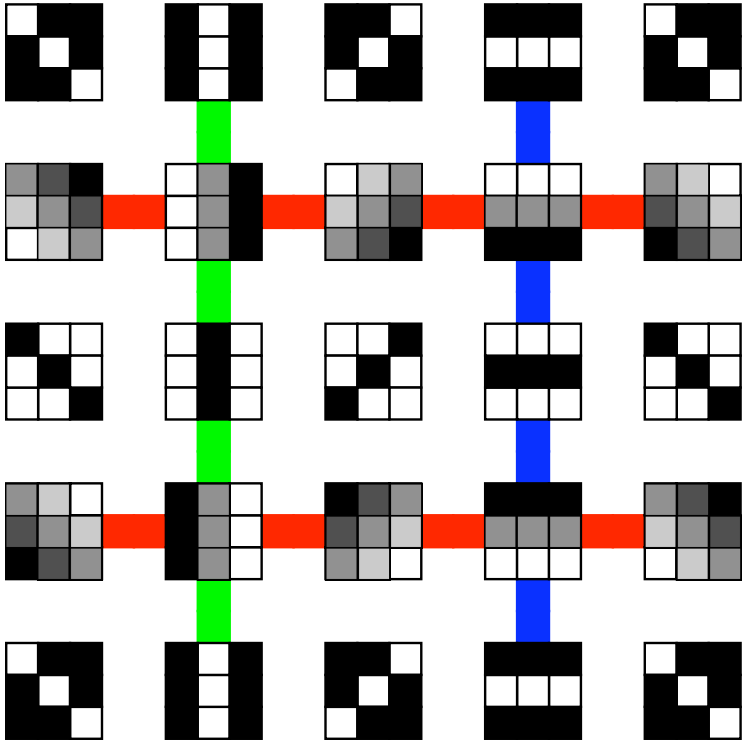
An Introduction to Computational Topology



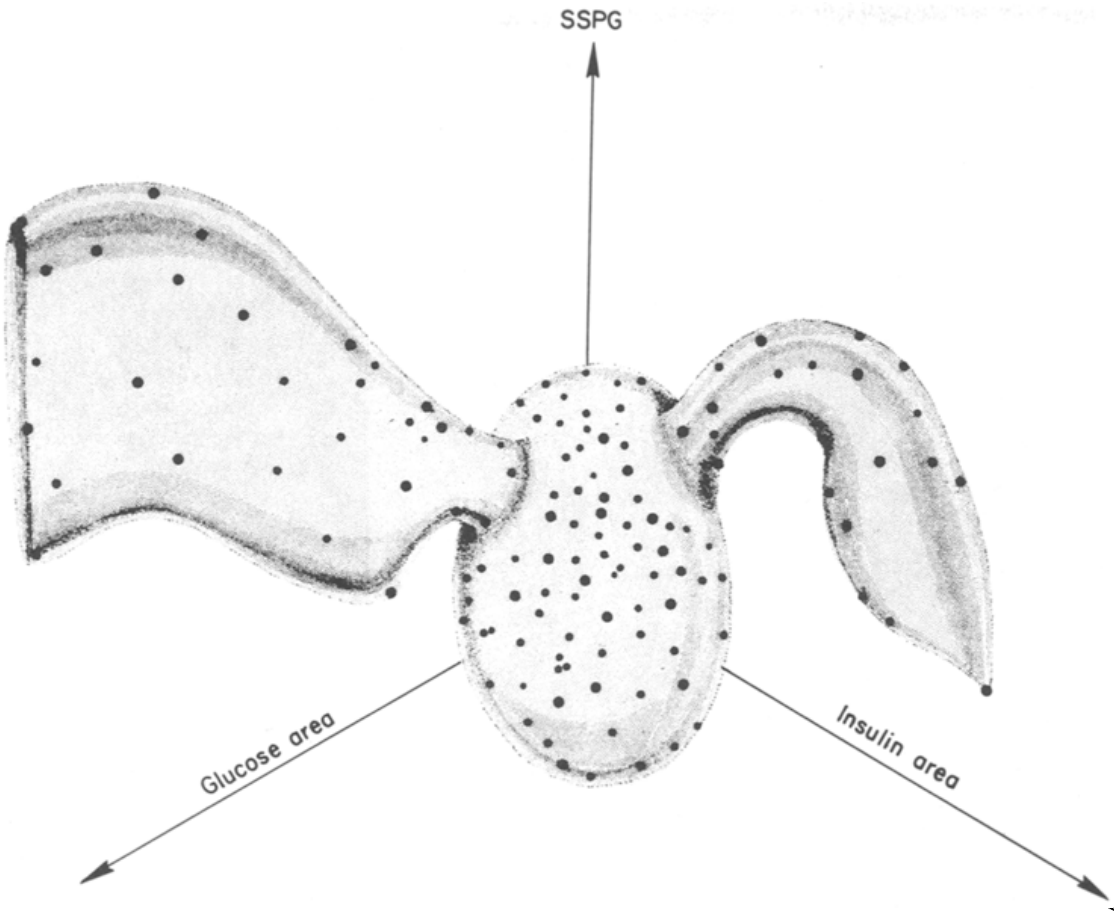
Henry Adams, Colorado State University, Mathematics Department

