An Introduction to Vietoris-Rips Complexes

Henry Adams, Colorado State University

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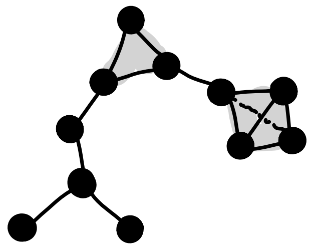
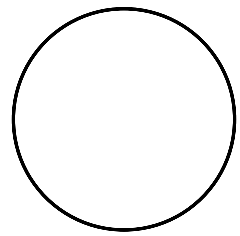
X metric space, $r \geq 0$.

<u>Def</u> The <u>Vietoris-Rips simplicial complex $VR(X,r)$</u> has

- vertex set X
- finite simplex $\sigma \subseteq X$ when $\text{diameter}(\sigma) \leq r$.



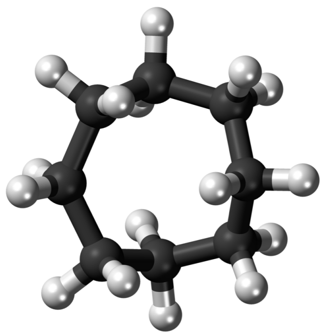


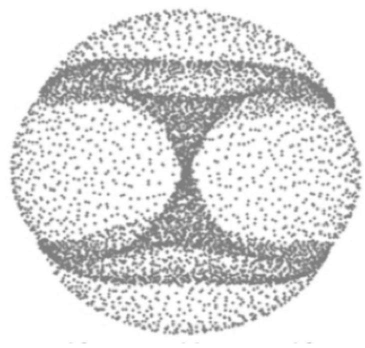
<u>History</u>

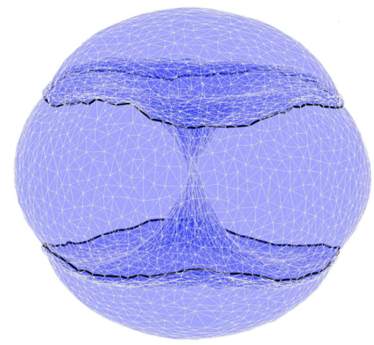
- Cohomology theory for metric spaces
- Geometric group theory
- Applied topology

Example Cyclo-octane molecule C_8H_{16}

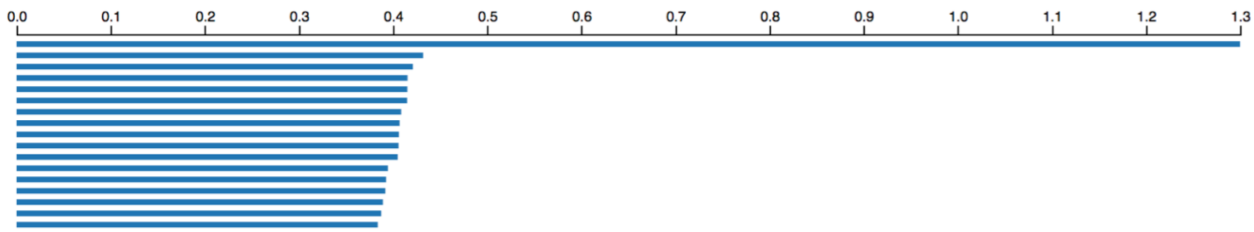
Martin, Thompson, Coutsias, Watson 2010



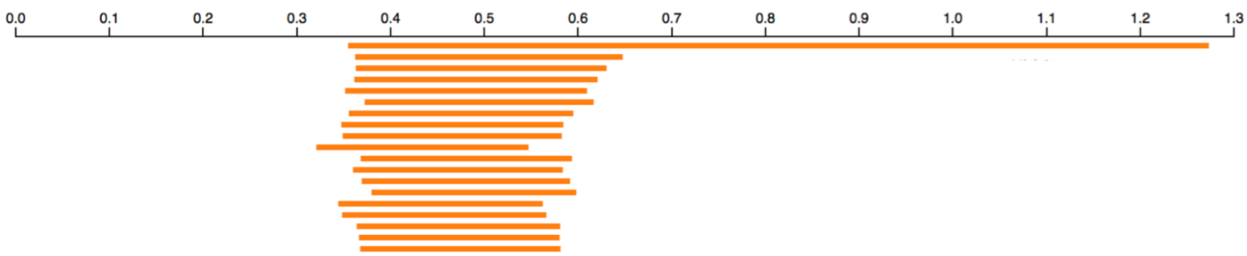




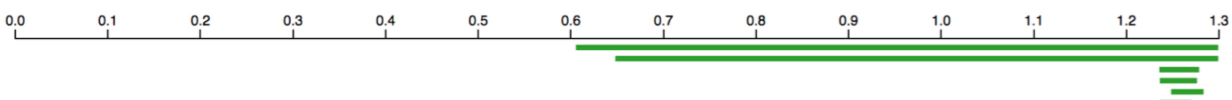
Persistence intervals in dimension 0:



Persistence intervals in dimension 1:



Persistence intervals in dimension 2:



X metric space, $r \geq 0$.

