4/24/18

An introduction to homotopy colimits

(Following notes by Rubén Sanchez-García and Daniel Dugger)

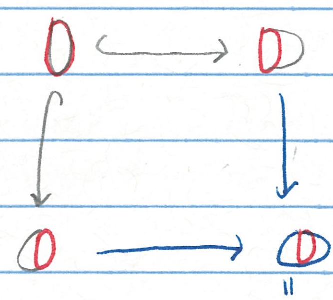
Colimits Colimits exist in the category Top of topological spaces. IE, given a diagram D (small category) and a functor $F : D \to \mathrm{Top}$ $\exists!$ object $\mathrm{colim}(F) \in \mathrm{Top}$ satisfying some universal property.

Ex1 D is
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$
F is
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd}$$
$X, Y, Z \in \mathrm{Top}$

The colimit $\mathrm{colim}(F)$ satisfies a universal property:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & \mathrm{colim}(F) \arrow[dr, dashed, "\exists!h"] & \\ & & W \end{tikzcd}$$

Ex1A F is
$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$
$S^n = \mathrm{colim}(F)$

$\underline{n=2}$ 
$\mathrm{colim}(F)$

Ex 1B F' is

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \\[-20pt] & \text{colim}(F') \arrow[u, phantom, "="] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \\[-20pt] & \text{colim}(F') \arrow[u, phantom, "="] \end{tikzcd}$$

For this shape of D, we have

$$\text{colim}(F) = (Z \amalg X \amalg Y) \Big/ \begin{pmatrix} x \sim f(x) & \text{for } x \in X \\ x \sim g(x) & \text{for } x \in X \end{pmatrix}$$

> This is called a "pushout"

Ex 2 D is $\bullet \quad \bullet \quad \bullet \quad \bullet$ $\qquad$ F is $X_1 \quad X_2 \quad X_3 \quad X_4$

Here $\text{colim}(F) = X_1 \amalg X_2 \amalg X_3 \amalg X_4$

$$\begin{tikzcd} X_1 \arrow[dr] \arrow[ddr, bend right] & X_2 \arrow[d] \arrow[dd, bend right] & X_3 \arrow[dl] \arrow[ddl, bend left] & X_4 \arrow[dll] \arrow[ddll, bend left] \\ & \text{colim}(F) = X_1 \amalg X_2 \amalg X_3 \amalg X_4 \arrow[d, dashed, "\exists!"] & & \\ & W & & \end{tikzcd}$$

Ex 3 D is $\bullet \to \bullet \to \bullet$ $\qquad$ F is $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$

Here $\text{colim}(F) = (X_1 \amalg X_2 \amalg X_3) \big/ (x_i \sim f_i(x_i) \text{ for } x_i \in X_i)$

> This is called a "direct limit" | Countable unions are an example

More generally, for any diagram D,

$$\text{colim}(F) = \left(\coprod_{d \in D} F(d) \right) \Big/ \begin{pmatrix} x_0 \sim x_1 \quad \text{for } x_0 \in F(d_0) \\ \text{and } x_1 \in F(d_1) \text{ when } \exists\, \alpha : d_0 \to d_1 \\ \text{with } x_1 = F(\alpha)(x_0) \end{pmatrix}$$

This is equipped with the colimit topology — the most open sets such that the maps from the spaces $F(d)$ in are continuous.

Unfortunately, colimits are not satisfying from the perspective of homotopy theory.
Consider F and F' from Ex 1A and 1B

$$
\begin{tikzcd}
S^{n-1} \arrow[rr] \arrow[dr, "="] \arrow[dd] & & D^n \arrow[dr, "\simeq"] \arrow[dd] & \\
& S^{n-1} \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
D^n \arrow[rr] \arrow[dr, "\simeq"] & & \mathrm{colim}(F)=S^n \arrow[dr, "\not\simeq"] & \\
& * \arrow[rr] & & \mathrm{colim}(F')=*
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[d, "\eta"] \\
F'
\end{tikzcd}
$$

The above is a natural transformation from F to F' that's a homotopy equivalence everywhere.