Computer Science Colloquium

October 10, 2016

Datasets have shapes

Example: Diabetes study
145 points in 5-dimensional space

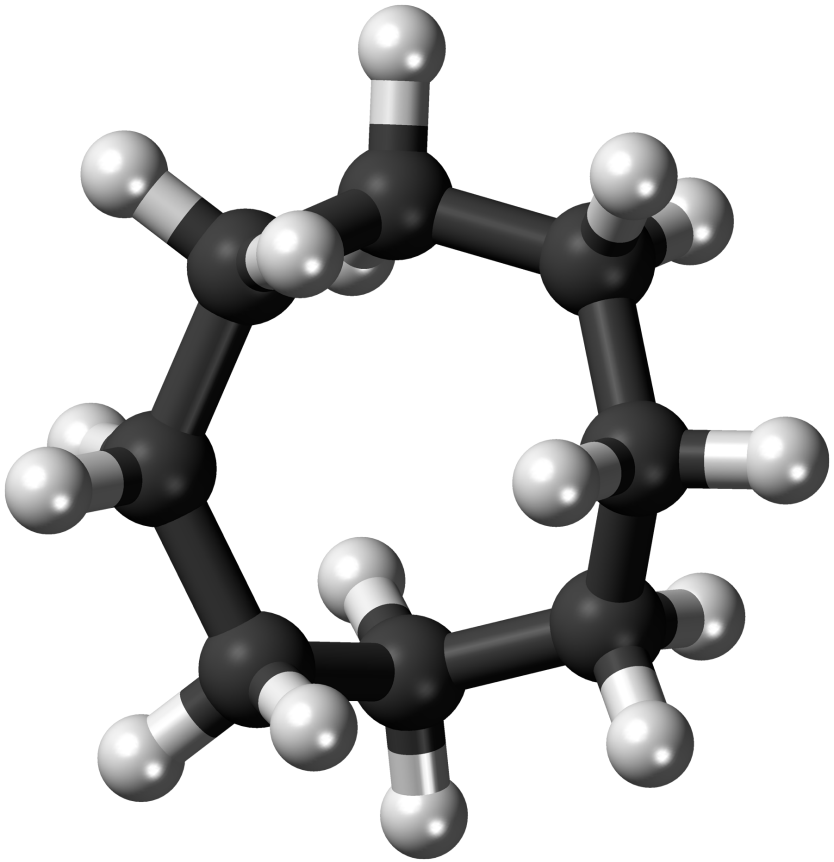


An Attempt to Define the Nature of Chemical Diabetes Using a Multidimensional Analysis by G. M. Reaven and R. G. Miller, 1979.

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

1,031,644 points in 72-dimensional space



Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data by Shawn Martin and Jean-Paul Watson, 2010.

Datasets have shapes

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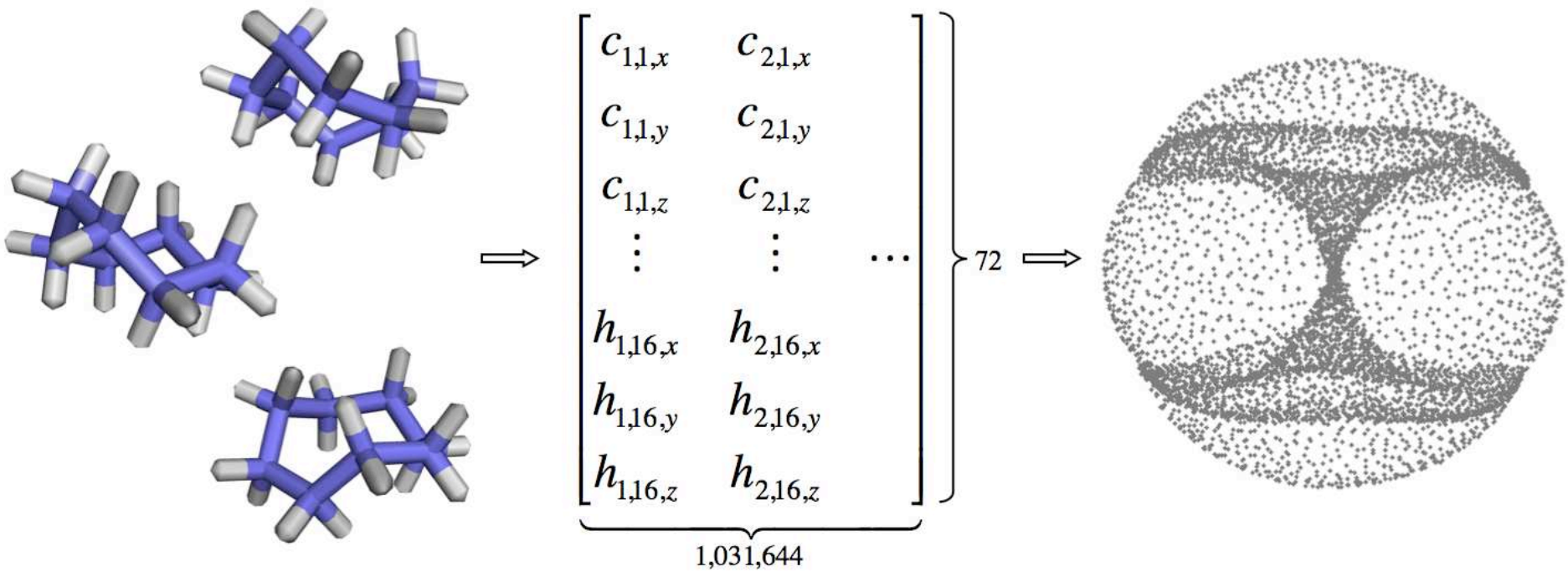

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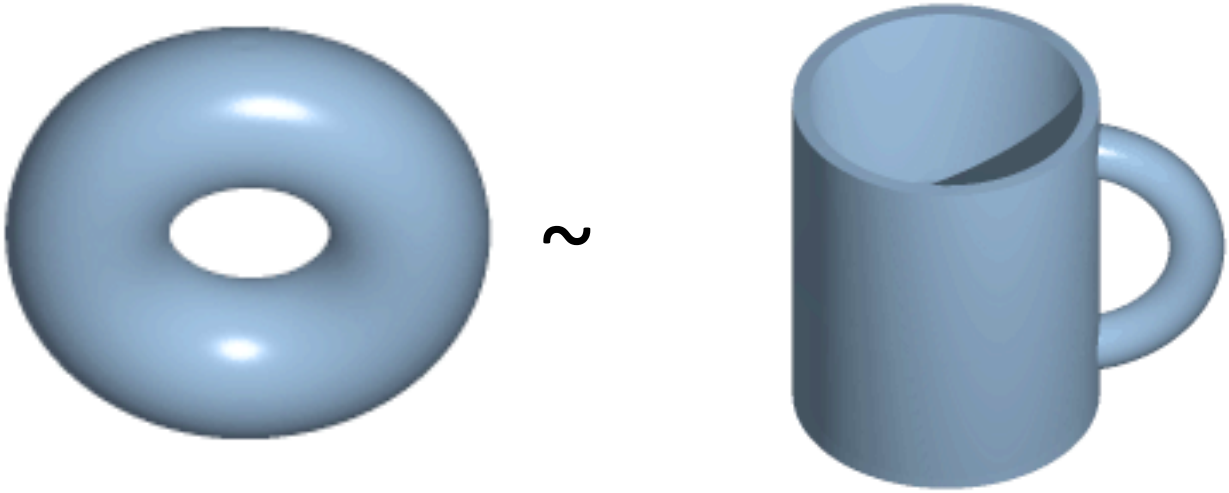
Topology studies shapes

A donut and coffee mug are “homotopy equivalent”, and considered to be the same shape. You can bend and stretch (but not tear) one to get the other.



