Bridging applied and quantitative topology

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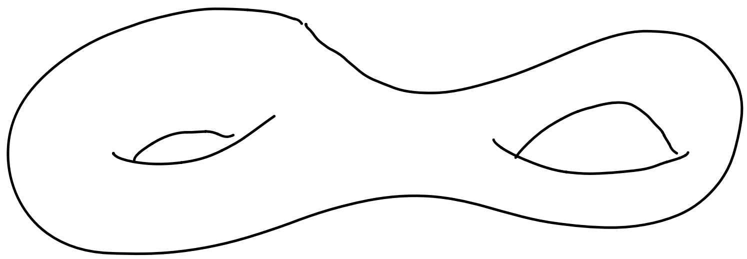
Gromov, 1983, "Filling Riemannian manifolds"

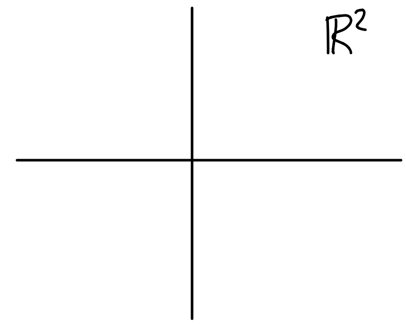
<u>Isosystolic inequality</u> For M an essential n-dimensional Riemannian manifold, $\text{systole}(M) \leq C \text{ vol}(M)^{1/n}$.



The systole of M is

"Essential" rules out counterexamples like $S^1 \times S^2$.

<u>Proof</u> $\text{sys}(M) \leq \qquad\qquad \leq C \text{ vol}(M)^{1/n}$



Filling radius

X a metric space

Kuratowski embedding $X \hookrightarrow L^\infty(X)$

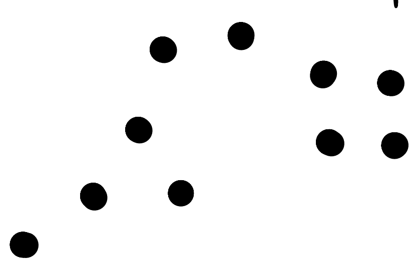
$x \longmapsto$

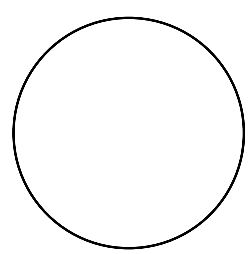
Def The filling radius of manifold M^n is the infimum r such that $M \hookrightarrow B_{L^\infty(M)}(M; r)$ induces a map on n-dimensional homology killing the (nonzero) fundamental class.

Lim, Mémoli, Okutan, 2020, "Vietoris-Rips persistent homology, injective metric spaces, and the filling radius"

Thm For X a metric space, $B_{L^\infty(X)}(X; \frac{r}{2}) \simeq$

Def For X a metric space and $r > 0$, the Vietoris-Rips simplicial complex has vertex set X and σ as a simplex if $\mathrm{diam}(\sigma) \le r$.

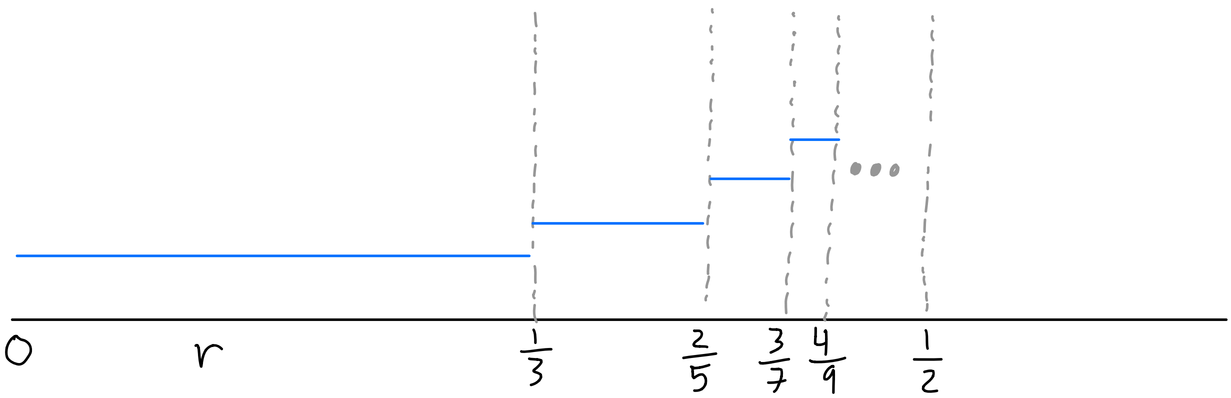


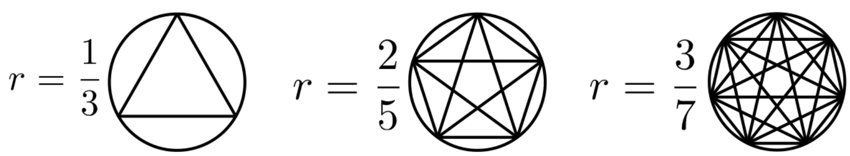


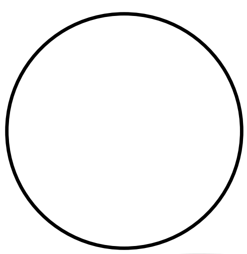
Adamaszek, A, 2017, "The Vietoris-Rips complexes of a circle"

