Borsuk-Ulam Theorems into Higher-Dimensional Codomains

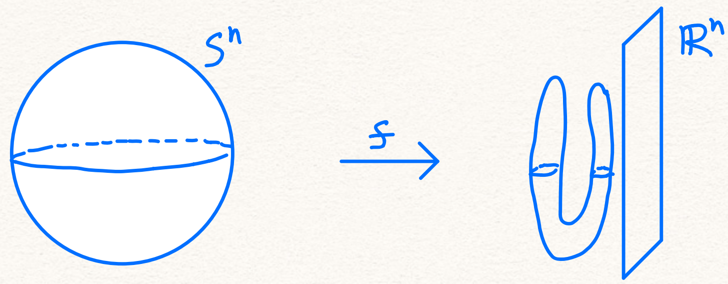
Joint with Johnathan Bush and Florian Frick

Mathematika 2020

History: Stanislaw Ulam, CU Boulder 1961-1962, 1965-1975

Erdös-Ulam problem, Collatz conjecture, cellular automaton, Monte Carlo, nuclear pulse propulsion, Teller-Ulam design.

(I) Borsuk-Ulam Theorem



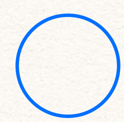
(i) Given $f: S^n \to \mathbb{R}^n$, $\exists\ x \in S^n$ with

(ii) Given $f: S^n \to \mathbb{R}^n$ odd ( ),

(i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (i)

Proof $n=1$: IVT.

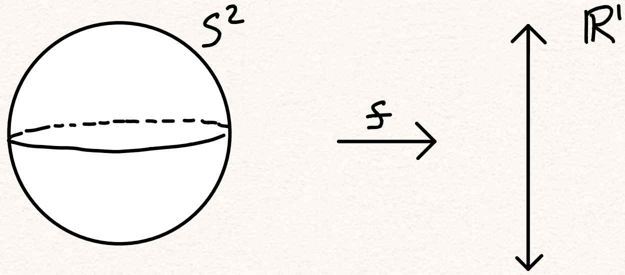


$n \geq 1$: $\quad S^n \longrightarrow S^{n-1}$

$$\mathbb{RP}^n \longrightarrow \mathbb{RP}^{n-1}$$

$$\mathbb{F}_2[a]/a^{n+1} \cong H^*(\mathbb{RP}^n) \longleftarrow H^*(\mathbb{RP}^{n-1}) \cong \mathbb{F}_2[b]/b^n$$

(II) What about $S^n \to \mathbb{R}^k$ with $k \le n$?

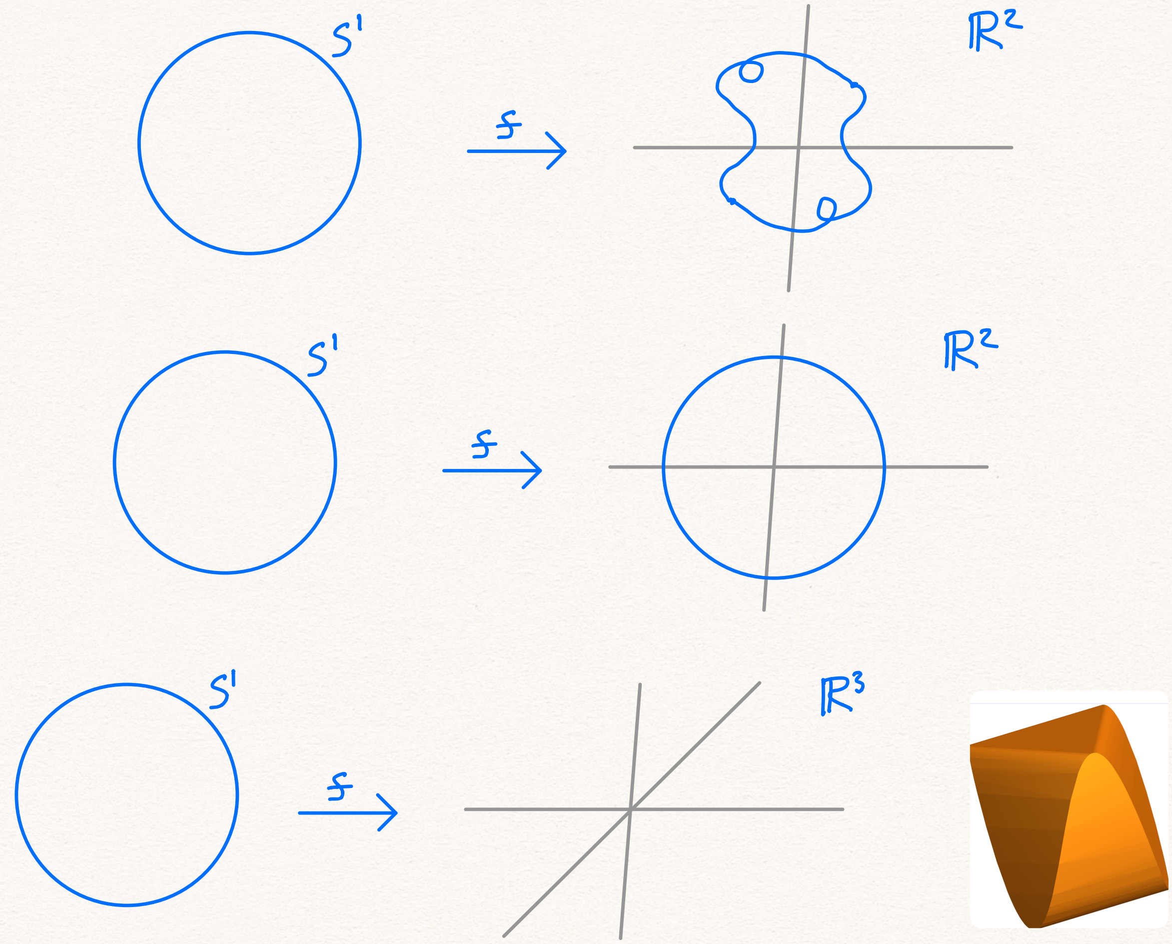


<u>Gromov's Waist Inequality</u> For $f: S^n \to \mathbb{R}^k$ with $k \le n$, $\exists y \in \mathbb{R}^k$ with

<u>Proof</u> $k = 1$: IVT plus isoperimetric inequality.
$k \ge 1$:

<u>Remark</u>

(III) What about $f: S^n \to \mathbb{R}^k$ odd with $k \geq n$?