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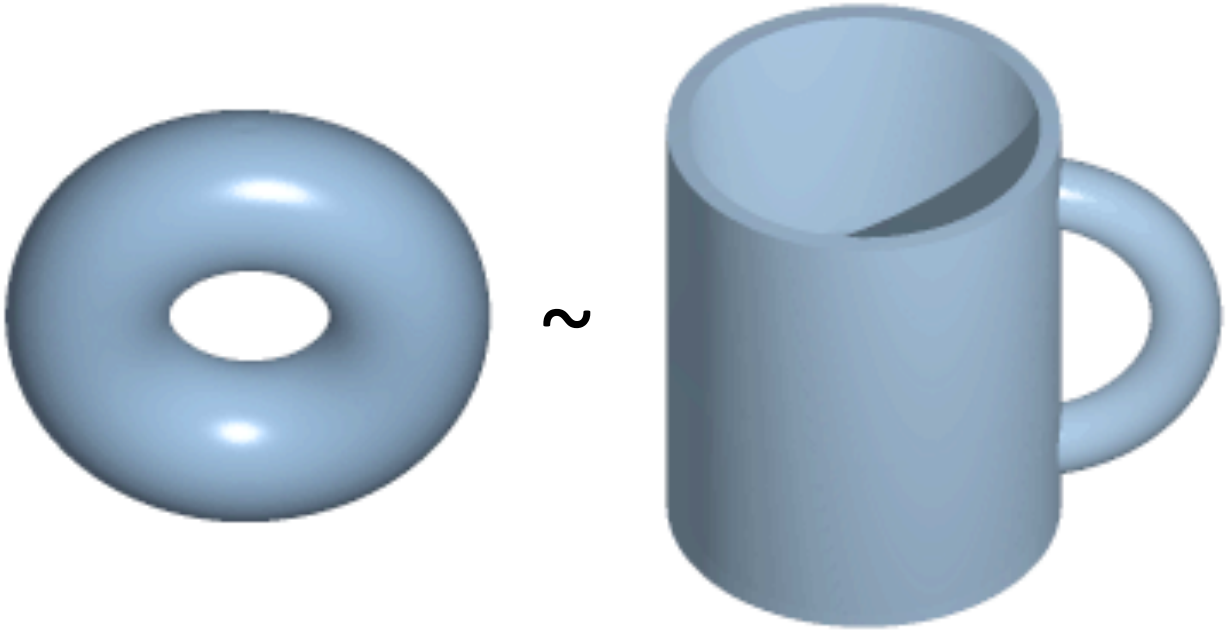
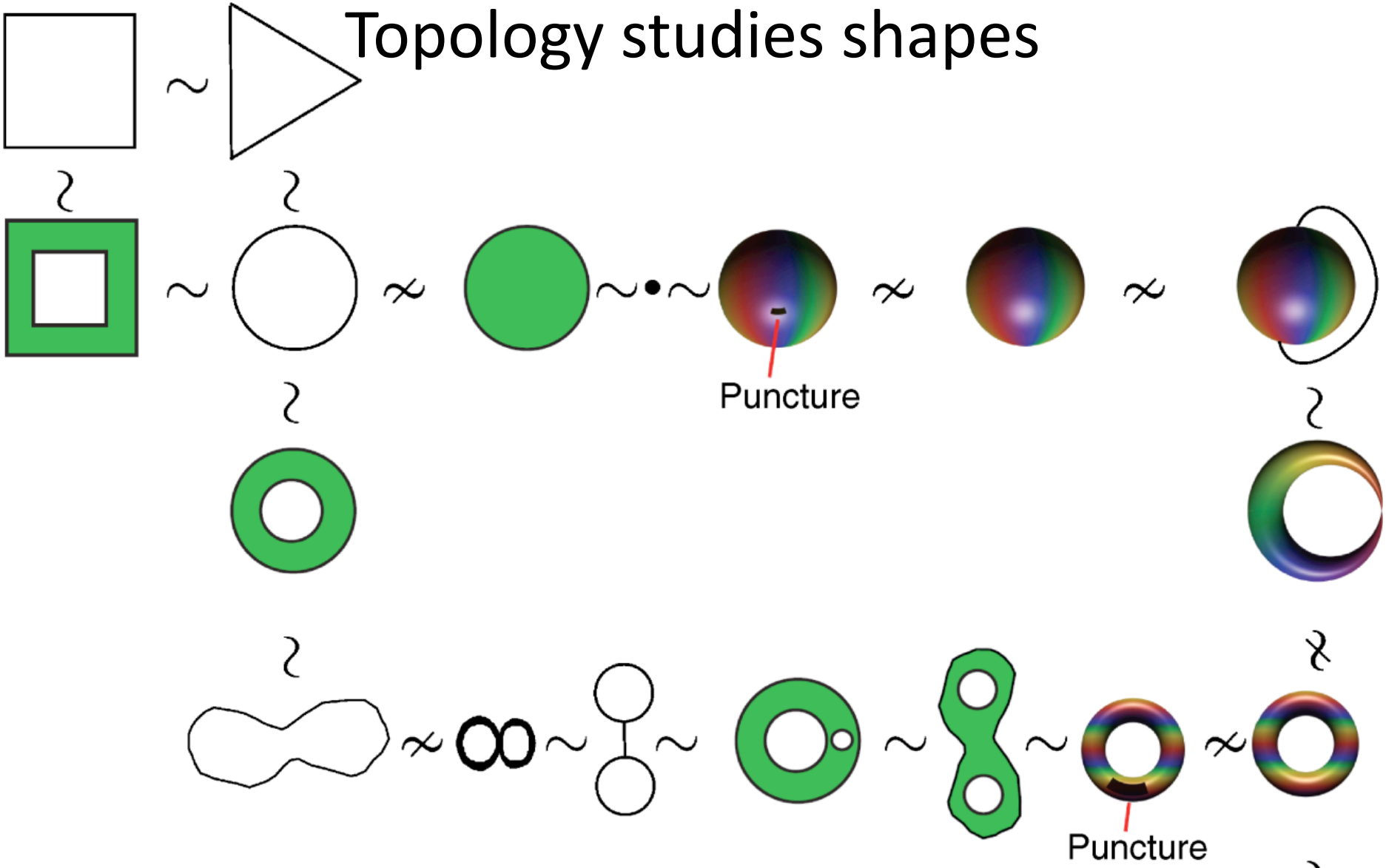
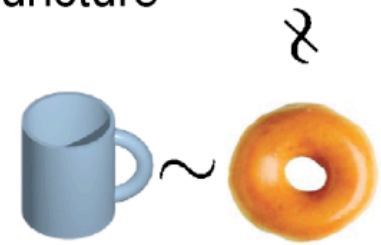


