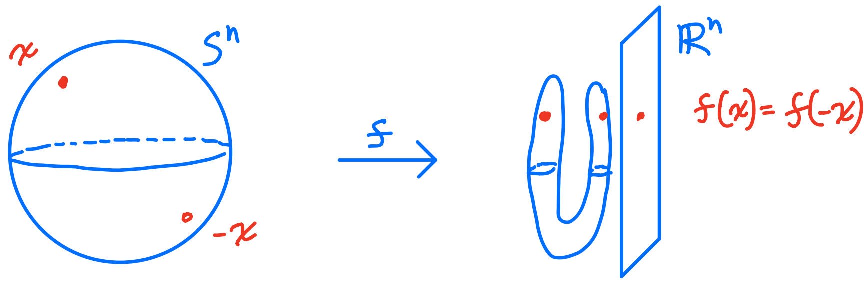
Borsuk-Ulam Theorems and Vietoris-Rips complexes

Joint with Johnathan Bush and Florian Frick
Mathematika 2020

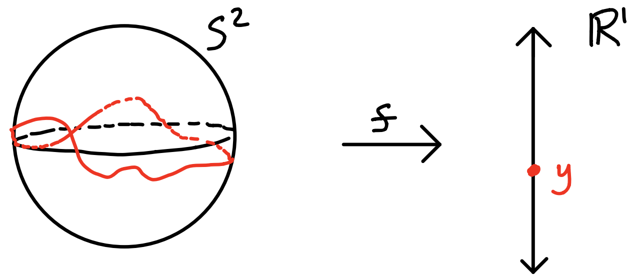
Emphasis: Connections to quantitative topology

(I) Borsuk-Ulam Theorem



Given $f: S^n \to \mathbb{R}^n$, $\exists\ x \in S^n$ with $f(x) = f(-x)$.

(II) What about $S^n \to \mathbb{R}^k$ with $k \leq n$?



<u>Gromov's Waist Inequality</u> For $f: S^n \to \mathbb{R}^k$ with $k \leq n$, $\exists y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \geq \mathrm{Vol}_{n-k}(S^{n-k})$.

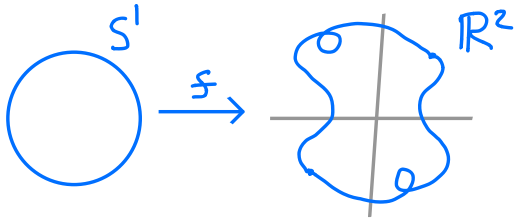
<u>Proof</u> Almgren: 100 pages of geometric measure theory

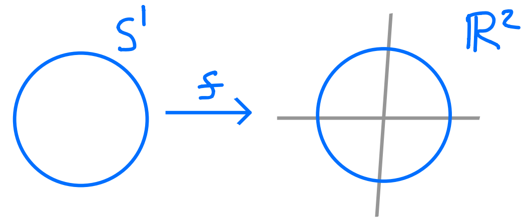
Gromov: The filling radius, or characteristic classes.

Lim, Mémoli, Okutan connect to Vietoris-Rips complexes

<u>Remark</u> Implies invariance of dimension: $\mathbb{R}^k \cong \mathbb{R}^{k'} \Leftrightarrow k = k'$.

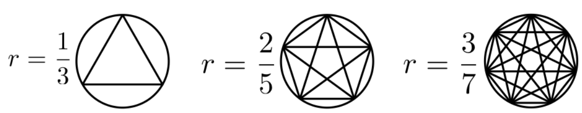
(III) What about $f: S^n \to \mathbb{R}^k$ with $k \geq n$?

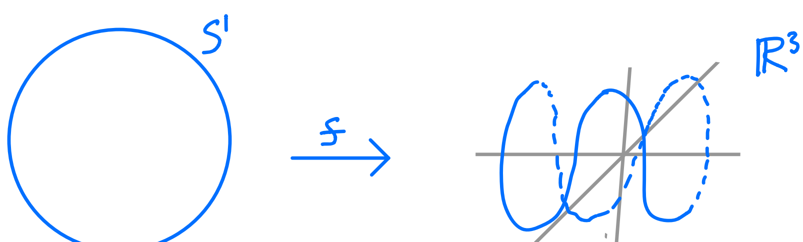


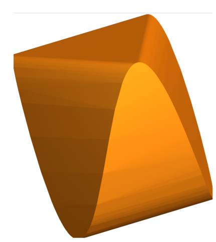


S^1 with path-length metric, unit circumference.

