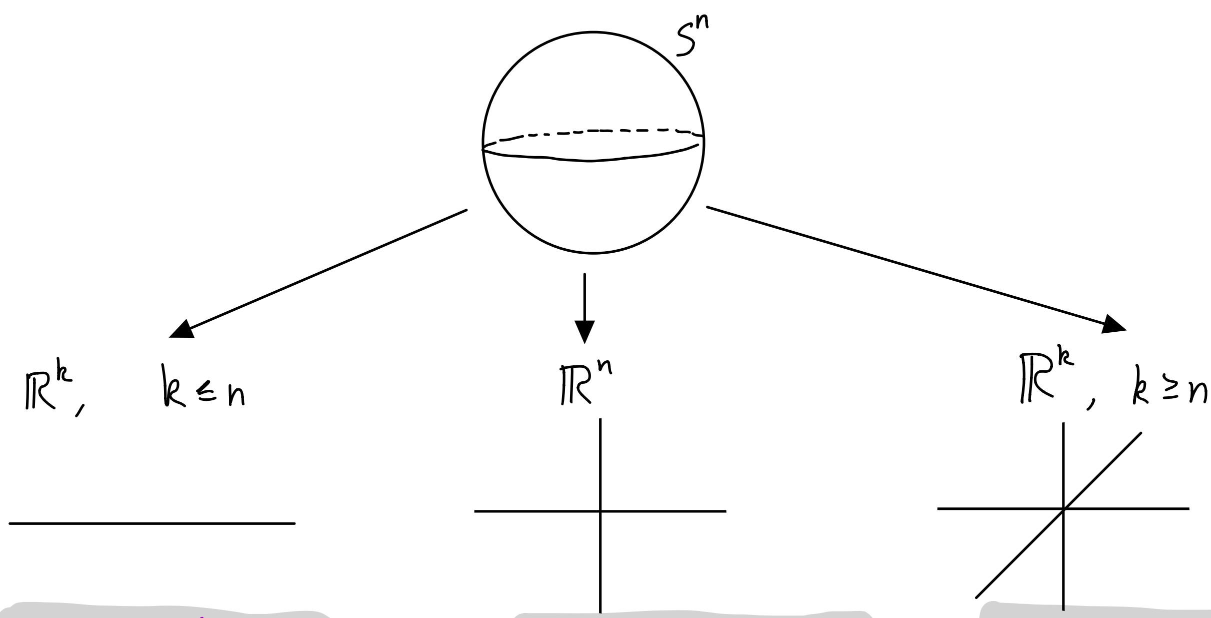
$x \longmapsto$

Def The filling radius of manifold M^n is the infimum r such that $M \hookrightarrow B_{L^\infty(M)}(M;r)$ induces a map on n-dimensional homology killing the (nonzero) fundamental class.

Lim, Mémoli, Okutan, 2020, "Vietoris-Rips persistent homology, injective metric spaces, and the filling radius"

Thm For X a metric space, $B_{L^\infty(X)}(X;\frac{r}{2}) \simeq \mathrm{VR}(X;r)$.

Bridge #2: Borsuk-Ulam theorems



Almgren (1965): Geometric measure theory
Gromov (2003): Filling radius, characteristic classes

Borsuk-Ulam (1933)

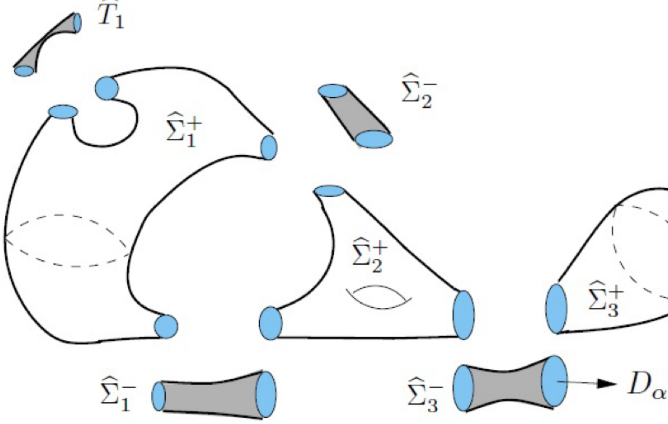
A, Bush, Frick (2020)

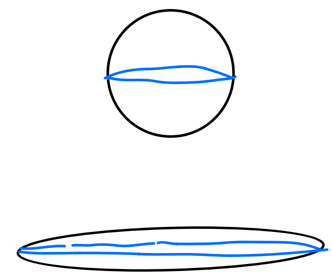
"Waist of sphere" theorem For $f: S^n \to \mathbb{R}^k$ with $k \le n$, $\exists y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \ge \mathrm{Vol}_{n-k}(S^{n-k})$.

