Bridging applied and quantitative topology

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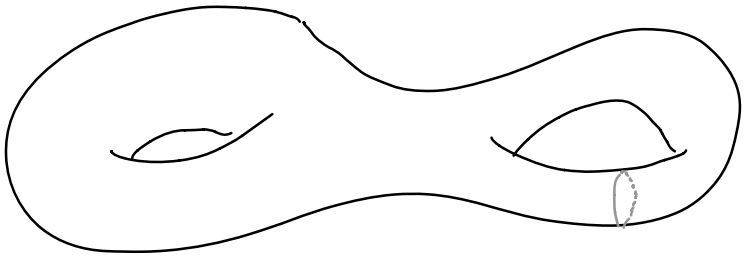
Gromov, 1983, "Filling Riemannian manifolds"

Isosystolic inequality For M an essential n-dimensional Riemannian manifold, $\text{systole}(M) \leq C \, \text{vol}(M)^{1/n}$.



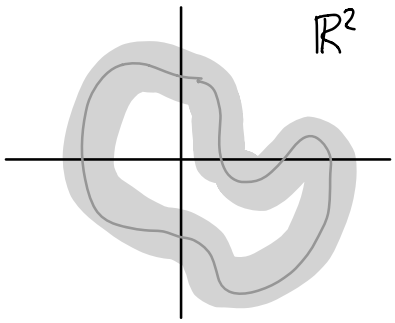
The systole of M is the length of the shortest non-contractible loop.

"Essential" rules out counterexamples like $S^1 \times S^2$.

Proof $\text{sys}(M) \leq 6 \cdot \text{FillingRadius}(M) \leq C \, \text{vol}(M)^{1/n}$

($\leq$: essential; $\leq$: all)

Filling radius



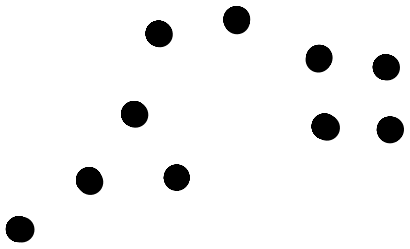
X a metric space
Kuratowski embedding $X \hookrightarrow L^\infty(X)$
$x \longmapsto d(x, \cdot)$

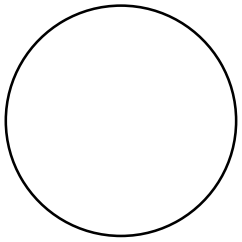
Def The **filling radius** of manifold M^n is the infimum r such that $M \hookrightarrow B_{L^\infty(M)}(M; r)$ induces a map on n-dimensional homology killing the (nonzero) fundamental class.

Lim, Mémoli, Okutan, 2020, "Vietoris-Rips persistent homology, injective metric spaces, and the filling radius"

Thm For X a metric space, $B_{L^\infty(X)}(X; \frac{r}{2}) \simeq VR(X; r)$.

Def For X a metric space and $r > 0$, the Vietoris-Rips simplicial complex has vertex set X and σ as a simplex if $\operatorname{diam}(\sigma) \leq r$.