__Theorem__ (A, Bush, Frick) For $f: S^1 \to \mathbb{R}^{2k+1}$, $\exists\ X \subset S^1$ of diameter at most $\frac{k}{2k+1}$ such that $\operatorname{conv}(f(X)) \cap \operatorname{conv}(f(-X)) \neq \emptyset$.







$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta)$

__Sharpness of diameter bound__

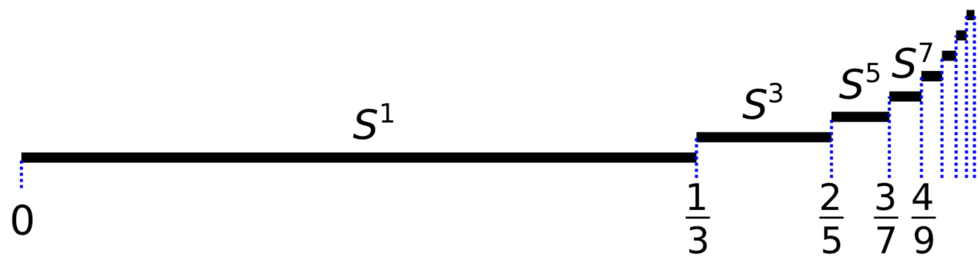
$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$

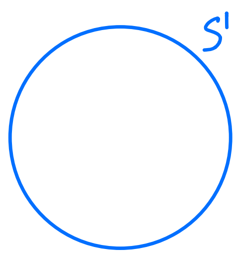
$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

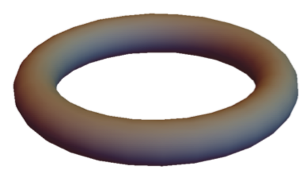
__Proof__ $S^1 \xrightarrow{f} \mathbb{R}^{2k+1}$ induces

$$S^{2k+1} \simeq \underbrace{VR\left(S^1; \tfrac{k}{2k+1}\right)} \xrightarrow{f} \mathbb{R}^{2k+1}$$

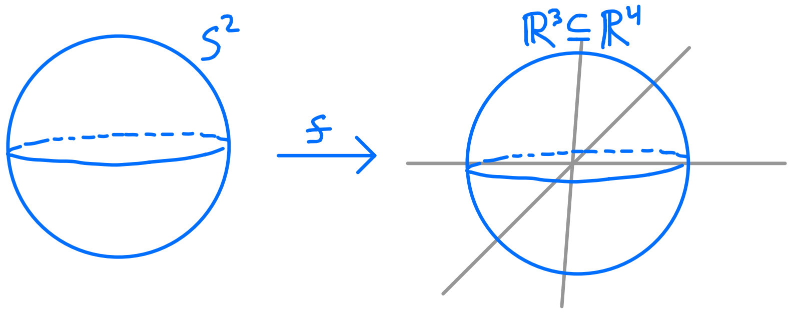
Vietoris-Rips simplicial complex with vertex set S^1 containing all simplices of diameter at most $\frac{k}{2k+1}$.

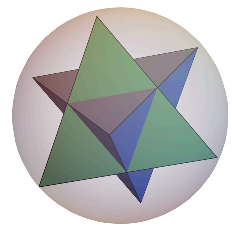






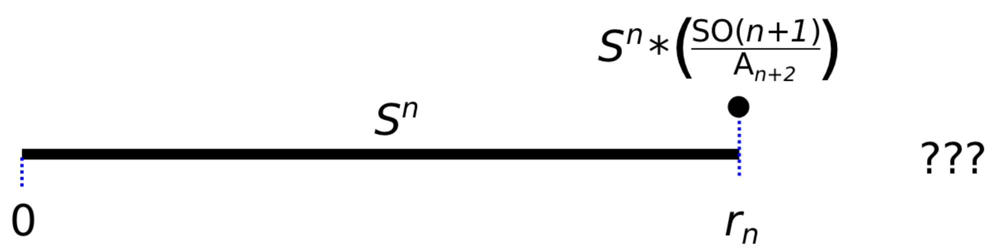
Theorem (A, Bush, Frick) For $f: S^n \to \mathbb{R}^{n+2}$, $\exists\ X \subset S^n$ of diameter at most r_n such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.



