Bridging applied and geometric topology

Henry Adams, Colorado State University
Joint with Baris Coskunuzer

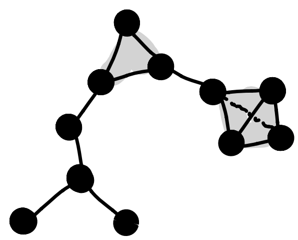
arXiv:2103.06408 "Geometric approaches on persistent homology"

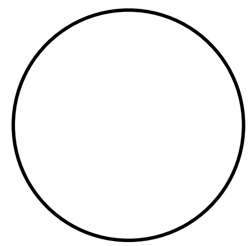
AATRN: 1 or 2 live talks per week
YouTube: 2,400 subscribers, 300+ videos,
20 hours watched per day

X metric space, $r \geq 0$.

<u>Def</u> The <u>Vietoris–Rips simplicial complex</u> has

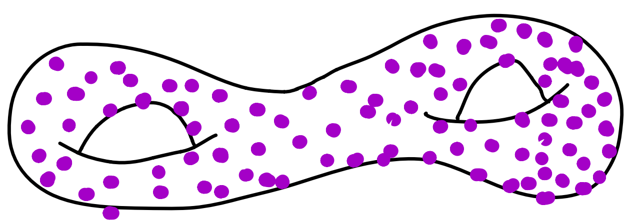
- vertex set X
- finite simplex $\sigma \subseteq X$ when $\mathrm{diameter}(\sigma) \leq r$.



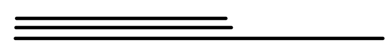


<u>History</u>

- Cohomology theory for metric spaces
- Geometric group theory
- Applied topology



$PH_1(VR(M;r))$



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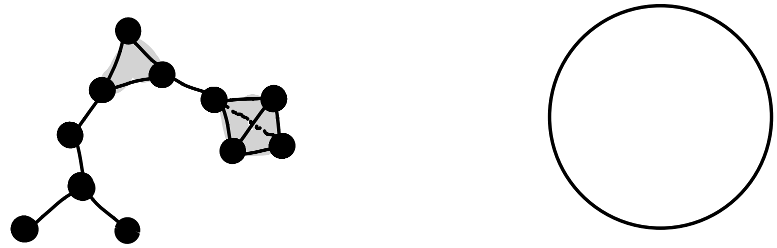
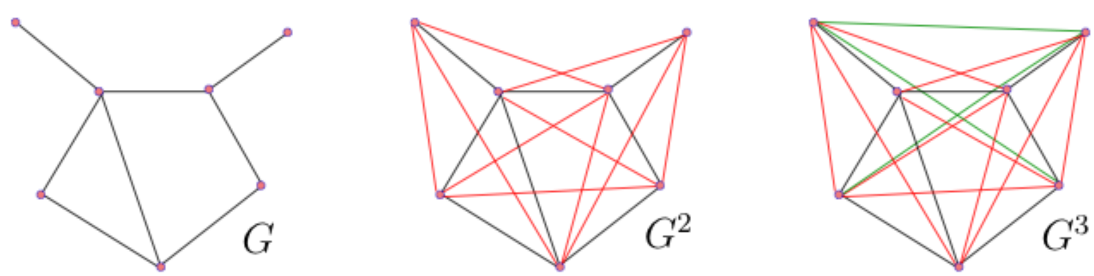


For this paper: X is vertex set of finite simple graph with shortest-path metric.



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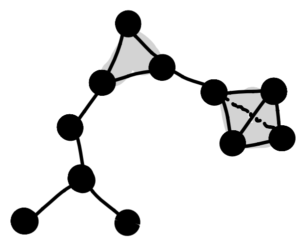