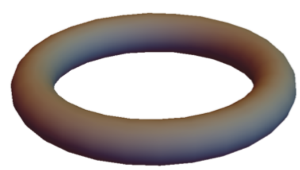
S^1 is circle with geodesic metric, unit circumference.

<u>Thm</u> $VR(S^1;r) \simeq$





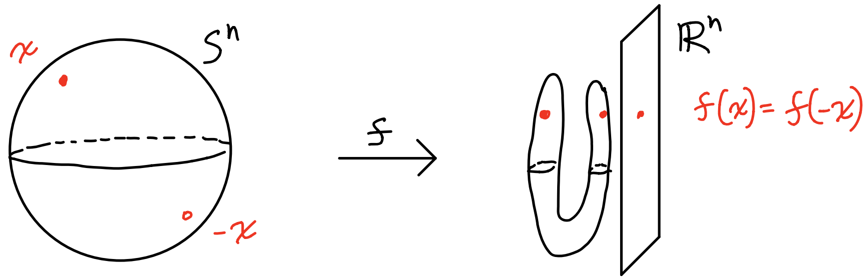




Katz, 1991, "On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional"

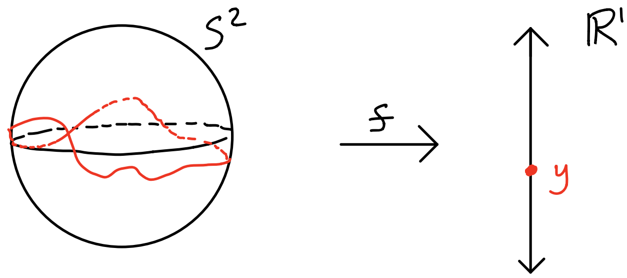
<u>Thm</u> $B_{L^\infty(X)}(S^1;\frac{r}{2}) \simeq \begin{cases} S^1 & 0<r<\frac{1}{3} \\ S^3 & \frac{1}{3}<r<\frac{2}{5} \end{cases}$

Borsuk-Ulam Theorem



Given $f: S^n \to \mathbb{R}^n$, $\exists\ x \in S^n$ with $f(x) = f(-x)$.

- What about $S^n \to \mathbb{R}^k$ with $k \leq n$?



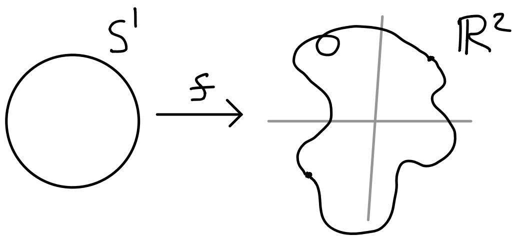
Gromov's Waist Inequality For $f: S^n \to \mathbb{R}^k$ with $k \leq n$, $\exists\ y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \geq$.

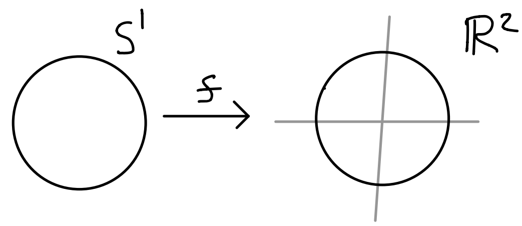
Remark Implies invariance of dimension: $\mathbb{R}^k \cong \mathbb{R}^{k'} \Leftrightarrow k = k'$.

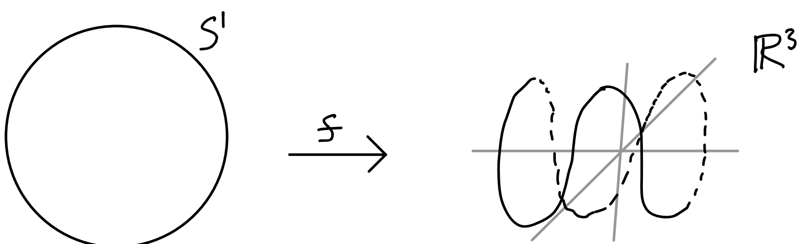
Proof Almgren: 100 pages of geometric measure theory
Gromov: The filling radius, or characteristic classes.