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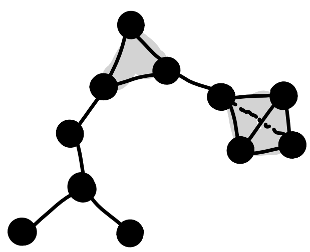
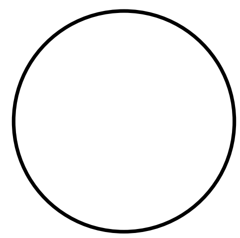
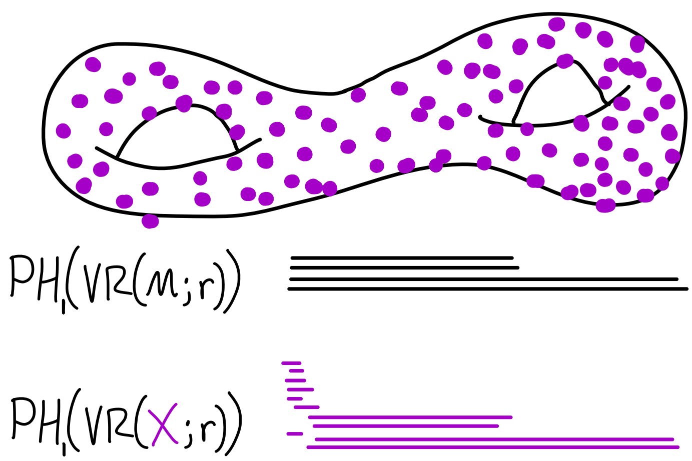
- vertex set X
- finite simplex $\sigma \subseteq X$ when $\text{diameter}(\sigma) \leq r$.





<u>Stability</u>



Chazal, de Silva, Oudot, 2014
Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, 2009

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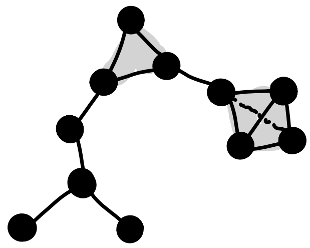
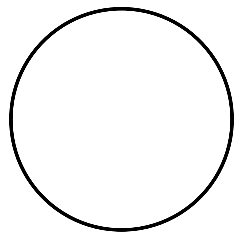
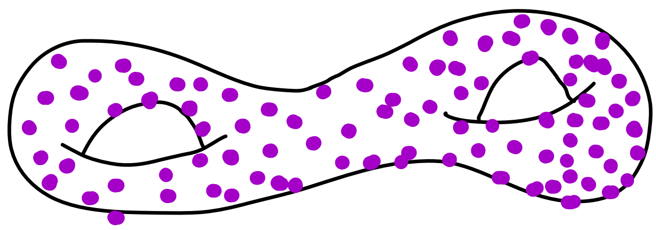
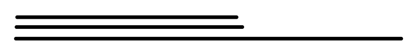




<u>Stability</u>



$PH_1(VR(M;r))$



$PH_1(VR(X;r))$

<u>Proof</u>

$$VR(M;r) \hookrightarrow VR(M;r+\varepsilon) \hookrightarrow VR(M;r+2\varepsilon)$$

$$VR(X;r) \hookrightarrow VR(X;r+\varepsilon) \hookrightarrow VR(X;r+2\varepsilon)$$

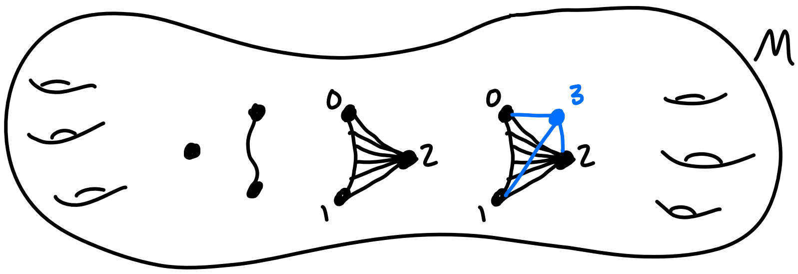
Chazal, de Silva, Oudot, 2014
Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, 2009

Thm (Hausmann 1995)

M compact Riemannian manifold.
Then $\exists\ r_0 > 0$ such that $VR(M;r) \simeq M\ \ \forall r < r_0$.