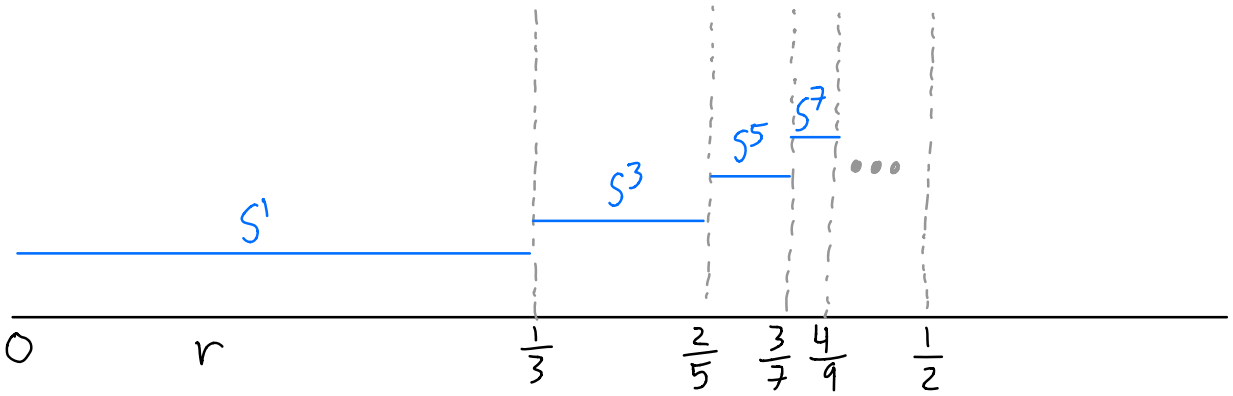


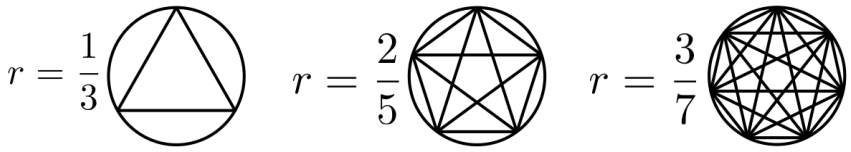


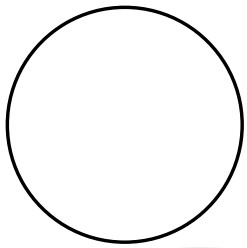
Adamaszek, A, 2017, "The Vietoris-Rips complexes of a circle"

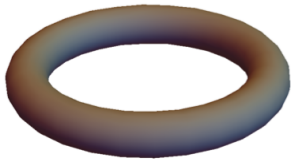
S^1 is circle with geodesic metric, unit circumference.

Thm $VR(S^1;r) \simeq S^{2k+1}$ for $\frac{k}{2k+1} < r < \frac{k+1}{2k+3}$, $k \in \mathbb{N}$





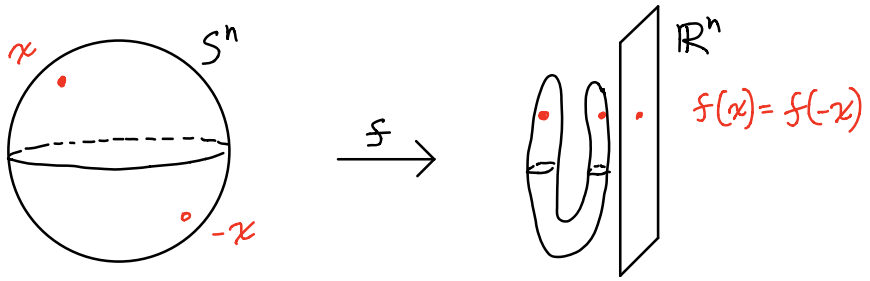




Katz, 1991, "On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional"

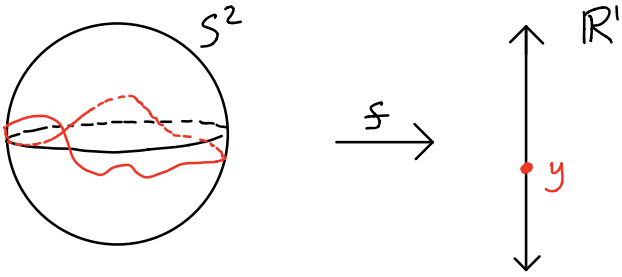
Thm $B_{L^\infty(X)}(S^1; \frac{r}{2}) \simeq \begin{cases} S^1 & 0 < r < \frac{1}{3} \\ S^3 & \frac{1}{3} < r < \frac{2}{5} \end{cases}$

Borsuk-Ulam Theorem



Given $f: S^n \to \mathbb{R}^n$, $\exists\ x \in S^n$ with $f(x) = f(-x)$.