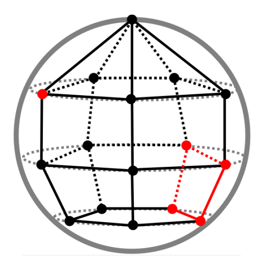


Proof $S^n \xrightarrow{f} \mathbb{R}^{n+2}$ induces

$(n+1)$-connected $VR(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$



Remark Lovász' strongly self-dual polytopes.



Remark Michael Crabb uses characteristic classes to get extensions

$$S^{2^r+t} \longrightarrow \mathbb{R}^{2^{r+1}+t}$$

Remark Versions of the ham sandwich theorem with more "fixings" than the dimension!

Remark Quantitative topology, esp. filling radius

Isosystolic Inequality:

$$\mathrm{sys}(M) \le 6 \cdot \mathrm{FillingRadius}(M) \le C_n \, \mathrm{vol}(M)^{1/n}$$

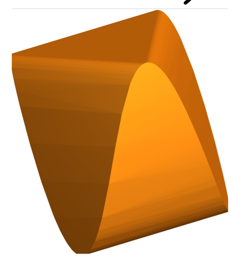
1983 Mikhael Gromov
1983-1991 Mikhail Katz

2020 Lim, Mémoli, Okutan
2017 Adamaszek, A
2018 Adamaszek, A, Frick
2020 A, Bush, Frick

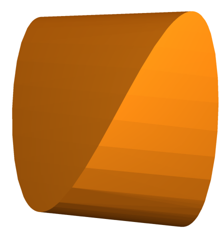
(IV) Orbitopes and Schur polynomials

Def$^{\underline{n}}$ The <u>Barvinok-Novik orbitope</u> $B_{2k} \subseteq \mathbb{R}^{2k}$ is the convex hull of the curve $(\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots, \cos(2k-1)\theta, \sin(2k-1)\theta)$.

Its faces are known only for $k=1,2$.



Def$^{\underline{n}}$ The <u>Carathéodory orbitope</u> $C_{2k} \subseteq \mathbb{R}^{2k}$ is the convex hull of the curve $(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos k\theta, \sin k\theta)$.



Its faces are known for all k.

	Odd Barvinok-Novik	Not odd Carathéodory
(scale) What diameter hits origin?	$\begin{matrix} \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 \\ \cos 3t_1 & \cos 3t_2 & \cos 3t_3 & \cos 3t_4 \\ \sin 3t_1 & \sin 3t_2 & \sin 3t_3 & \sin 3t_4 \end{matrix}$	$\begin{matrix} \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 \\ \cos 2t_1 & \cos 2t_2 & \cos 2t_3 & \cos 2t_4 \\ \sin 2t_1 & \sin 2t_2 & \sin 2t_3 & \sin 2t_4 \end{matrix}$
What are the orbitope's faces?	$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 & \cos t_5 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 & \sin t_5 \\ \cos 3t_1 & \cos 3t_2 & \cos 3t_3 & \cos 3t_4 & \cos 3t_5 \\ \sin 3t_1 & \sin 3t_2 & \sin 3t_3 & \sin 3t_4 & \sin 3t_5 \end{matrix}$	$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ \cos t_1 & \cos t_2 & \cos t_3 & \cos t_4 & \cos t_5 \\ \sin t_1 & \sin t_2 & \sin t_3 & \sin t_4 & \sin t_5 \\ \cos 2t_1 & \cos 2t_2 & \cos 2t_3 & \cos 2t_4 & \cos 2t_5 \\ \sin 2t_1 & \sin 2t_2 & \sin 2t_3 & \sin 2t_4 & \sin 2t_5 \end{matrix}$

Changing to exponentials gives

- A Vandermonde matrix, whose determinant is easy to factor, answering the question.

- A generalized Vandermonde matrix, whose determinant contains a Schur polynomial in its factorization, whose sign is hard to analyze.

Top left: $\det = C_1 \prod_{i<j} \sin(t_j - t_i)$

Bottom right: $\det = C_1 \prod_{i<j} \sin\left(\frac{t_j - t_i}{2}\right)$

Bottom left: $\det = C_1 \underbrace{\left(\prod_{1\le i<j\le 5} \sin\left(\frac{t_j-t_i}{2}\right)\right)}_{\text{Vandermonde}} \underbrace{\left(2 + \sum_{1\le i<j\le 5} \cos(t_j - t_i)\right)}_{\text{Schur}}$