Proof Sketch

$$\begin{array}{c} VR(M;r) \\ \downarrow \\ M \end{array}$$

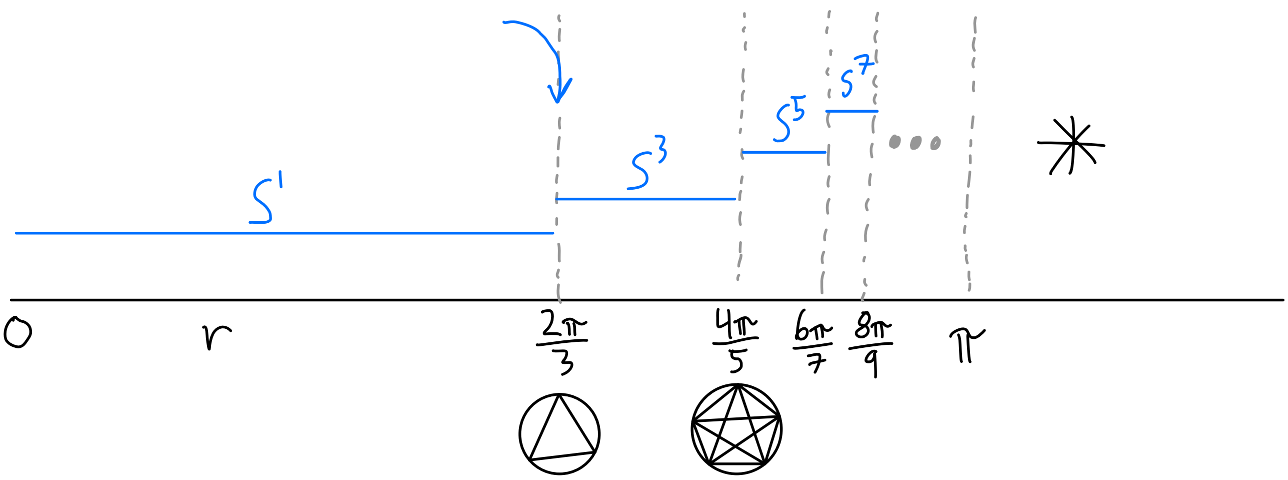


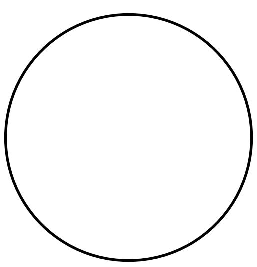
- Not canonical
- $M \hookrightarrow VR(M;r)$ not continuous.

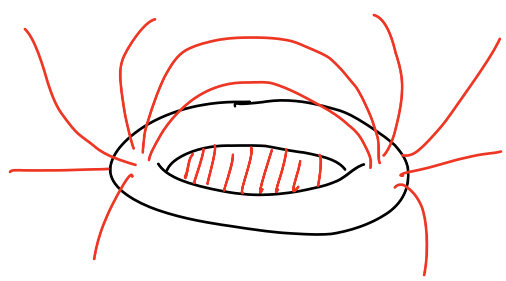
A, Adamaszek, "The Vietoris-Rips complexes of a circle", 2017

S^1 is circle with geodesic metric.

<u>Thm</u> $VR(S^1;r) \simeq \begin{cases} & \text{if } \frac{2\pi k}{2k+1} < r < \frac{2\pi(k+1)}{2k+3} \\ & \text{if } r = \frac{2\pi k}{2k+1} \end{cases}$

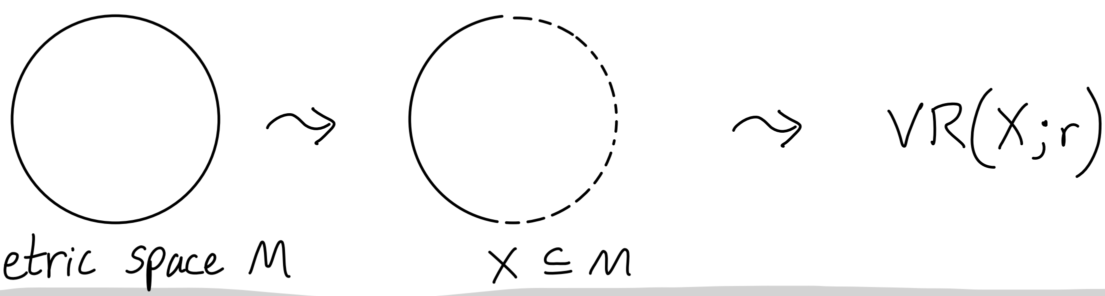






Metric Reconstruction

A simplicial complex whose vertex set is a metric space should often be equipped with an optimal transport metric (instead of the simplicial complex topology).