The universal property gives a map $\mathrm{colim}\, F \to \mathrm{colim}\, F'$, which sadly is not a homotopy equivalence since $\mathrm{colim}(F) = S^n \not\simeq * = \mathrm{colim}(F')$

<u>Homotopy colimits</u> will satisfy <u>homotopy invariance</u>:
Given two functors $F, F' : D \to Top$ and a natural transformation $\eta : F \to F'$ such that $\eta_d : F(d) \to F'(d)$ is a homotopy equivalence $\forall d \in D$, then

$$\mathrm{hocolim}(F) \simeq \mathrm{hocolim}(F') \quad \in Top$$

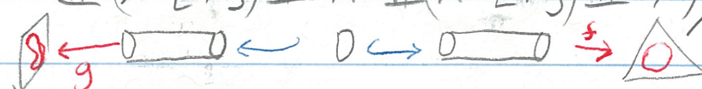
Ex 1 D is

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

F is

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array}$$

Here $\operatorname{hocolim}(F) = \left(Z \amalg (X \times [-2,-1]) \amalg X \amalg (X \times [1,2]) \amalg Y \right) / \sim$

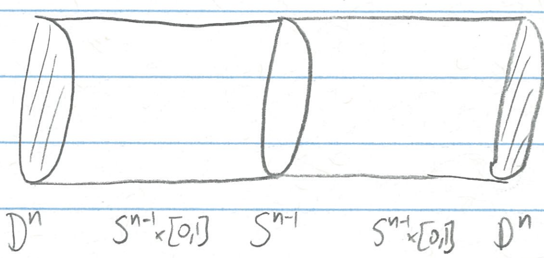


with $(x,-2) \sim g(x)$
$(x,2) \sim f(x)$
$(x,-1) \sim x \sim (x,1)$

Ex 1A F is

$$\begin{array}{ccc} S^{n-1} & \overset{f}{\hookrightarrow} & D^n \\ {\scriptstyle g}\downarrow & & \\ D^n & & \end{array}$$

$\operatorname{hocolim}(F)$ is

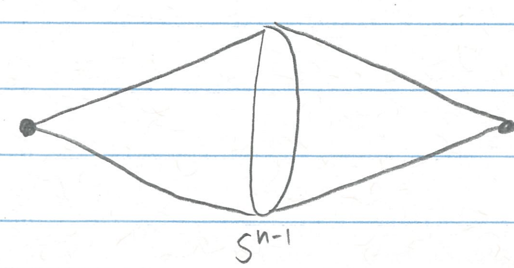


$\simeq S^n = \operatorname{colim}(F)$

Ex 1B F' is

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & * \\ {\scriptstyle g}\downarrow & & \\ * & & \end{array}$$

$\operatorname{hocolim}(F')$ is



$\simeq S^n \not\simeq * = \operatorname{colim}(F')$

sSet has a model category structure where the cofibrations are monomorphisms, the fibrations are Kan fibrations, and weak equivalences are maps whose geometric realizations are weak equivalences.

<u>Main point</u> $\text{hocolim}(F) \simeq \text{hocolim}(F')$
whereas $\text{colim}(F) \not\simeq \text{colim}(F')$.

<u>Side point</u> We have $\text{hocolim}(F) \simeq \text{colim}(F)$ (but not $\text{hocolim}(F') \simeq \text{colim}(F')$) since every map in F is a <u>cofibration</u> (think "nice inclusion").