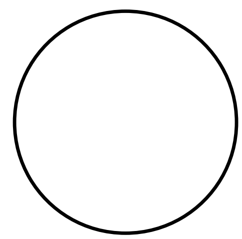

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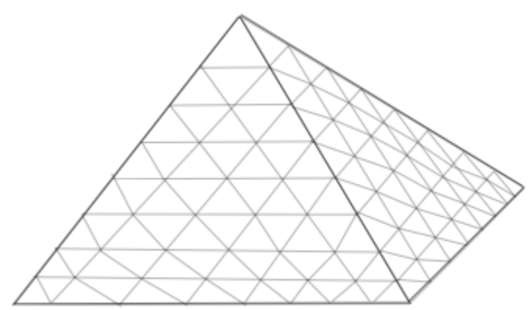
(A) Thin long box



(B) Tetrahedron

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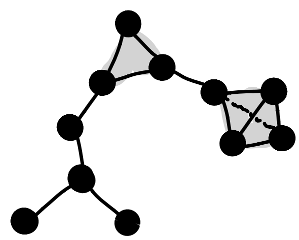

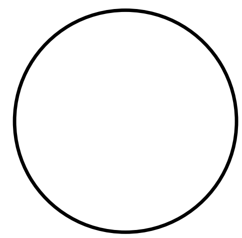

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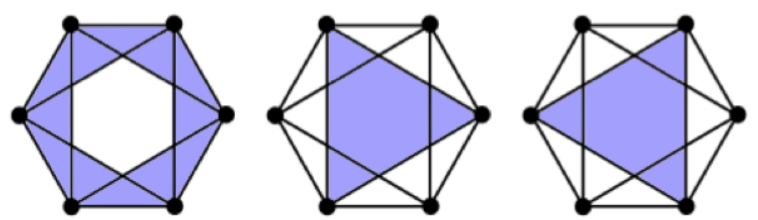

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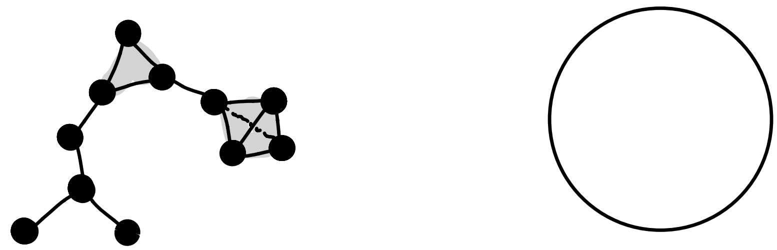

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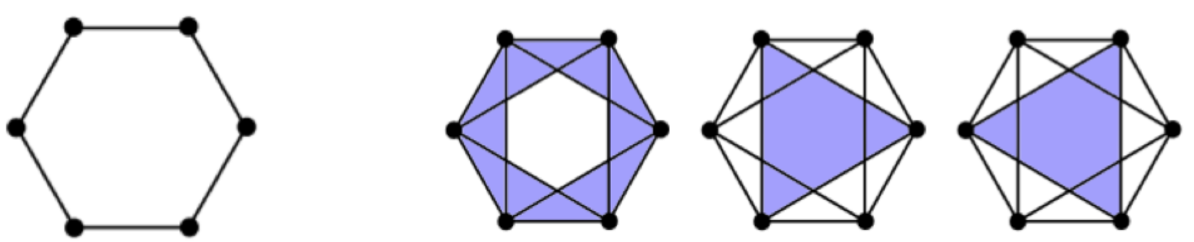
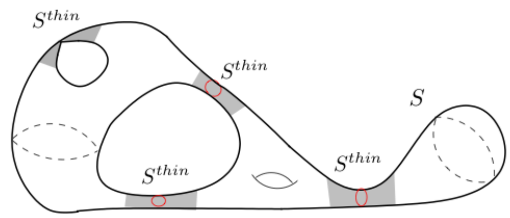


Any simplicial complex is, up to homeomorphism, the Vietoris-Rips complex of some graph.