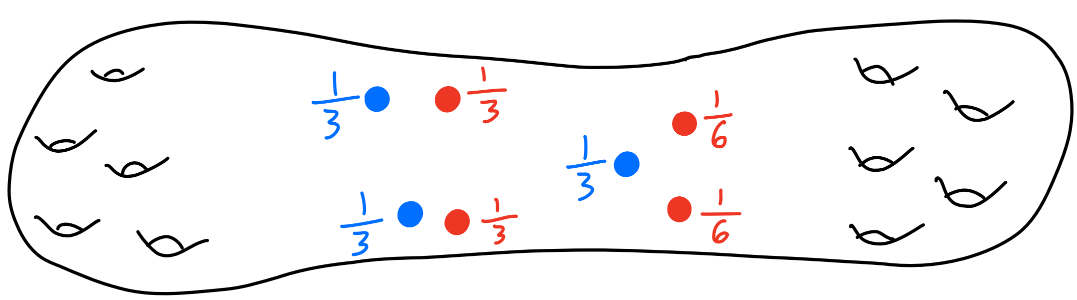
Adamaszek, A, Frick, 2018, "Metric reconstruction via optimal transport"

<u>Def</u> X metric space, $r \geq 0$.

The <u>Vietoris-Rips metric thickening</u> is

$$VR(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \;\middle|\; \begin{array}{l} x_i \in X, \quad \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r, \\ \lambda_i \geq 0, \quad \sum \lambda_i = 1 \end{array} \right\},$$

equipped with the optimal transport metric.

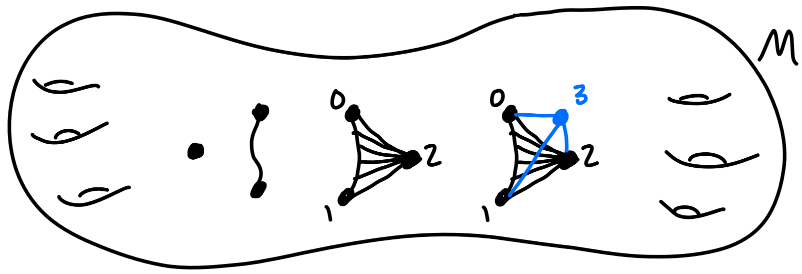


Thm (Hausmann 1995)

M compact Riemannian manifold.
Then $\exists\ r_0 > 0$ such that $VR(M;r) \simeq M\ \forall r < r_0$.

Proof Sketch

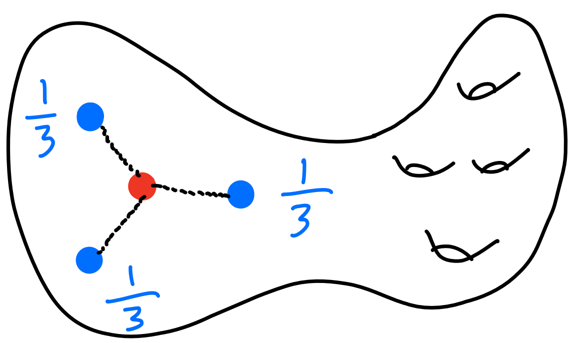
$$VR(M;r) \to M$$



- Not canonical
- $M \hookrightarrow VR(M;r)$ not continuous.

Our Proof Sketch

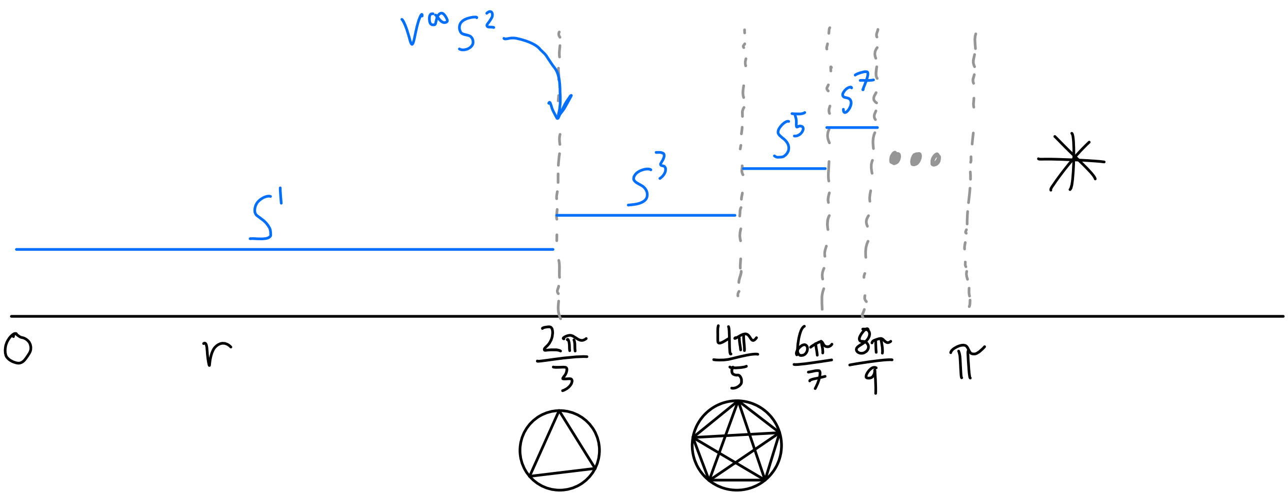
$$VR^m(M;r) \rightleftarrows M \qquad \sum \lambda_i \delta_{x_i} \mapsto \text{Karcher or Frechét mean}$$



