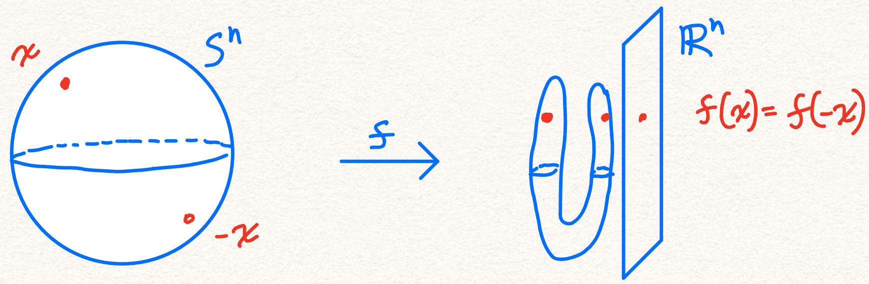
Borsuk-Ulam Theorems into Higher-Dimensional Codomains

Joint with Johnathan Bush and Florian Frick

Mathematika 2020

History: Stanislaw Ulam, CU Boulder 1961-1962, 1965-1975.
Erdös-Ulam problem, Collatz conjecture, cellular automaton,
Monte Carlo, nuclear pulse propulsion, Teller-Ulam design.

(I) Borsuk-Ulam Theorem



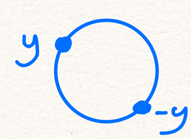
(i) Given $f: S^n \to \mathbb{R}^n$, $\exists\, x \in S^n$ with $f(x) = f(-x)$.

(ii) Given $f: S^n \to \mathbb{R}^n$ odd $\left(f(-y) = -f(y)\ \forall y \in S^n \right)$, $\exists\, x \in S^n$ with $f(x) = \vec{0}$.

(i) $\Rightarrow$ (ii) $\quad f(x) = f(-x) = -f(x)$ so $f(x) = \vec{0}$.

(ii) $\Rightarrow$ (i) $\quad$ Apply (ii) to $f(x) - f(-x)$.

Proof $n = 1$: IVT.



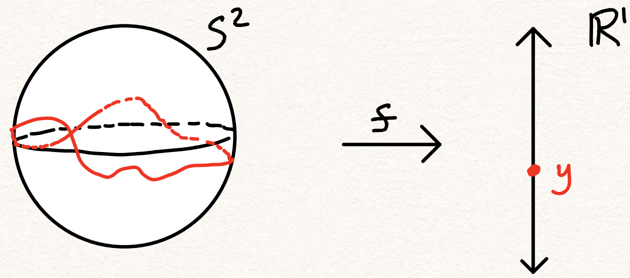
$n \geq 1$:

$$\begin{array}{ccc} S^n & \longrightarrow & S^{n-1} \\ x & \longmapsto & \dfrac{f(x)}{\| f(x) \|} \end{array}$$

$$\mathbb{RP}^n \longrightarrow \mathbb{RP}^{n-1}$$

$$\begin{array}{ccc} \mathbb{F}_2[a]/a^{n+1} \cong H^*(\mathbb{RP}^n) & \longleftarrow & H^*(\mathbb{RP}^{n-1}) \cong \mathbb{F}_2[b]/b^n \\ 0 \neq a^n & \longleftarrow\!\shortmid & b^n = 0 \end{array}$$

(II) What about $S^n \to \mathbb{R}^k$ with $k \leq n$?

