An Introduction to Computational Topology

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Computer Science Colloquium
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Datasets have shapes

Example: Diabetes study
145 points in 5-dimensional space

Datasets have shapes

Example: Cyclo-Octane ($\text{C}_8\text{H}_{16}$) data

1,031,644 points in 72-dimensional space

*Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data* by Shawn Martin and Jean-Paul Watson, 2010.
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Topology studies shapes

A donut and coffee mug are “homotopy equivalent”, and considered to be the same shape. You can bend and stretch (but not tear) one to get the other.
Topology studies shapes

A donut and coffee mug are “homotopy equivalent”, and considered to be the same shape. You can bend and stretch (but not tear) one to get the other.
A torus has a Betti sequence \((1, 2, 1, 0, 1)\), since it has a single connected component, two different loops that cannot be deformed into a point (shown in red in the bottom panel of Figure 2c), and there is a two-dimensional surface that cannot be deformed into a point (shown in orange in Figure 2c). The Klein bottle has the same sequence as the torus \((1, 2, 1, 0, 1)\). This shows that while two objects that are equivalent must have the same Betti sequences, two objects that are not equivalent do not necessarily have different sequences. Finally, a sphere has a sequence \((1, 0, 1, 0, 1)\), as any one-dimensional loop on its surface can be deformed into a point. The Betti sequence therefore provides a signature (albeit not unique) of the underlying topology of the object.

These definitions work for smooth continuous objects. But suppose now that instead of a continuous rubbery object we are faced with a finite set of (noisy) points sampled from it, which may represent actual experimental data. How can one estimate the Betti numbers of the original object from these samples? The proposed method...

Topology studies shapes

Torus
Topology studies shapes

Klein bottle
Topology studies shapes

Klein bottle

https://plus.maths.org/content/imaging-maths-inside-klein-bottle
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Topology studies shapes

Homology \( (\mathbb{Z}/2\mathbb{Z}) \)
Topology studies shapes

Homology groups $H_0, H_1, H_2, H_3, \ldots$  
(over $\mathbb{Z}/2\mathbb{Z}$)

$H_k$ “counts the number of $k$-dimensional holes”.

Homotopy equivalent shapes have the same homology.

$H_0$ has rank 6.
$H_1$ has rank 0.
$H_2$ has rank 0.

$H_0$ has rank 1.
$H_1$ has rank 1.
$H_2$ has rank 0.

$H_0$ has rank 1.
$H_1$ has rank 3.
$H_2$ has rank 0.
Topology studies shapes

Homology groups $H_0, H_1, H_2, H_3, \ldots \quad \text{(over } \mathbb{Z}/2\mathbb{Z})$

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Homotopy equivalent shapes have the same homology.

$H_0$ has rank 1.
$H_1$ has rank 0.
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$H_0$ has rank 1.
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$H_1$ has rank 2.
$H_2$ has rank 1.
Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

0-simplex  1-simplex  2-simplex  3-simplex
Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

0-simplex  1-simplex  2-simplex  3-simplex

Simplicial complexes
Topology studies shapes

Homology ( $\mathbb{Z}/2\mathbb{Z}$ )

0-simplex  1-simplex  2-simplex  3-simplex

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Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

0-simplex  1-simplex  2-simplex  3-simplex
Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

0-simplices  1-simplices  2-simplices
Topology studies shapes

Homology \( \mathbb{Z}/2\mathbb{Z} \)

0-simplices 1-simplices 2-simplices

\[
\begin{align*}
R^2 & \times R^2 \\
\mathbb{Z} & / 2\mathbb{Z}
\end{align*}
\]
Topology studies shapes

Homology (\(\mathbb{Z}/2\mathbb{Z}\))

0-cycle

1-cycle

2-cycle

A cycle has no boundary.
Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

- **0-cycle**
- **1-cycle**
- **2-cycle**

A cycle has no boundary.
Topology studies shapes

Homology \((\mathbb{Z}/2\mathbb{Z})\)

Two cycles are equivalent if they differ by a boundary. \(H_k\) measures equivalence classes of \(k\)-cycles.
“Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone ...”