Topology studies shapes

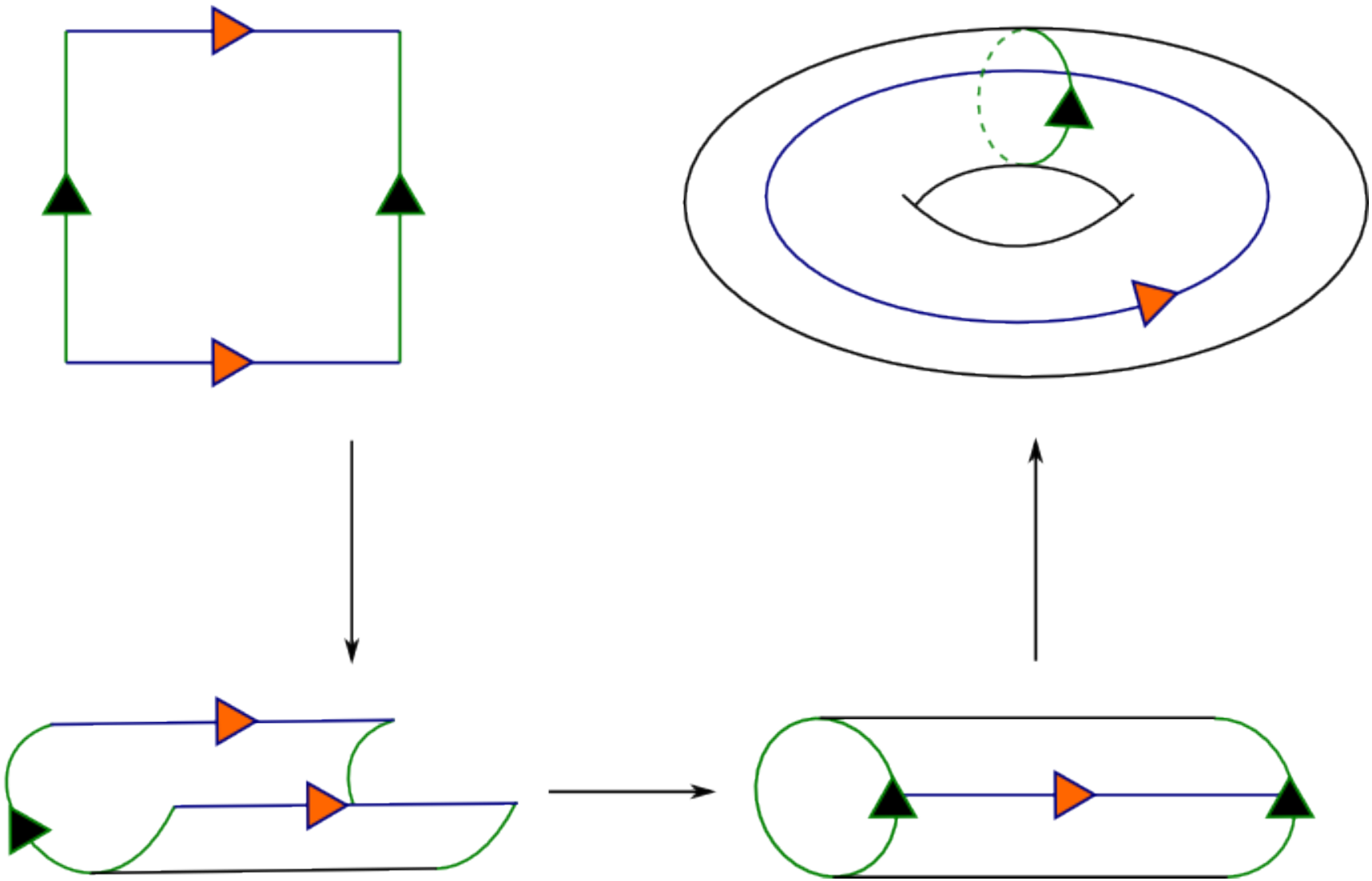


Topological Analysis of Population Activity in Visual Cortex by Singh, Memoli, Ishkhanov, Sapiro, Carlsson, and Ringach, 2008.



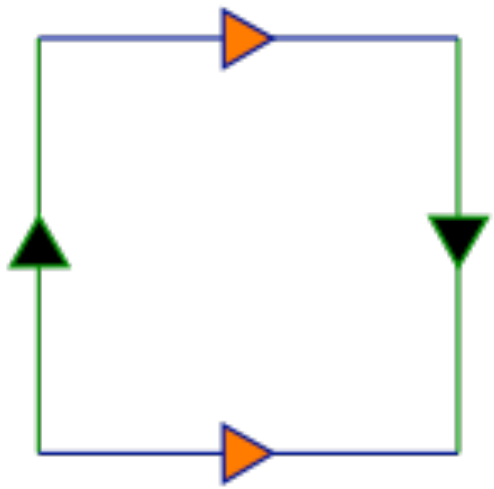
Topology studies shapes

Torus



Topology studies shapes

Klein bottle



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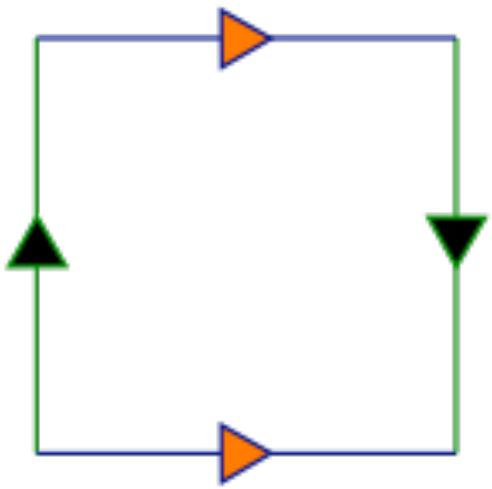