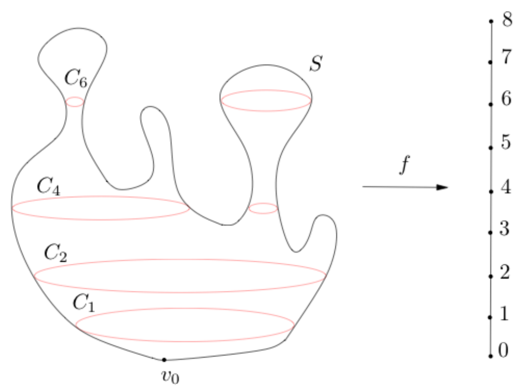
Theorem 1 The persistence of a 2-dimensional homology class σ is at most $\lceil \sqrt{\text{area}(\sigma)} \rceil + 1$.

Proof: Thick-thin decompositions



Theorem 2 The persistence of a k-dimensional homology class σ is at most width$(\sigma) + 1$.

Proof: Sweepouts (min-max)



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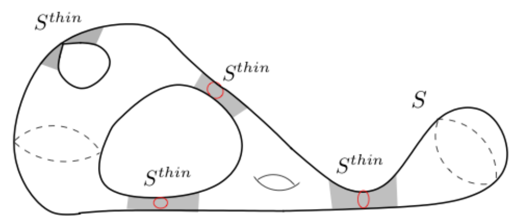

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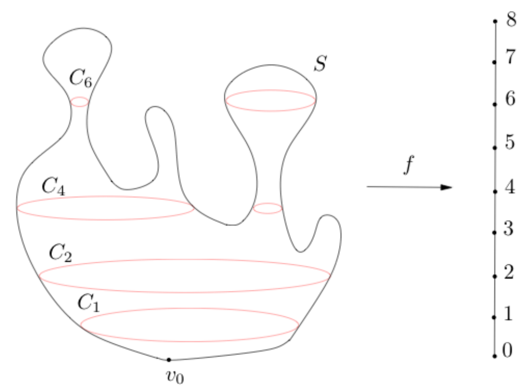

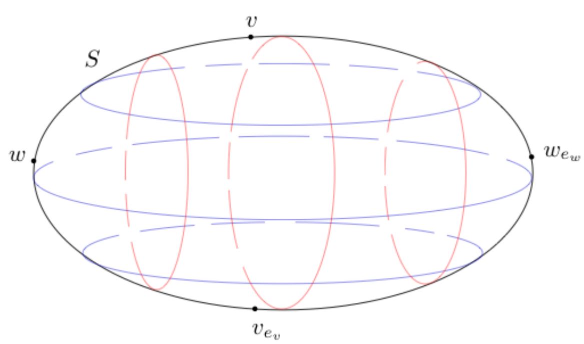

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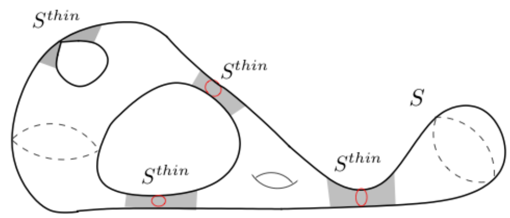

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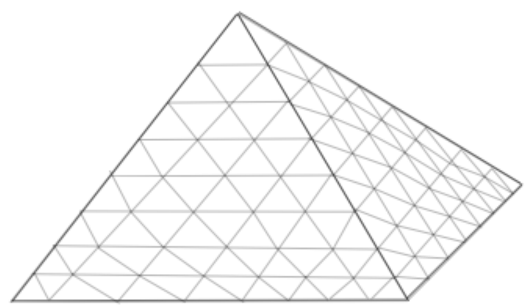

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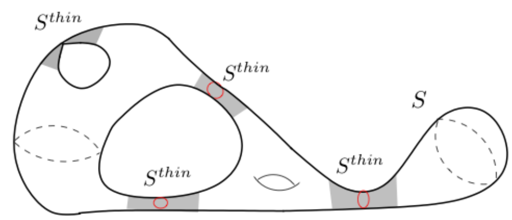

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Question Theorem 1 for k-dimensional homology?
Question Lower bounds?
Question More general metric spaces?