- What about $S^n \to \mathbb{R}^k$ with $k \leq n$?



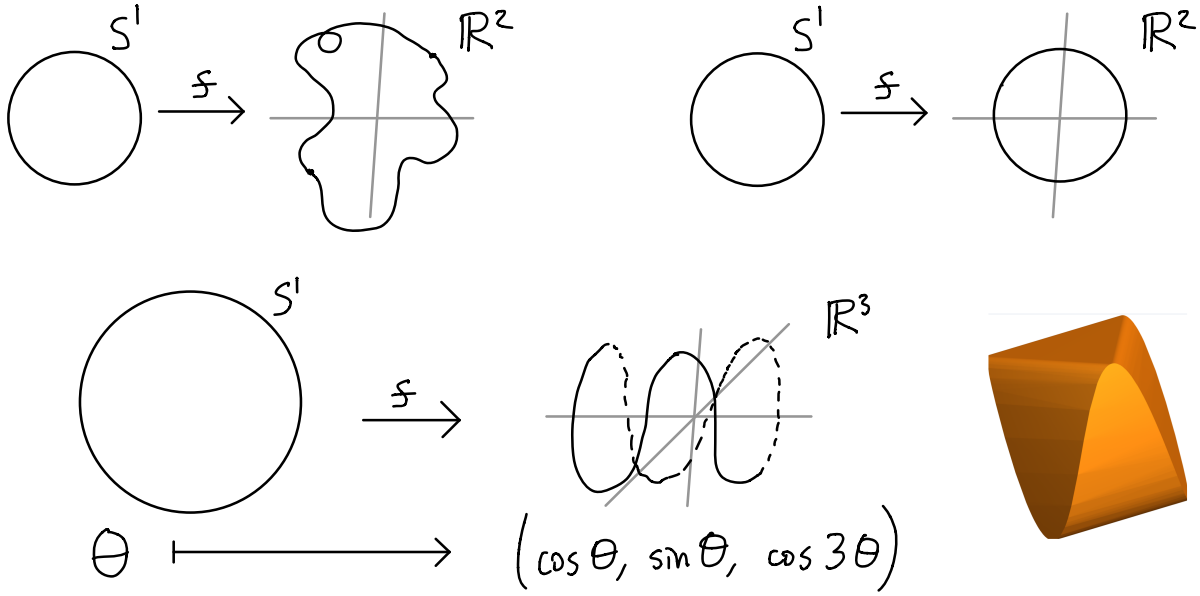
Gromov's Waist Inequality For $f: S^n \to \mathbb{R}^k$ with $k \leq n$, $\exists\ y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \geq \mathrm{Vol}_{n-k}(S^{n-k})$.

Remark Implies invariance of dimension: $\mathbb{R}^k \cong \mathbb{R}^{k'} \iff k = k'$.

Proof Almgren: 100 pages of geometric measure theory
Gromov: The filling radius, or characteristic classes.

- What about $f: S^n \to \mathbb{R}^k$ with $k \geq n$?

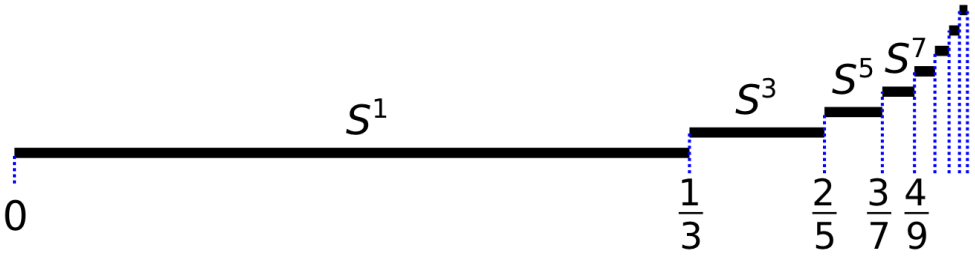
A, Bush, Frick, 2020, "Metric thickenings, Borsuk-Ulam theorems, and orbitopes



Thm For $f: S^1 \to \mathbb{R}^{2k+1}$, $\exists\ X \subset S^1$ of diameter at most $\frac{k}{2k+1}$ such that $\operatorname{conv}(f(X)) \cap \operatorname{conv}(f(-X)) \neq \emptyset$.