Invariance of dimension.

Bridge #3: Thick-thin decompositions, sweep-outs

A, Coskunuzer, 2021, "Geometric approaches on persistent homology





X vertex set of unweighted graph.

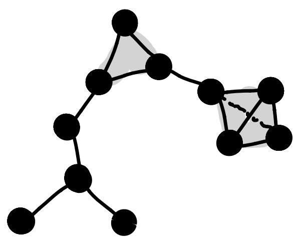
Thm A 2-dimensional homology class σ in $VR(X;r)$ has persistence $\leq \sqrt{\text{area}(\sigma)} + 1$.

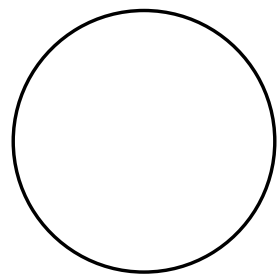
Thm A k-dimensional homology class σ in $VR(X;r)$ has persistence $\leq \text{width}(\sigma) + 1$.

X metric space, $r \geq 0$.

<u>Def</u> The <u>Vietoris-Rips simplicial complex</u> has

- vertex set X
- finite simplex $\sigma \subseteq X$ when $\mathrm{diameter}(\sigma) \leq r$.

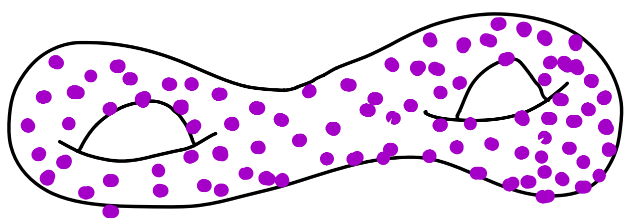


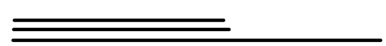


<u>History</u>

- Cohomology theory for metric spaces
- Geometric group theory
- Applied topology

<u>Stability</u>



$PH_1(VR(M;r))$ 

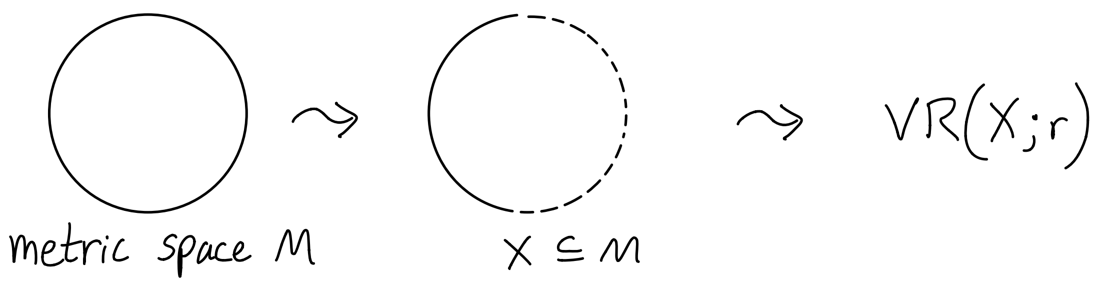
$PH_1(VR(X;r))$ 

Chazal, de Silva, Oudot, 2014

Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, 2009

Metric Reconstruction

A simplicial complex whose vertex set is a metric space should often be equipped with an optimal transport metric (instead of the simplicial complex topology).

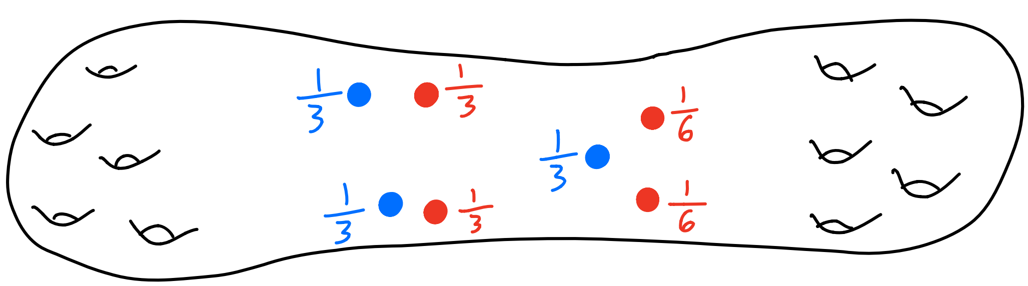


Adamaszek, A, Frick, 2018, "Metric reconstruction via optimal transport"

Def X metric space, $r \geq 0$.
The <u>Vietoris-Rips metric thickening</u> is

$$\mathrm{VR}(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; \begin{array}{l} x_i \in X, \quad \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r, \\ \lambda_i \geq 0, \quad \sum \lambda_i = 1 \end{array} \right\},$$

equipped with the p-Wasserstein metric.