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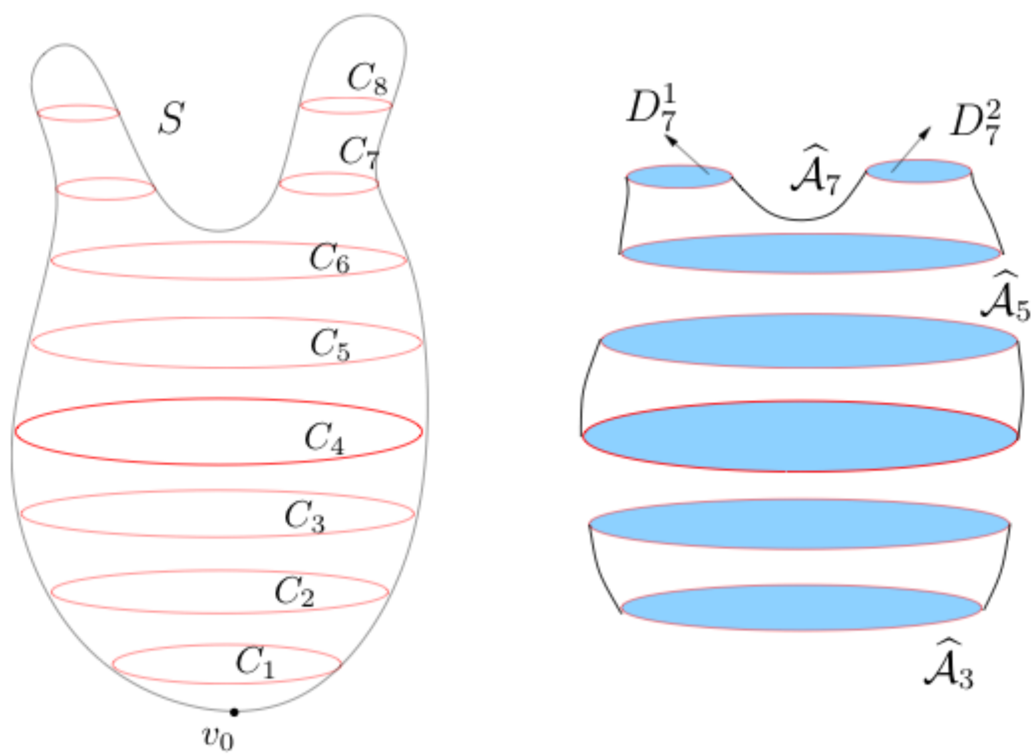

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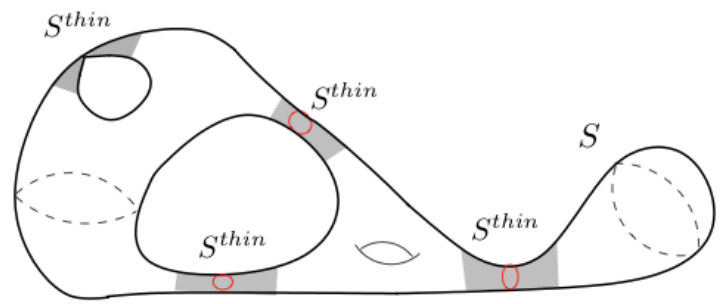
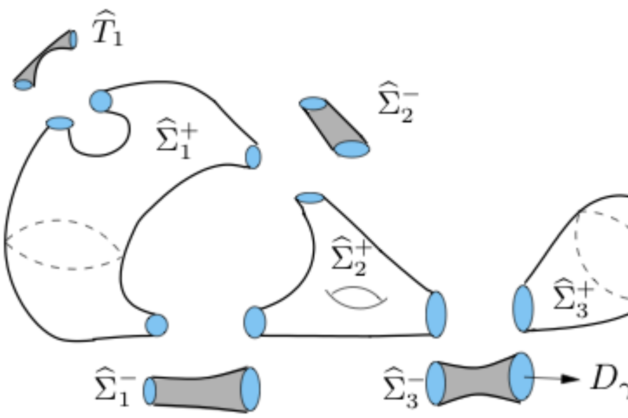
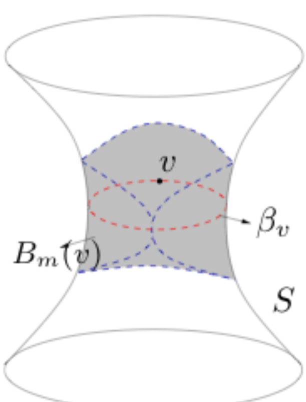