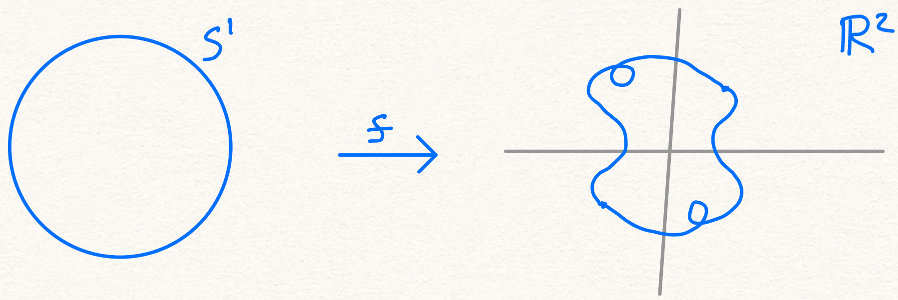


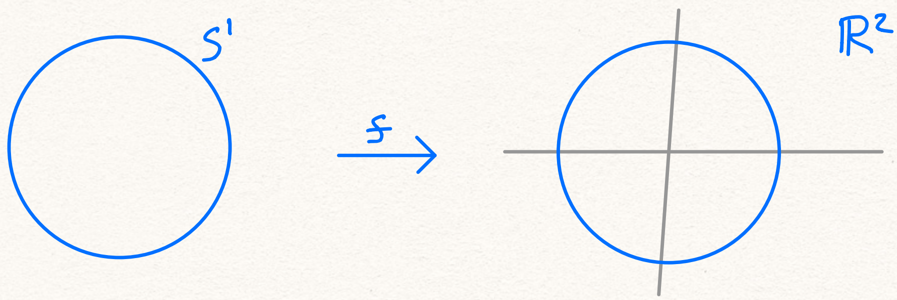
Gromov's Waist Inequality For $f: S^n \to \mathbb{R}^k$ with $k \leq n$, $\exists y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \geq \mathrm{Vol}_{n-k}(S^{n-k})$.

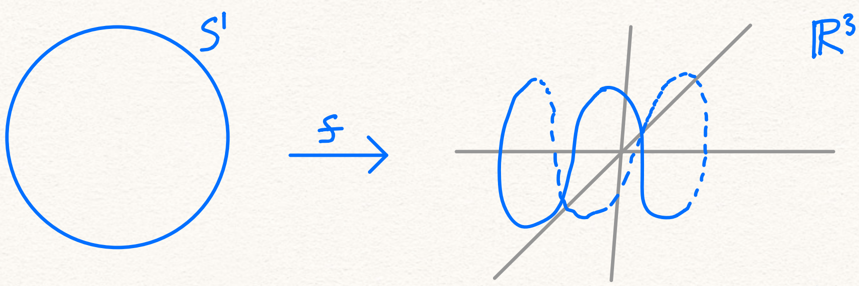
Proof $k=1$: IVT plus isoperimetric inequality.
$k \geq 1$: 100 pages of geometric measure theory or characteristic classes.

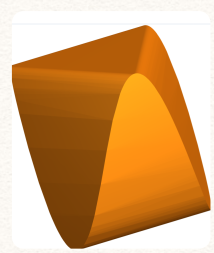
Remark Implies invariance of dimension: $\mathbb{R}^k \cong \mathbb{R}^{k'} \Leftrightarrow k = k'$.

(III) What about $f: S^n \to \mathbb{R}^k$ odd with $k \geq n$?









$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta)$$

S^1 with path-length metric, unit circumference.

<u>Theorem</u> (A, Bush, Frick) For $f: S^1 \to \mathbb{R}^{2k+1}$ odd, $\exists$ $X \subset S^1$ of diameter at most $\frac{k}{2k+1}$ such that $\vec{0} \in \mathrm{conv}(f(X))$.

Sharpness of diameter bound

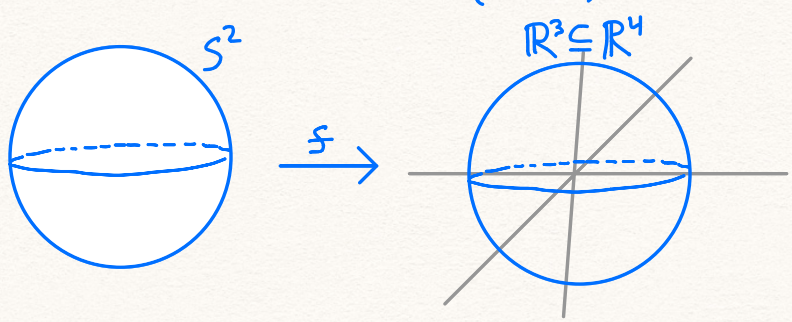
$$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$$
$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \dots)$$

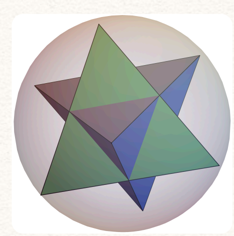
Proof $\quad S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$ induces

$$S^{2k+1} \simeq \underbrace{VR(S^1; \tfrac{k}{2k+1})} \xrightarrow{f} \mathbb{R}^{2k+1}$$

Vietoris-Rips simplicial complex with vertex set S^1 containing all simplices of diameter at most $\frac{k}{2k+1}$.