$\theta \longmapsto$

S^1 with path-length metric, unit circumference.

<u>Theorem</u> (A, Bush, Frick) For $f: S^1 \to \mathbb{R}^{2k+1}$ odd,

$\exists$

such that

Sharpness of diameter bound

$$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$$

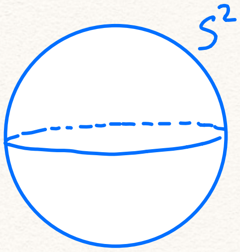
Proof $\quad S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$ induces

$$S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$$

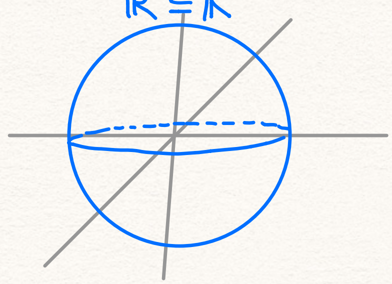
Theorem (A, Bush, Frick) For $f: S^n \to \mathbb{R}^{n+2}$ odd,

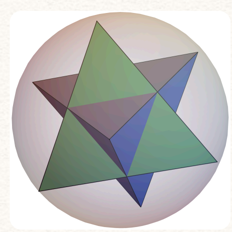
$\exists$

such that



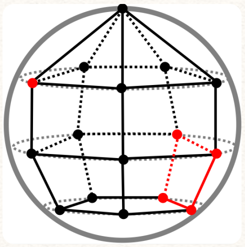
$\xrightarrow{f}$





Proof $\quad S^n \xrightarrow{f} \mathbb{R}^{n+2}$ induces

$$S^n \xrightarrow{f} \mathbb{R}^{n+2}$$

Remark Lovász' strongly self-dual polytopes. 

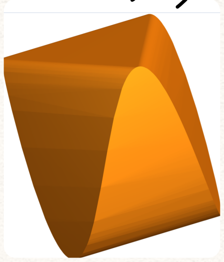
Remark Michael Crabb uses characteristic classes to get extensions

Remark Versions of the ham sandwich theorem with more "fixings" than the dimension!

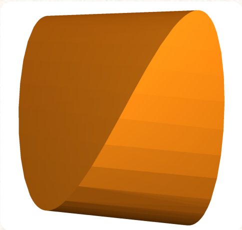
(IV) Orbitopes and Schur polynomials

<u>Def$^{\underline{n}}$</u> The <u>Barvinok-Novik orbitope</u> $B_{2k} \subseteq \mathbb{R}^{2k}$ is the convex hull of the curve

$$(\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots, \cos(2k-1)\theta, \sin(2k-1)\theta).$$



<u>Def$^{\underline{n}}$</u> The <u>Carathéodory orbitope</u> $C_{2k} \subseteq \mathbb{R}^{2k}$ is the convex hull of the curve



	Odd Barvinok-Novik	Not odd Carathéodory
What diameter (scale) hits origin?	$\begin{matrix} \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 \\ \cos 3t_1 & \cos 3t_2 & \cos 3t_3 & \cos 3t_4 \\ \sin 3t_1 & \sin 3t_2 & \sin 3t_3 & \sin 3t_4 \end{matrix}$	$\begin{matrix} \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 \\ \cos 2t_1 & \cos 2t_2 & \cos 2t_3 & \cos 2t_4 \\ \sin 2t_1 & \sin 2t_2 & \sin 2t_3 & \sin 2t_4 \end{matrix}$
What are the orbitope's faces?	$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 & \cos t_5 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 & \sin t_5 \\ \cos 3t_1 & \cos 3t_2 & \cos 3t_3 & \cos 3t_4 & \cos 3t_5 \\ \sin 3t_1 & \sin 3t_2 & \sin 3t_3 & \sin 3t_4 & \sin 3t_5 \end{matrix}$	$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 & \cos t_5 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 & \sin t_5 \\ \cos 2t_1 & \cos 2t_2 & \cos 2t_3 & \cos 2t_4 & \cos 2t_5 \\ \sin 2t_1 & \sin 2t_2 & \sin 2t_3 & \sin 2t_4 & \sin 2t_5 \end{matrix}$

Changing to exponentials gives

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- A generalized Vandermonde matrix, whose determinant contains a Schur polynomial in its factorization, Top left: $\det = C_1 \prod_{i<j} \sin(t_j - t_i)$

Bottom right: $\det = C_1 \prod_{i<j} \sin\left(\frac{t_j - t_i}{2}\right)$

Bottom left: $\det = C_1 \left(\underbrace{\prod_{1\le i<j\le 5} \sin\left(\frac{t_j - t_i}{2}\right)}_{\text{Vandermonde}}\right)\left(\underbrace{2 + \sum_{1\le i<j\le 5} \cos(t_j - t_i)}_{\text{Schur}}\right)$