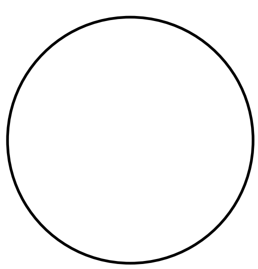
A. Adamaszek, "The Vietoris-Rips complexes of a circle", 2017

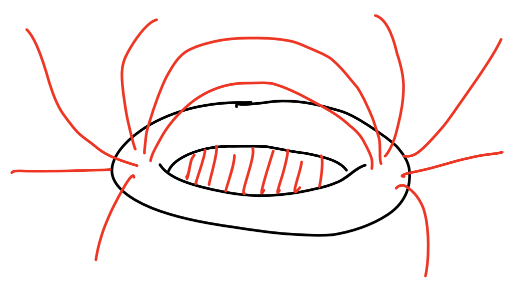
S^1 is circle with geodesic metric.

<u>Thm</u> $\quad VR(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{2\pi k}{2k+1} < r < \frac{2\pi(k+1)}{2k+3} \\ \bigvee^{\infty} S^{2k} & \text{if } r = \frac{2\pi k}{2k+1} \end{cases}$



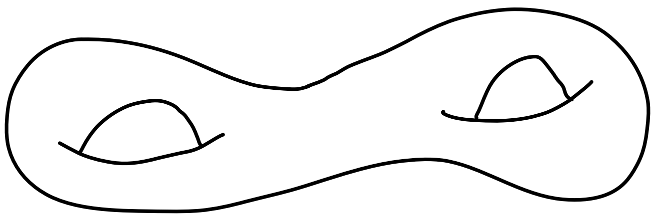
By contrast, $VR^m(S^1; \frac{1}{3}) \simeq S^3$.





A, Mémoli, Moy, Wang, 2021, "The persistent homology of optimal transport based metric thickenings"

<u>Thm</u> For X totally bounded, $VR^m(X;r)$ and $VR(X;r)$ have the same (undecorated) persistence diagrams.



$PH_1(VR(M;r))$ 

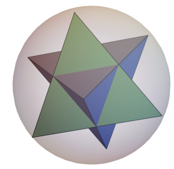
<u>Proof</u>

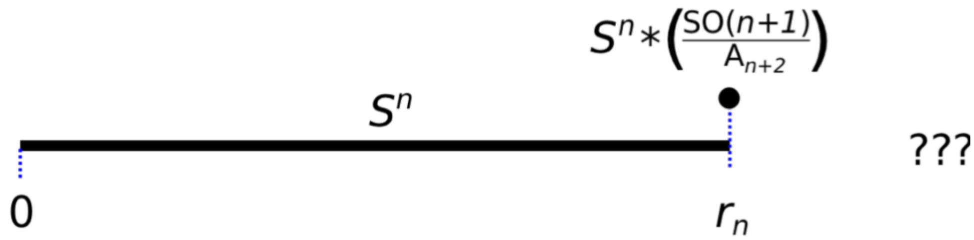
$$\begin{tikzcd}
VR(M;r) \arrow[r, hook] \arrow[dr] & VR(M;r+\varepsilon) \arrow[r, hook] \arrow[dr] & VR(M;r+2\varepsilon) \\
VR^m(M;r) \arrow[r, hook] \arrow[ur] & VR^m(M;r+\varepsilon) \arrow[r, hook] \arrow[ur] & VR^m(M;r+2\varepsilon)
\end{tikzcd}$$

<u>Question</u> Is $VR^m_<(X;r) \simeq VR_<(X;r)$?