Theorem (A, Bush, Frick) For $f: S^n \to \mathbb{R}^{n+2}$ odd, $\exists\ X \subset S^n$ of diameter at most r_n such that $\vec{0} \in \mathrm{conv}(f(X))$.

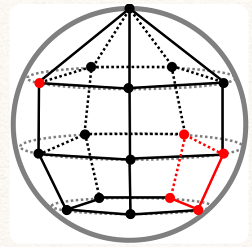




Proof $\quad S^n \xrightarrow{f} \mathbb{R}^{n+2}$ induces

$$(n+1)\text{-connected } VR(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$$

Remark Lovász' strongly self-dual polytopes.



Remark Michael Crabb uses characteristic classes to get extensions

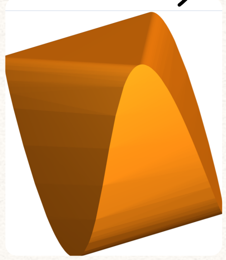
$$S^{2^r+t} \longrightarrow \mathbb{R}^{2^{r+1}+t}$$

Remark Versions of the ham sandwich theorem with more "fixings" than the dimension!

(IV) Orbitopes and Schur polynomials

<u>Def$^{\underline{n}}$</u> The <u>Barvinok-Novik orbitope</u> $B_{2k} \subseteq \mathbb{R}^{2k}$ is the convex hull of the curve
$$(\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots, \cos(2k-1)\theta, \sin(2k-1)\theta).$$

Its faces are known only for $k=1,2$.



<u>Def$^{\underline{n}}$</u> The <u>Carathéodory orbitope</u> $C_{2k} \subseteq \mathbb{R}^{2k}$ is the convex hull of the curve
$$(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos k\theta, \sin k\theta).$$

Its faces are known for all k.

	Odd Barvinok-Novik	Not odd Carathéodory
(scale) What diameter hits origin?	$\begin{matrix} \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 \\ \cos 3t_1 & \cos 3t_2 & \cos 3t_3 & \cos 3t_4 \\ \sin 3t_1 & \sin 3t_2 & \sin 3t_3 & \sin 3t_4 \end{matrix}$	$\begin{matrix} \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 \\ \cos 2t_1 & \cos 2t_2 & \cos 2t_3 & \cos 2t_4 \\ \sin 2t_1 & \sin 2t_2 & \sin 2t_3 & \sin 2t_4 \end{matrix}$
What are the orbitope's faces?	$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 & \cos t_5 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 & \sin t_5 \\ \cos 3t_1 & \cos 3t_2 & \cos 3t_3 & \cos 3t_4 & \cos 3t_5 \\ \sin 3t_1 & \sin 3t_2 & \sin 3t_3 & \sin 3t_4 & \sin 3t_5 \end{matrix}$	$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 & \cos t_5 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 & \sin t_5 \\ \cos 2t_1 & \cos 2t_2 & \cos 2t_3 & \cos 2t_4 & \cos 2t_5 \\ \sin 2t_1 & \sin 2t_2 & \sin 2t_3 & \sin 2t_4 & \sin 2t_5 \end{matrix}$

Changing to exponentials gives

- A Vandermonde matrix, whose determinant is easy to factor, answering the question.
- A generalized Vandermonde matrix, whose determinant contains a Schur polynomial in its factorization, whose sign is hard to analyze.

Top left: $\det = C_1 \prod_{i<j} \sin(t_j - t_i)$

Bottom right: $\det = C_1 \prod_{i<j} \sin\left(\frac{t_j - t_i}{2}\right)$

Bottom left: $\det = C_1 \underbrace{\left(\prod_{1\le i<j\le 5} \sin\left(\frac{t_j - t_i}{2}\right)\right)}_{\text{Vandermonde}} \underbrace{\left(2 + \sum_{1\le i<j\le 5} \cos(t_j - t_i)\right)}_{\text{Schur}}$