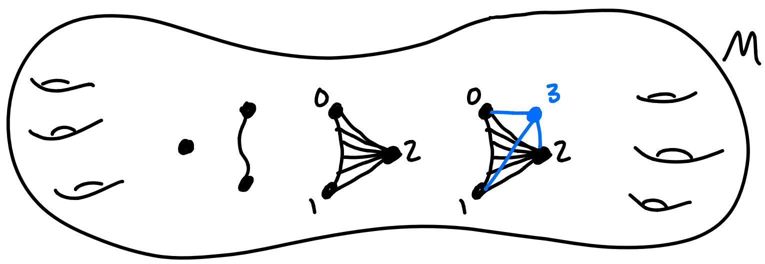


Thm (Hausmann 1995)

M compact Riemannian manifold.
Then $\exists\ r_0 > 0$ such that $VR(M;r) \simeq M\ \ \forall r < r_0$.

Proof Sketch

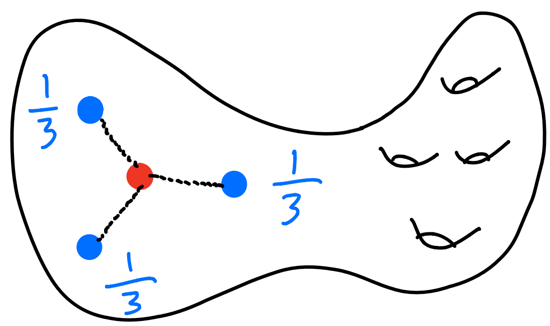
$$VR(M;r) \to M$$



- Not canonical
- $M \hookrightarrow VR(M;r)$ not continuous.

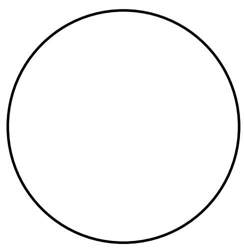
Our Proof Sketch

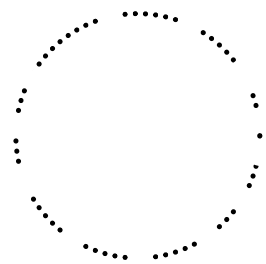
$$\begin{tikzcd} VR^m(M;r) \arrow[d, bend right] \\ M \arrow[u, bend right] \end{tikzcd} \qquad \sum \lambda_i \delta_{x_i} \mapsto \text{Karcher or Frechét mean}$$



A, Mémoli, Moy, Wang, 2021+

<u>Thm</u> For X totally bounded, $\mathrm{VR}^m(X;r)$ and $\mathrm{VR}(X;r)$ have the same (undecorated) persistence diagrams.





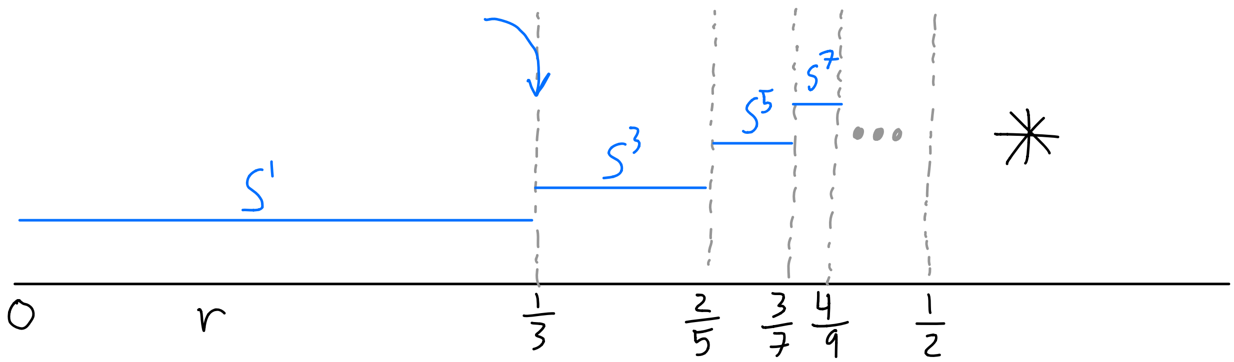
<u>Cor</u> $\mathrm{VR}^m(X;r)$ is stable.