More generally,

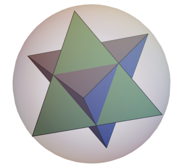
<u>Thm</u> $VR^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$

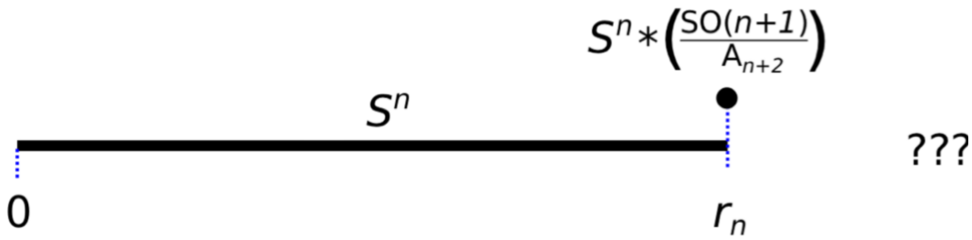




More generally,

Thm $\mathrm{VR}^m(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$



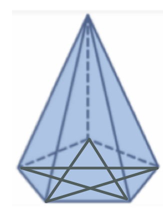


Sketch $\mathrm{VR}^m(S^n;r_n)$

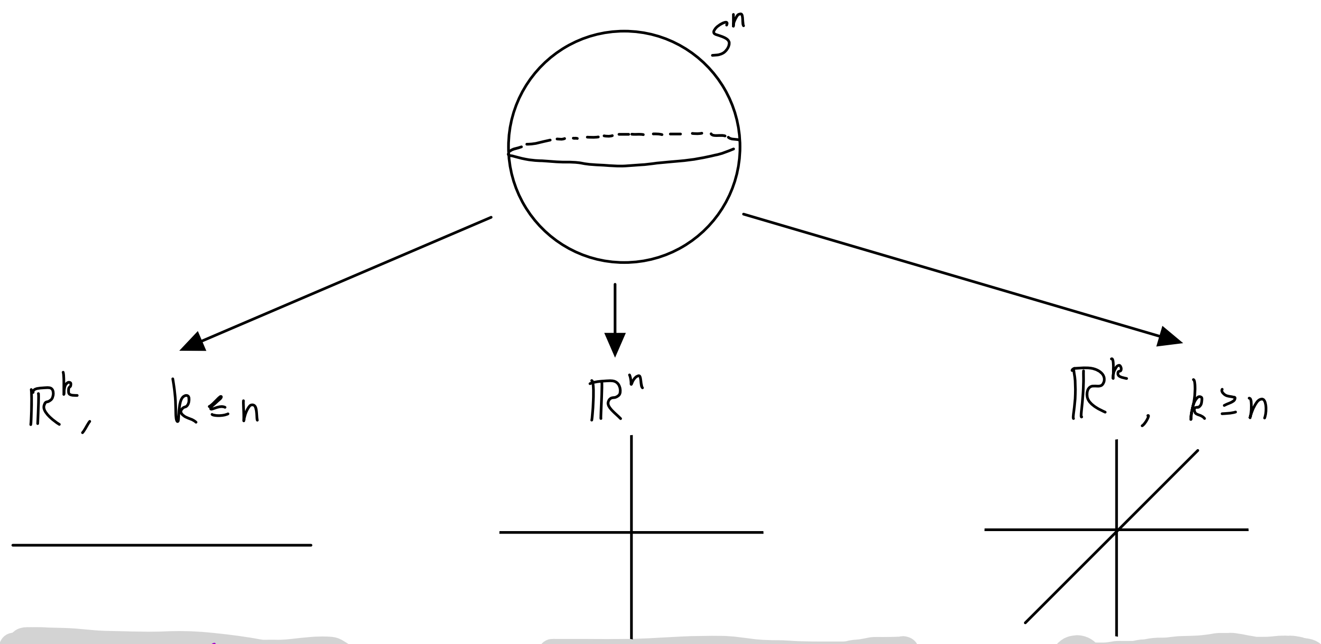
$= \mathrm{VR}^m(S^n;r_n) \setminus \left(\text{interiors of regular } \Delta^{n+1} \right) \cup \Delta^{n+1} \times \left(\frac{SO(n+1)}{A_{n+2}} \right)$

$\simeq S^n \times C\left(\frac{SO(n+1)}{A_{n+2}} \right) \cup C(S^n) \times \left(\frac{SO(n+1)}{A_{n+2}} \right)$

$= S^n * \frac{SO(n+1)}{A_{n+2}}$



Application: Borsuk-Ulam theorems



Almgren (1965):
Geometric measure theory
Gromov (2003):
Filling radius, characteristic classes

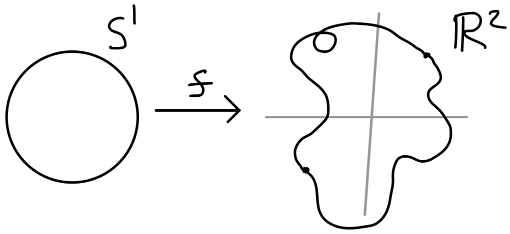
Borsuk-Ulam (1933)