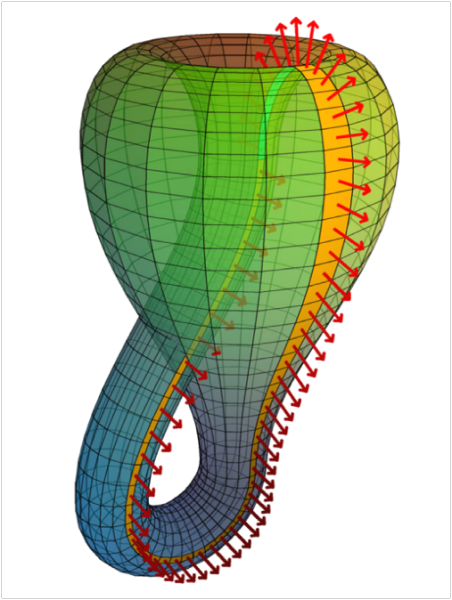

https://plus.maths.org/content/imaging-maths-inside-klein-bottle

Topology studies shapes

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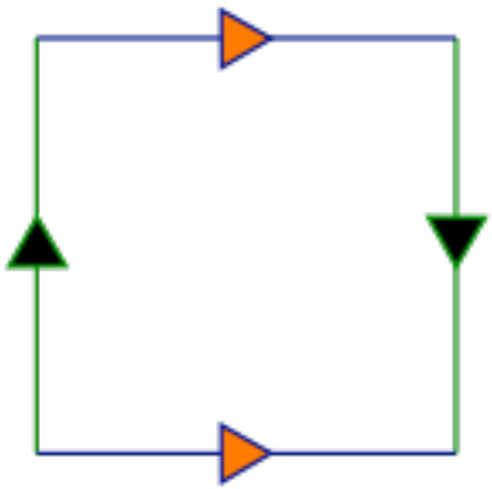

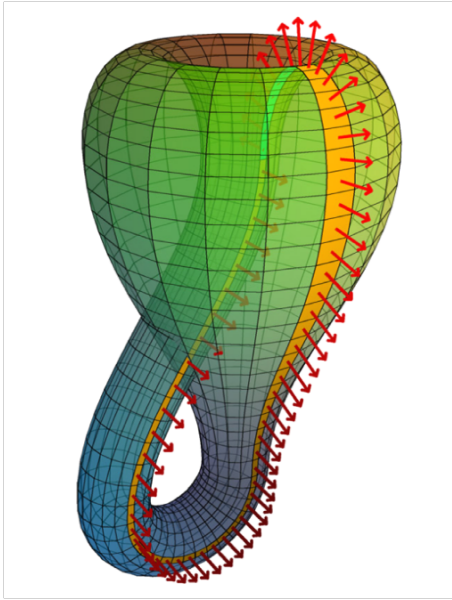

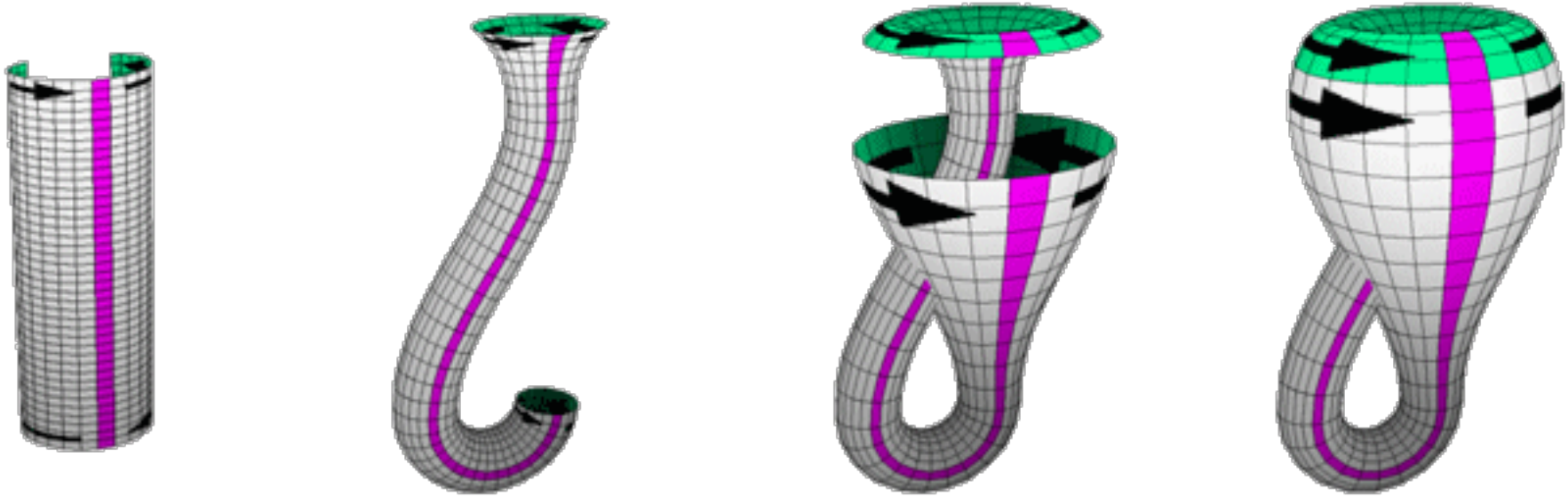

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Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\alpha_{n+1}"] \arrow[d, "\partial_{n+1}"'] & B_{n+1} \arrow[r, "\beta_{n+1}"] \arrow[d, "\partial'_{n+1}"'] & C_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"'] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\alpha_n"] \arrow[d, "\partial_n"'] & B_n \arrow[r, "\beta_n"] \arrow[d, "\partial'_n"'] & C_n \arrow[r] \arrow[d, "\partial''_n"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