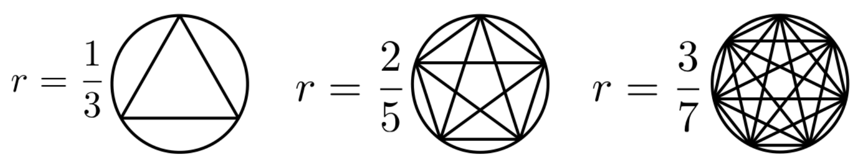
<u>Question</u> Is $\mathrm{VR}^m_<(X;r) \simeq \mathrm{VR}_<(X;r)$?

A, Adamaszek, "The Vietoris-Rips complexes of a circle", 2017

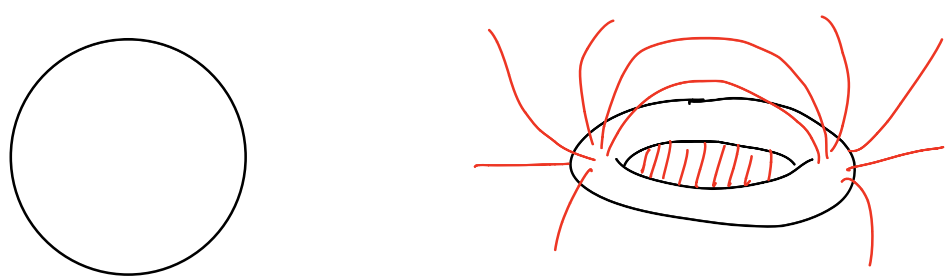
S^1 is circle with geodesic metric, unit circumference.

<u>Thm</u> $VR(S^1;r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{if } r = \frac{k}{2k+1} \end{cases}$ $k \in \mathbb{N}$



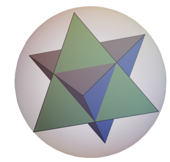


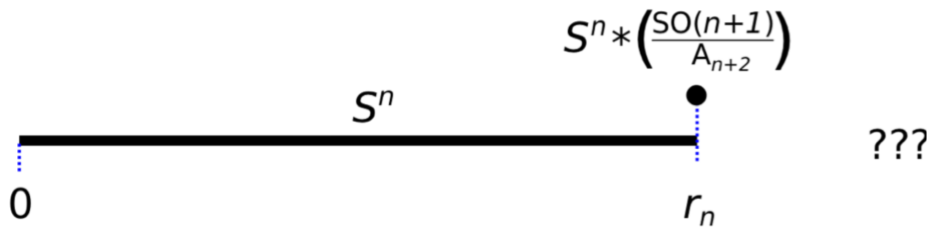
By contrast, $VR^m(S^1; \frac{1}{3}) \simeq S^3$. <u>Why?</u>



More generally,

<u>Thm</u> $VR^m(S^n; r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$





Katz, 1991, "On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional"

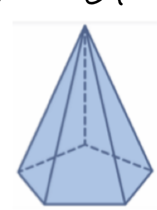
<u>Thm</u> $B_{L^\infty(X)}(S^2; \frac{r}{2}) \simeq$ first S^2, then $S^2 * \frac{SO(3)}{A_4}$

$$\parallel$$

$$S^2 * \frac{S^3}{E_6}$$

<u>Conjecture</u> The next change in homotopy type for $VR^m(S^2; r)$ occurs at the diameter of a pentagonal pyramid, with homotopy type an 8-dimensional CW complex

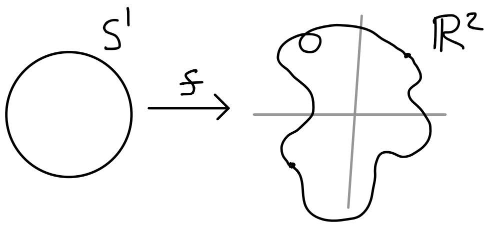
$$\left(S^2 * \frac{SO(3)}{A_4}\right) \cup_f \left(\Delta^5 \times \frac{SO(3)}{\mathbb{Z}/5\mathbb{Z}}\right).$$