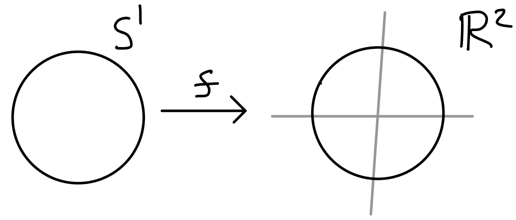
A, Bush, Frick (2020)
"Metric thickenings, Borsuk-Ulam theorems, and orbitopes"

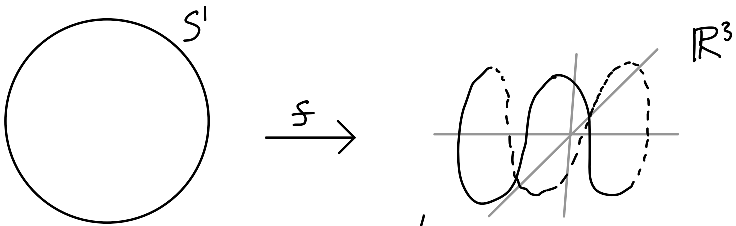
"Waist of sphere" theorem: For $f: S^n \to \mathbb{R}^k$ with $k \leq n$, $\exists y \in \mathbb{R}^k$ with $\mathrm{Vol}_{n-k}(f^{-1}(y)) \geq \mathrm{Vol}_{n-k}(S^{n-k})$.

Invariance of dimension.

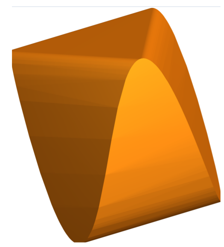
Borsuk-Ulam theorems for $f: S^n \to \mathbb{R}^k$ with $k \geq n$?







$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta)$$



Thm For $f: S^1 \to \mathbb{R}^{2k+1}$, $\exists\ X \subset S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.

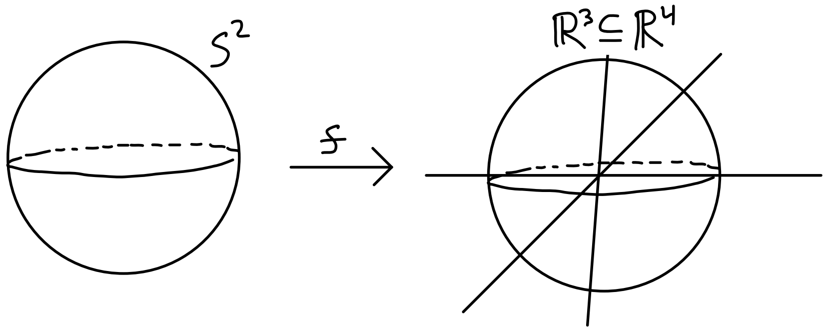
Proof

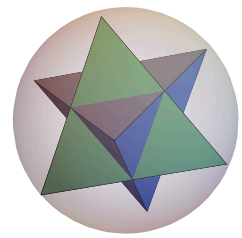
$$S^1 \xrightarrow{f} \mathbb{R}^{2k+1} \quad \text{induces}$$
$$VR(S^1; r) \xrightarrow{f} \mathbb{R}^{2k+1}$$

Sharpness of diameter bound

$$S^1 \longrightarrow \mathbb{R}^{2k} \subseteq \mathbb{R}^{2k+1}$$
$$\theta \longmapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$$

Thm For $f: S^n \to \mathbb{R}^{n+2}$, $\exists\ X \subset S^n$ of diameter at most r_n such that $\mathrm{conv}(f(X)) \cap \mathrm{conv}(f(-X)) \neq \emptyset$.





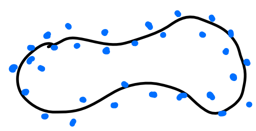
Proof

$$S^n \xrightarrow{f} \mathbb{R}^{n+2} \text{ induces}$$

$$S^n * \frac{SO(n+1)}{A_{n+2}} \simeq VR^m(S^n; r) \xrightarrow{f} \mathbb{R}^{n+2}$$

Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes

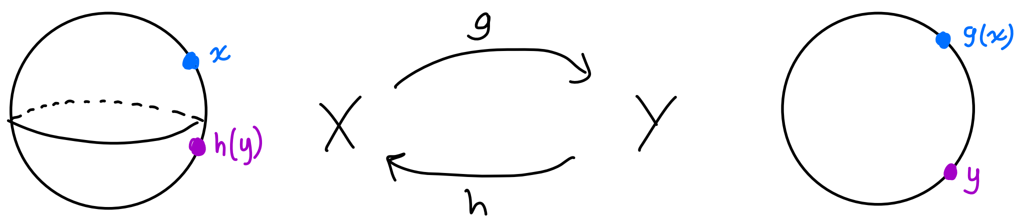
Joint with CSU, OSU, CMU, Berlin



Gromov-Hausdorff

X, Y compact metric spaces

Def $2 \cdot d_{GH}(X, Y) = \inf_{\substack{g: X \to Y \\ h: Y \to X}} \max \{ dis(g), dis(h), codis(g,h) \}$.