<u>Imprecise definition</u> For D an arbitrary diagram (small category), $\text{hocolim}(F)$ is constructed by taking the disjoint union of

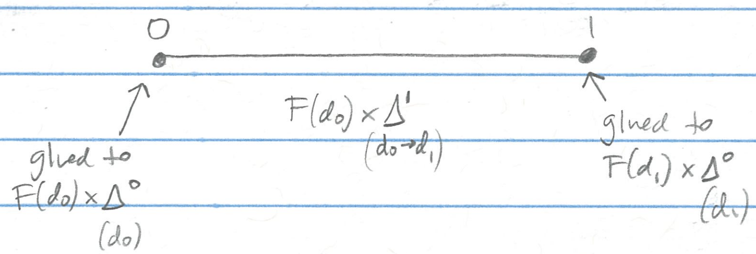
- a copy of $F(d_0)$ $(= F(d_0) \times \Delta^0)$ for each vertex $d_0 \in D$
- a copy of $F(d_0) \times \Delta^1$ for each arrow $d_0 \to d_1$ in D

$\vdots$

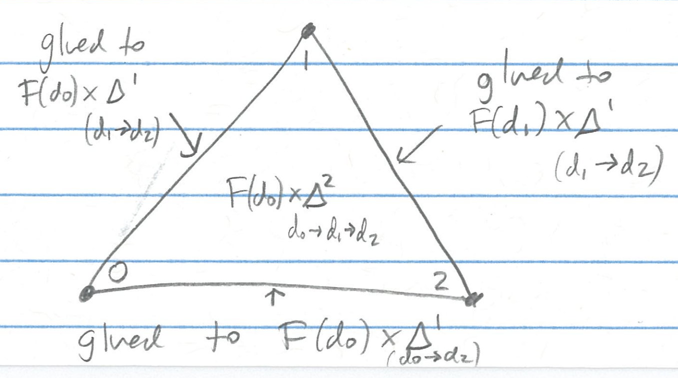
- a copy of $F(d_0) \times \Delta^n$ for each chain of n composable arrows $d_0 \to d_1 \to \cdots \to d_{n-1} \to d_n$ in D,

with identifications of $F(d_0) \times \partial\Delta^n$ with the <u>appropriate spaces</u>

<u>n=1</u>

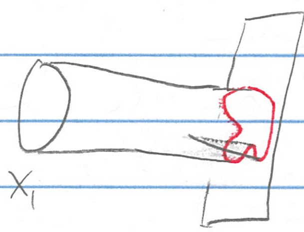


<u>n=2</u>



Ex D is $\bullet \longrightarrow \bullet$ F is $X_1 \xrightarrow{f} X_2$

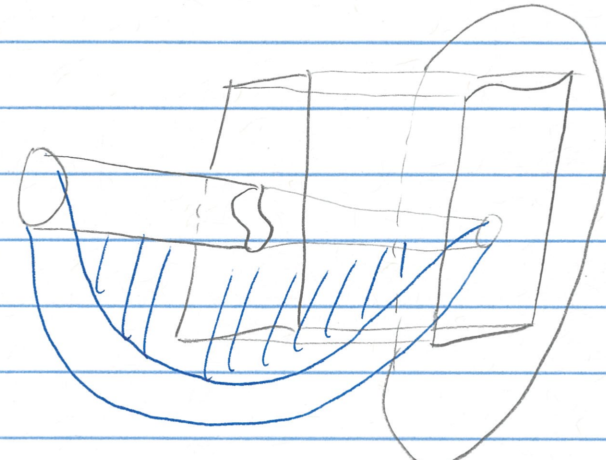
$\mathrm{hocolim}(F)$ is



X_1 X_2

Ex D is $\bullet \longrightarrow \bullet \longrightarrow \bullet$ F is $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$

$\mathrm{hocolim}(F)$ is



X_1 X_2 X_3

A precise definition of the homotopy colimit involves <u>simplicial sets</u>.

Rmk There's a natural map $\mathrm{hocolim}(F) \to \mathrm{colim}(F)$ obtained by projecting each $F(d_0) \times \Delta^n$ to $F(d_0)$.

(This map does not "contradict" the universality of $\mathrm{colim}(F)$, since the maps $\mathrm{hocolim}(F)$ admits from the spaces in the initial diagram do not satisfy commutativity)