Proof $S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$ induces

$$S^{2k+1} \simeq VR\left(S^1; \tfrac{k}{2k+1}\right) \xrightarrow{f} \mathbb{R}^{2k+1}$$

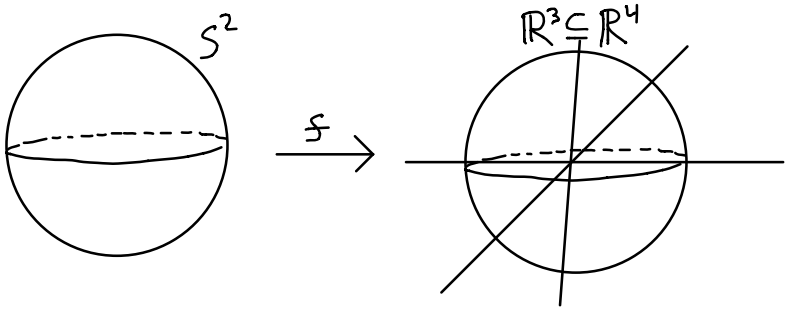


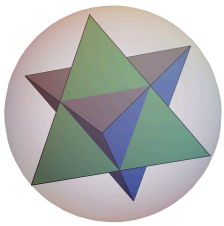
Sharpness of diameter bound

$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$

$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

Thm For $f: S^n \to \mathbb{R}^{n+2}$, $\exists\ X \subset S^n$ of diameter at most r_n such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.



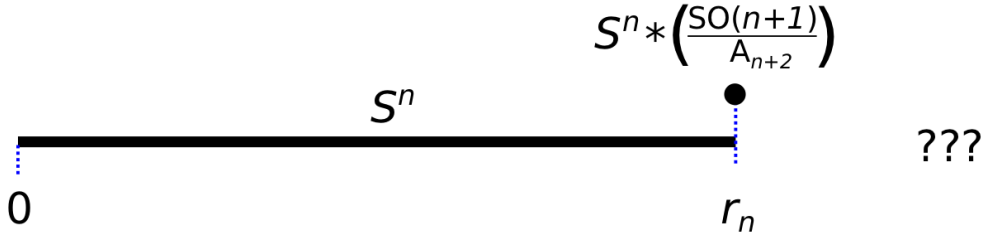


Proof ($n+1$)-connected

$$S^n \xrightarrow{s} \mathbb{R}^{n+2} \qquad \mathcal{VR}^m_{\leq}(S^n; r_n) \xrightarrow{s} \mathbb{R}^{n+2}$$

induces

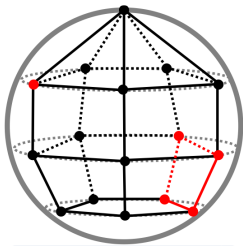
Adamaszek, A, Frick, 2018, "Metric reconstruction via optimal transport"



Katz, 1991, "On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional"

Thm $B_{L^\infty(X)}\left(S^2; \frac{r}{2}\right) \simeq$ first S^2, then $S^2 * \frac{SO(3)}{A_4}$

Questions



- Homotopy types of $VR(S^n; r)$ for larger r?
- Other connections between applied and quantitative topology?
- Are there connections between Optimal transport, flat norm, filling radius?
- Relations between thickenings
  $X \hookrightarrow B_{L^\infty(X)}(X; \frac{r}{2})$
  $X \hookrightarrow VR^m(X; r)$?
- Čech complex connections to quantitative topology?