$$dis(g) = \sup_{x, x' \in X} \left| d(x,x') - d(g(x), g(x')) \right|$$

$$codis(g,h) = \sup_{\substack{x \in X \\ y \in Y}} \left| d(x, h(y)) - d(g(x), y) \right|$$

Lim, Mémoli, Smith, 2021, "The Gromov-Hausdorff distance between spheres"

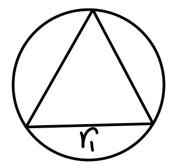
Sphere S^n, geodesic metric, diameter π.

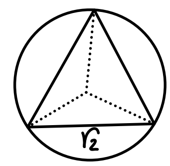
$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$				
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$
S^7							0

Symmetric matrix

Non-zero entries in $(\pi/2, \pi)$





Lim, Mémoli, Smith, 2021, "The Gromov-Hausdorff distance between spheres"

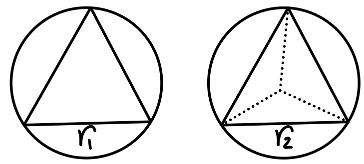
Sphere S^n, geodesic metric, diameter π.

$2 \cdot d_{GH}(S^n, S^k)$

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
S^1	0	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\geq \frac{4\pi}{5}$	$\geq \frac{4\pi}{5}$	$\geq \frac{6\pi}{7}$	$\geq \frac{6\pi}{7}$
S^2		0	r_2				
S^3			0	$\geq r_3$			
S^4				0	$\geq r_4$		
S^5					0	$\geq r_5$	
S^6						0	$\geq r_6$
S^7							0

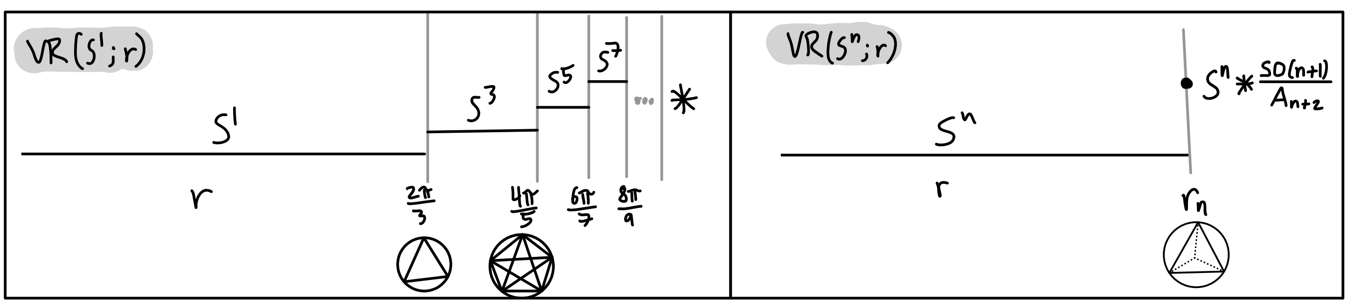
Symmetric matrix

Non-zero entries in $(\pi/2, \pi)$



Theorem (Oct, 2021)

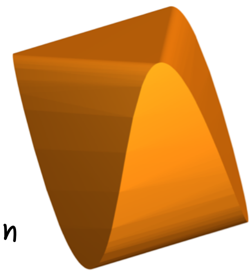
For $n<k$,
$$2 \cdot d_{GH}(S^n, S^k) \geq \inf\{r : \text{there exists cont. odd } S^k \to VR(S^n; r)\}.$$



Map $f : X \to Y$ induces a cont. map

$$f : VR(X, r) \longrightarrow VR(Y; \mathrm{dis}(f) + r)$$
$$x \longmapsto f(x)$$

Questions



(1) $VR^m(S^n;r)$ for larger r?

(2) $\check{C}ech^m(S^n;r)$?

(3) Other manifolds? Tori, ellipsoids, $\mathbb{RP}^n$, $\mathbb{CP}^n$

(4) $VR^m_<(X;r) \simeq VR_<(X;r)$?

(5) Morse and Morse-Bott theories

(6) Measures with infinite support

(8) Tighter connections between $VR^m(X;r)$ and $B_{L^\infty(X)}(X;r)$.

(7) In $VR^m(X;r)$ replace ∞-diam with p-diam.

In $\check{C}ech^m(X;r)$ replace ∞-variance with p-variance.