$$m = \left\lceil \frac{\sqrt{\text{area}(\sigma)}}{2} \right\rceil$$

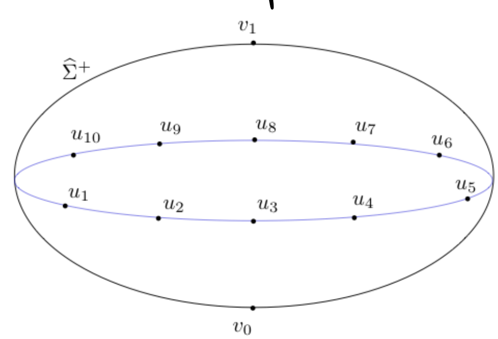


Thin parts



"Bracelets"

Thick parts



Isoperimetric inequality

$$|B_m(v_0)| \geq 2m^2$$

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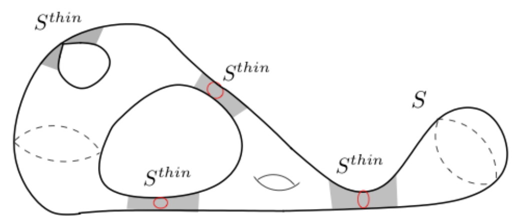

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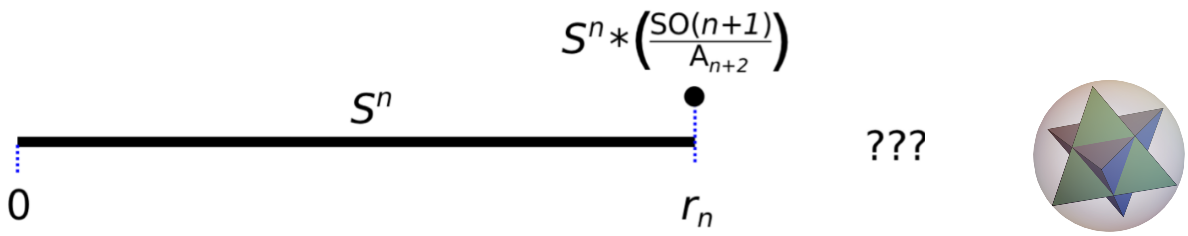
Vietoris-Rips complexes and other geometric topics

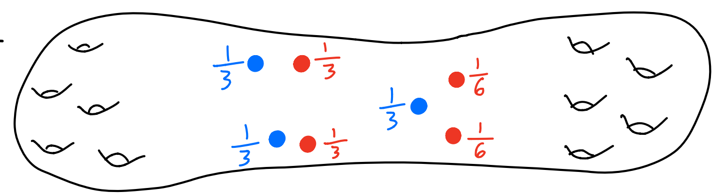
- Vietoris-Rips of spheres



- Morse and Morse-Bott theories
- Gromov's filling radius

Lim, Mémoli, Okutan, 2020

- Optimal transport



- Borsuk-Ulam and waist of sphere theorems
- Geometric group theory: Is $VR((\mathbb{Z}^n, L^1); r)$ contractible for $r \geq n$?
- Čech complexes

<u>Vietoris-Rips complexes and other geometric topics</u>

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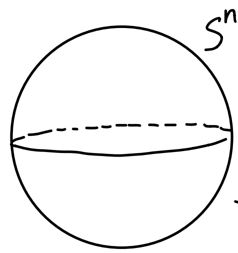

$\mathbb{R}^k$, $k \leq n$

Almgren (1965):
Geometric measure theory
Gromov (2003):
Filling radius, characteristic classes

$\mathbb{R}^n$

Borsuk-Ulam (1933)

$\mathbb{R}^k$, $k \geq n$

A, Bush,
Frick (2020)