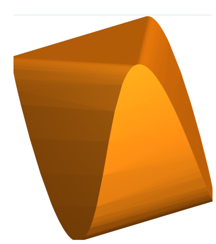
- What about $f : S^n \to \mathbb{R}^k$ with $k \geq n$?

A, Bush, Frick, 2020, "Metric thickenings, Borsuk-Ulam theorems, and orbitopes







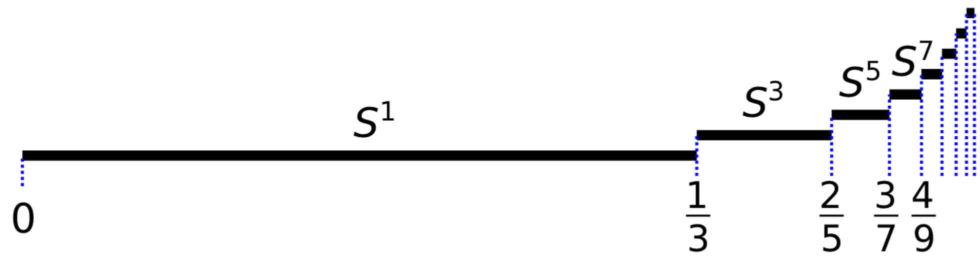


$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta)$$

<u>Thm</u> For $f : S^1 \to \mathbb{R}^{2k+1}$, $\exists\ X \subset S^1$ of diameter at most $\frac{k}{2k+1}$ such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.

<u>Proof</u> $S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$ induces

$$S^{2k+1} \simeq VR\left(S^1; \tfrac{k}{2k+1}\right) \xrightarrow{f} \mathbb{R}^{2k+1}$$

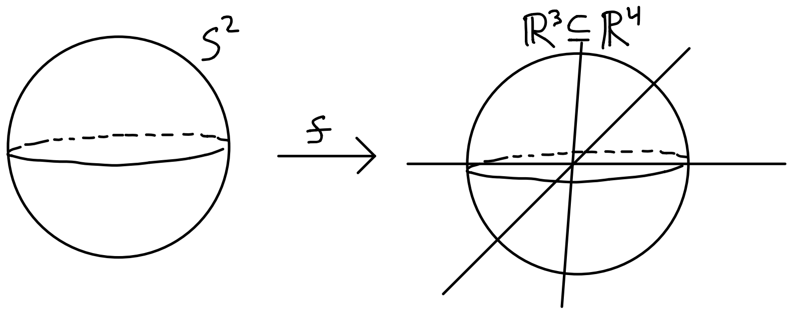


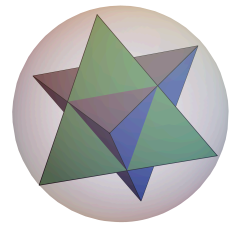
Sharpness of diameter bound

$$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$$

$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$$

Thm For $f: S^n \to \mathbb{R}^{n+2}$, $\exists\ X \subset S^n$ of diameter at most r_n such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.

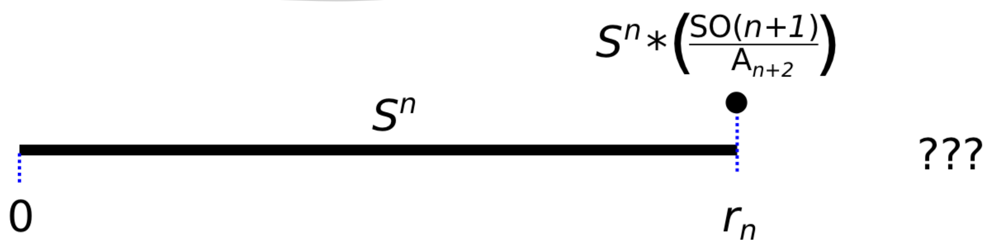




Proof $S^n \xrightarrow{f} \mathbb{R}^{n+2}$ induces

$VR(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$

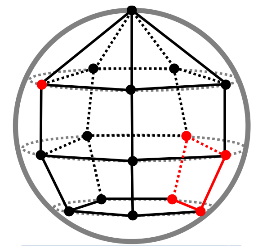
Adamaszek, A, Frick, 2018, "Metric reconstruction via optimal transport"



Katz, 1991, "On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional"

Thm $B_{L^\infty(X)}(S^2; \frac{r}{2}) \simeq$ first S^2, then $S^2 * \frac{SO(3)}{A_4}$

Questions



- Homotopy types of $VR(S^n; r)$ for larger r?
- Other connections between applied and quantitative topology?
- Are there connections between Optimal transport, flat norm, filling radius?
- Relations between thickenings
  $$X \hookrightarrow B_{L^\infty(X)}(X; \tfrac{r}{2})$$
  $$X \hookrightarrow VR^m(X; r) \qquad ?$$
- Čech complex connections to quantitative topology?