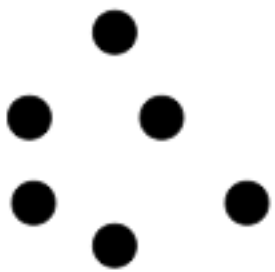
$$
\begin{tikzcd}
 & & \vdots \arrow[dl] & & \\
H_{n+1}(\mathcal{A}) \arrow[r, "\alpha_*"'] & H_{n+1}(\mathcal{B}) \arrow[r, "\beta_*"] & H_{n+1}(\mathcal{C}) \arrow[dll, "\delta_{n+1}"] & & \\
H_n(\mathcal{A}) \arrow[r, "\alpha_*"'] & H_n(\mathcal{B}) \arrow[r, "\beta_*"] & H_n(\mathcal{C}) \arrow[dll, "\delta_n"] & & \\
H_{n-1}(\mathcal{A}) \arrow[r, "\alpha_*"'] & H_{n-1}(\mathcal{B}) \arrow[r, "\beta_*"] & H_{n-1}(\mathcal{C}) \arrow[dl] & & \\
 & & \vdots & &
\end{tikzcd}
$$

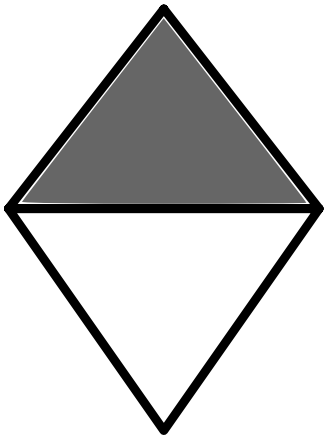
Topology studies shapes

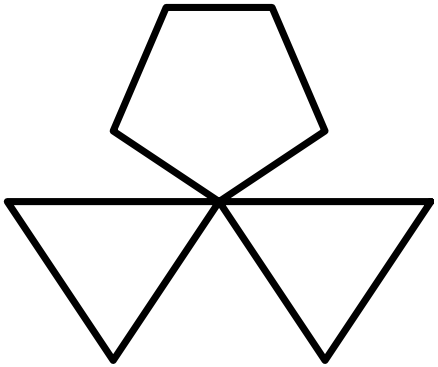
Homology groups $H_0, H_1, H_2, H_3, \ldots$ (over $\mathbb{Z}/2\mathbb{Z}$)

H_k "counts the number of k-dimensional holes".

Homotopy equivalent shapes have the same homology.







H_0 has rank 6.
H_1 has rank 0.
H_2 has rank 0.

H_0 has rank 1.
H_1 has rank 1.
H_2 has rank 0.

H_0 has rank 1.
H_1 has rank 3.
H_2 has rank 0.

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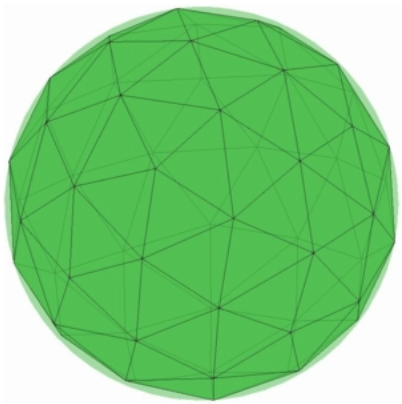
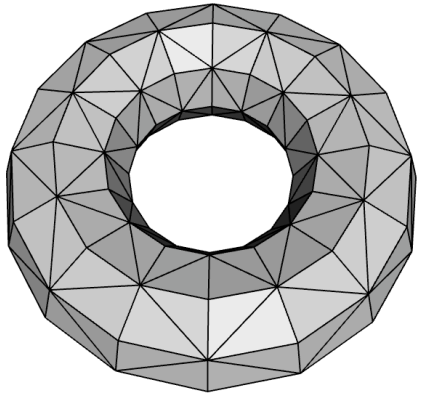


H_0 has rank 1.

H_1 has rank 0.

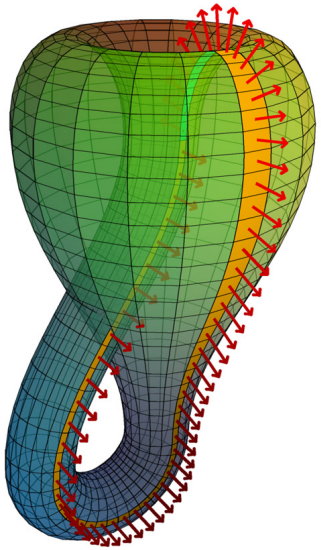
H_2 has rank 1.



H_0 has rank 1.

H_1 has rank 2.

H_2 has rank 1.



H_0 has rank 1.

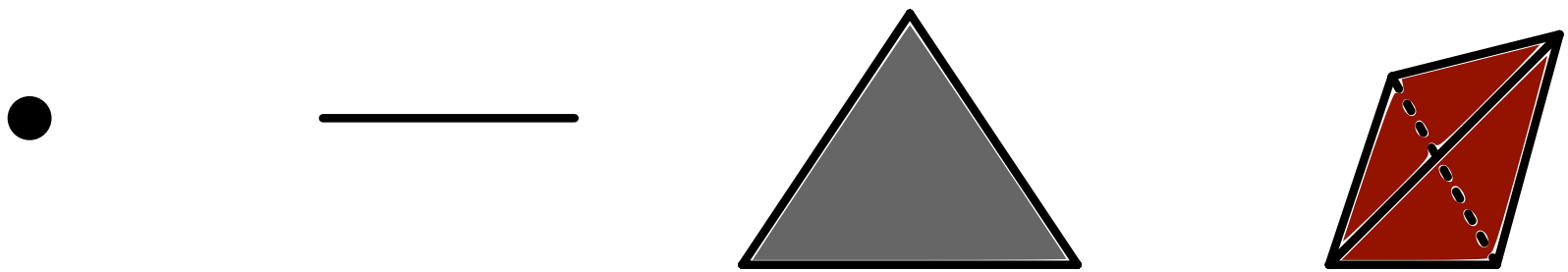
H_1 has rank 2.

H_2 has rank 1.

Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

0-simplex 1-simplex 2-simplex 3-simplex



Topology studies shapes

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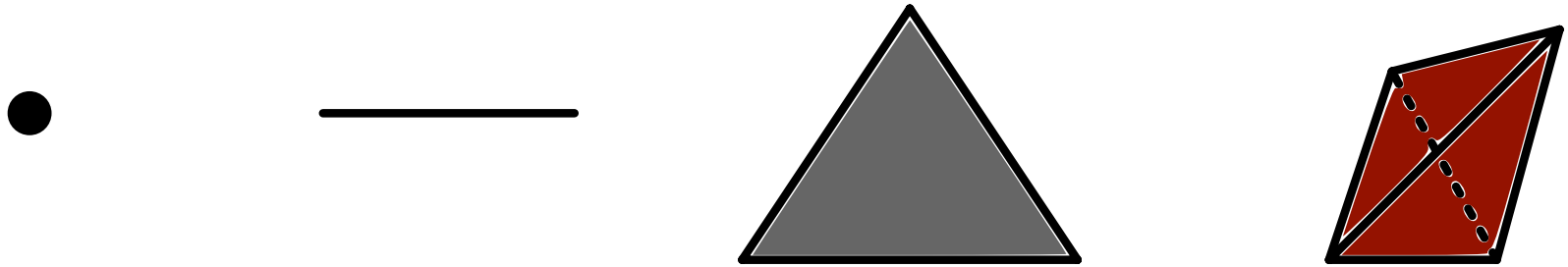
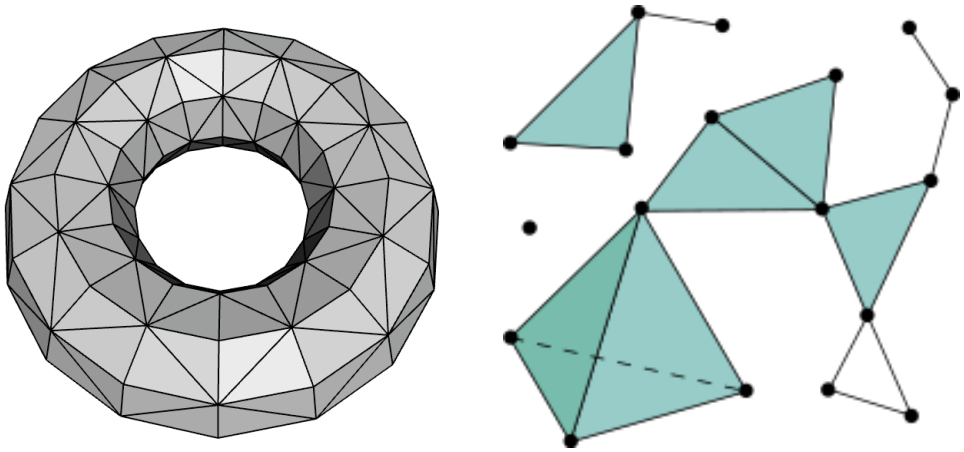


Simplicial complexes



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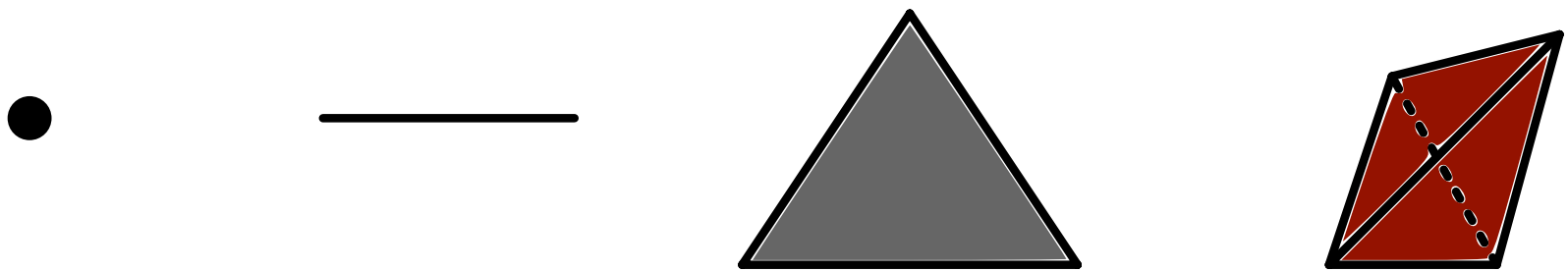

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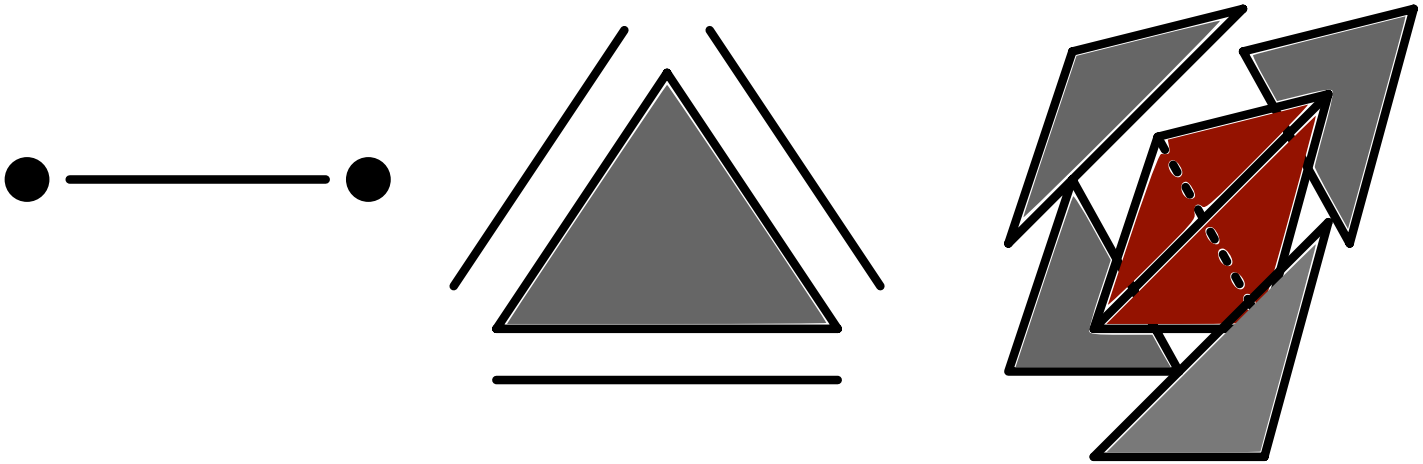
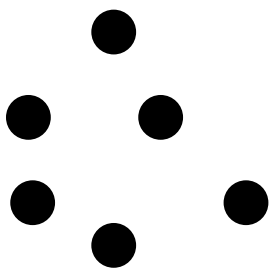