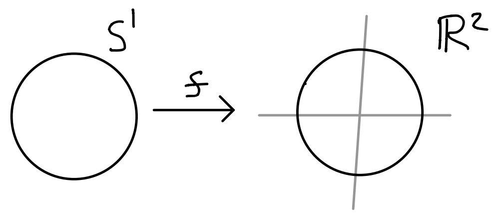


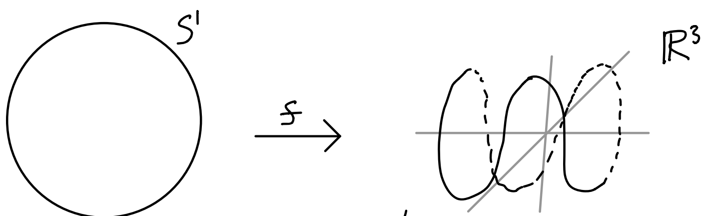
Here $\partial\Delta^5 \times \frac{SO(3)}{\mathbb{Z}/5\mathbb{Z}} \xrightarrow{f} S^2 * \frac{SO(3)}{A_4}$
with $\pi_4\left(S^2 * \frac{SO(3)}{A_4}\right) \cong \mathbb{Z}/3\mathbb{Z}$.

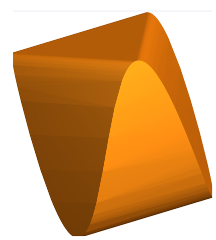
Borsuk-Ulam theorems for $f: S^n \to \mathbb{R}^k$ with $k \geq n$?

A, Bush, Frick, 2020, "Metric thickenings, Borsuk-Ulam theorems, and orbitopes"









$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta)$$

Thm For $f: S^1 \to \mathbb{R}^{2k+1}$, $\exists\ X \subset S^1$ of diameter at most $\frac{k}{2k+1}$ such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.

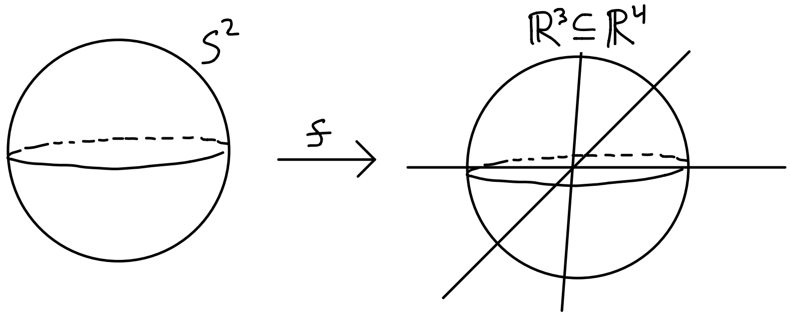
Proof

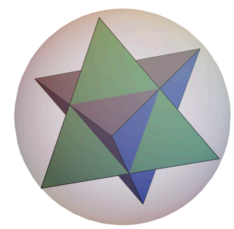
$$S^1 \xrightarrow{\quad f \quad} \mathbb{R}^{2k+1} \quad \text{induces}$$
$$VR(S^1; r) \xrightarrow{\quad f \quad} \mathbb{R}^{2k+1}$$

Sharpness of diameter bound

$$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$$
$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \dots)$$

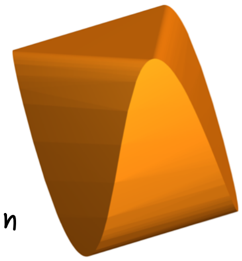
Thm For $f: S^n \to \mathbb{R}^{n+2}$, $\exists$ $X \subset S^n$ of diameter at most r_n such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.





Proof $S^n \xrightarrow{\ f\ } \mathbb{R}^{n+2}$ induces

$$S^n * \frac{SO(n+1)}{A_{n+2}} \simeq VR^m(S^n; r) \xrightarrow{\ f\ } \mathbb{R}^{n+2}$$



Questions

(1) $\mathrm{VR}^m(S^n;r)$ for larger r?

(2) $\check{\mathrm{C}}\mathrm{ech}^m(S^n;r)$?

(3) Other manifolds ? Tori, ellipsoids, $\mathbb{RP}^n$, $\mathbb{CP}^n$

(4) $\mathrm{VR}^m_<(X;r) \simeq \mathrm{VR}_<(X;r)$?

(5) Morse and Morse-Bott theories (Mirth PhD thesis)

(6) Measures with infinite support

(8) Tighter connections between $\mathrm{VR}^m(X;r)$ and $B_{L^\infty(X)}(X;r)$.

(7) In $\mathrm{VR}^m(X;r)$ replace ∞-diam with p-diam.
In $\check{\mathrm{C}}\mathrm{ech}^m(X;r)$ replace ∞-variance with p-variance.
(A, Mémoli, Moy, Wang)