-Aleksandr Solzhenitsyn, *The First Circle*
Topology applied to data: Persistent homology

What shape is this?
Topology applied to data: Persistent homology

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Čech simplicial complex ~ union of balls
Topology applied to data: Persistent homology

Čech simplicial complex \sim union of balls
Topology applied to data: Persistent homology

Čech simplicial complex $\sim$ union of balls
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Čech simplicial complex \( \sim \) union of balls
Topology applied to data: Persistent homology

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Topology applied to data: Persistent homology

Significant features persist.
Topology applied to data: Persistent homology

Example: 3x3 high-contrast patches from images
Points in 9-dimensional space

Topology applied to data: Persistent homology

1st densest group of patches
Topology applied to data: Persistent homology

1st densest group of patches

Interpretation: nature prefers linearity
Topology applied to data: Persistent homology

2\textsuperscript{nd} densest group of patches
Topology applied to data: Persistent homology

$2^{nd}$ densest group of patches

Interpretation: nature prefers horizontal and vertical directions
Topology applied to data: Persistent homology

3rd densest group of patches

Betti_0

Betti_1

Betti_2
Topology applied to data: Persistent homology

3rd densest group of patches
Topology applied to data: Persistent homology

3\textsuperscript{rd} densest group of patches
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3rd densest group of patches

https://plus.maths.org/content/imaging-maths-inside-klein-bottle
Topology applied to data: Persistent homology

3\textsuperscript{rd} densest group of patches

Interpretation: nature prefers linear and quadratic patches at all angles
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References


• Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data by Shawn Martin and Jean-Paul Watson, 2010.

• On the Local Behavior of Spaces of Natural Images by Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian, 2008.
Evasion Paths in Mobile Sensor Networks

Evasion problem

- Sensors move in a ball-shaped domain $B \subset \mathbb{R}^d$ over time interval $I$. Fixed sensors cover $\partial B$.
- Measure only the Čech complex.
- Is there an evasion path?

Coordinate-free Coverage in Sensor Networks with Controlled Boundaries via Homology by V. de Silva and R. Ghrist
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Evasion problem

- Let $X \subset B \times I$ be the covered region.
- **Evasion Problem.** Using the time-varying Čech complex, can we determine if an evasion path exists?

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Evasion problem

- **Theorem (de Silva, Ghrist).** If there is an $\alpha \in H_d(SC, \partial B \times I)$ with $0 \neq \delta \alpha \in H_{d-1}(\partial B \times I)$, then no evasion path exists.
Evasion problem

• Theorem (de Silva, Ghrist).
If there is an \( \alpha \in H_d(SC, \partial B \times I) \) with
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- Coordinate-free.

- Not sharp. Can it be sharpened?
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Coordinate-free Coverage in Sensor Networks with Controlled Boundaries via Homology by V. de Silva and R. Ghrist
Zigzag Persistence
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- Form zigzag module for $X \to I$ with $H_{d-1}$. 
Zigzag persistence

- Form zigzag module for $X \rightarrow I$ with $H_{d-1}$. 
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- **Hypothesis:** There is an evasion path $\iff$ there is a full-length bar.
Zigzag persistence

- Form zigzag module for $SC \rightarrow I$ with $H_{d-1}$.

- **Hypothesis:** There is an evasion path $\iff$ there is a full-length bar.
- **Theorem:** If there is an evasion path then there is a full-length bar.
- **Streaming computation.**
Dependence on embedding $X \hookrightarrow B \times I$

- The time-varying Čech complex alone does not determine if an evasion path exists.
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- Covered regions are fibrewise homotopy equivalent but their complements are not.
Planar sensors measuring cyclic orders

- Each vertex measures the cyclic ordering of adjacent edges.
- Equivalent to a set of boundary cycles.

Cyclic orderings

Boundary cycles
Planar sensors measuring cyclic orders

- **Theorem.** In a planar sensor network that remains connected, the time-varying alpha complex with rotation information determines if an evasion path exists.
Planar sensors measuring cyclic orders

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**Diagram**: Čech and Alpha complexes.
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- **Open question.** Is the Čech complex with rotation information sufficient?
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Conclusions

• Streaming zigzag persistence criterion.
• Čech complex insufficient.
  Alpha complex with rotation information suffices.
  What about the Čech complex with rotation information?


Thank you!