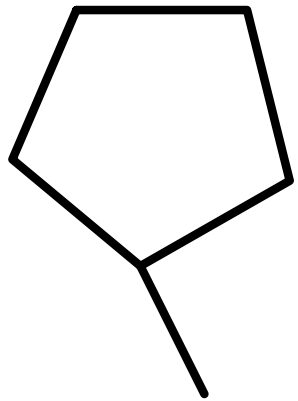
Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

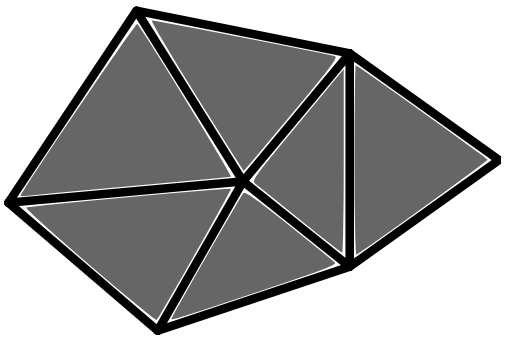
0-simplices



1-simplices



2-simplices

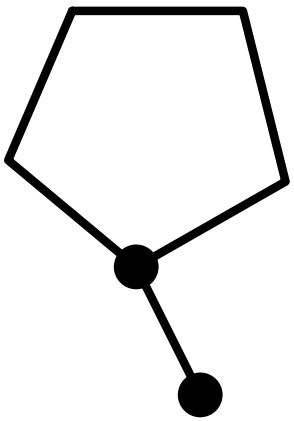


Topology studies shapes

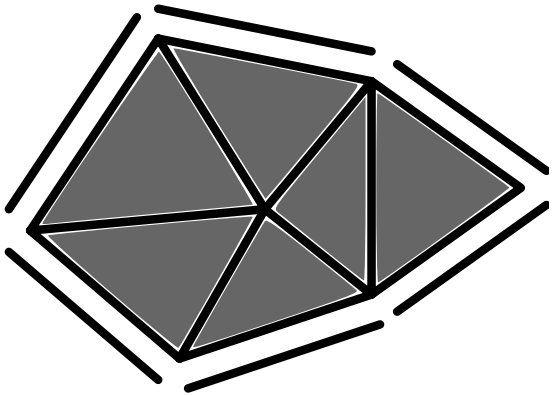
Homology ($\mathbb{Z}/2\mathbb{Z}$)

0-simplices

1-simplices



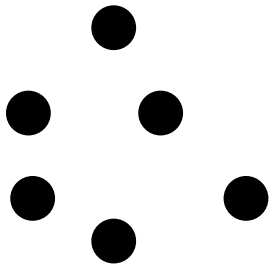
2-simplices



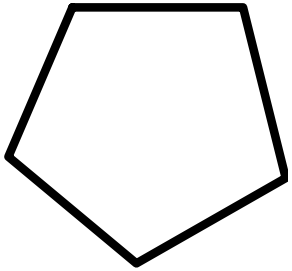
Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

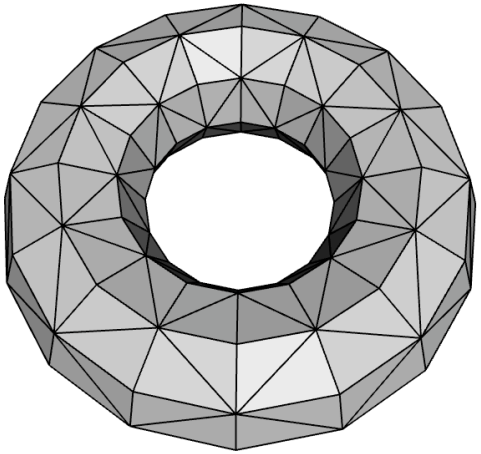
0-cycle



1-cycle



2-cycle

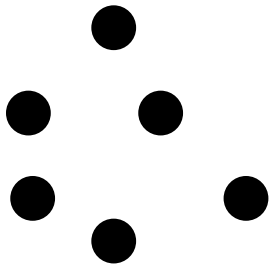


A cycle has no boundary.

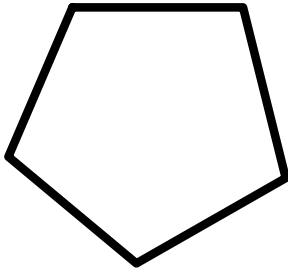
Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)

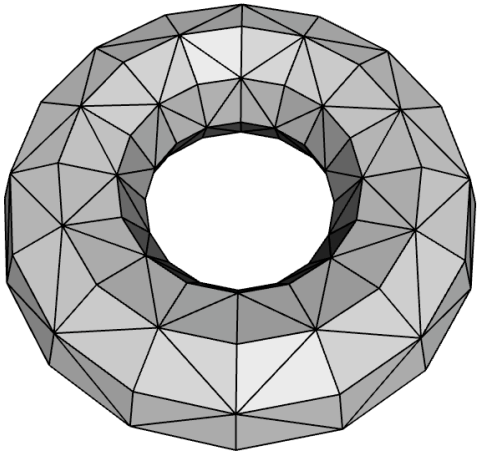
0-cycle



1-cycle



2-cycle



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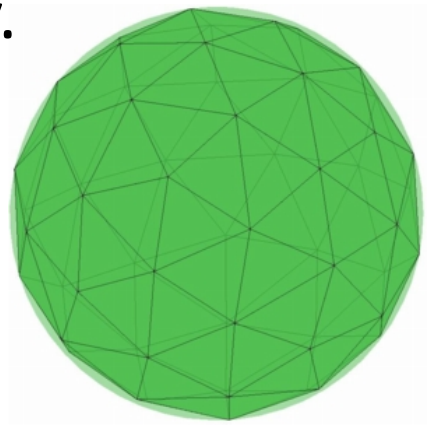
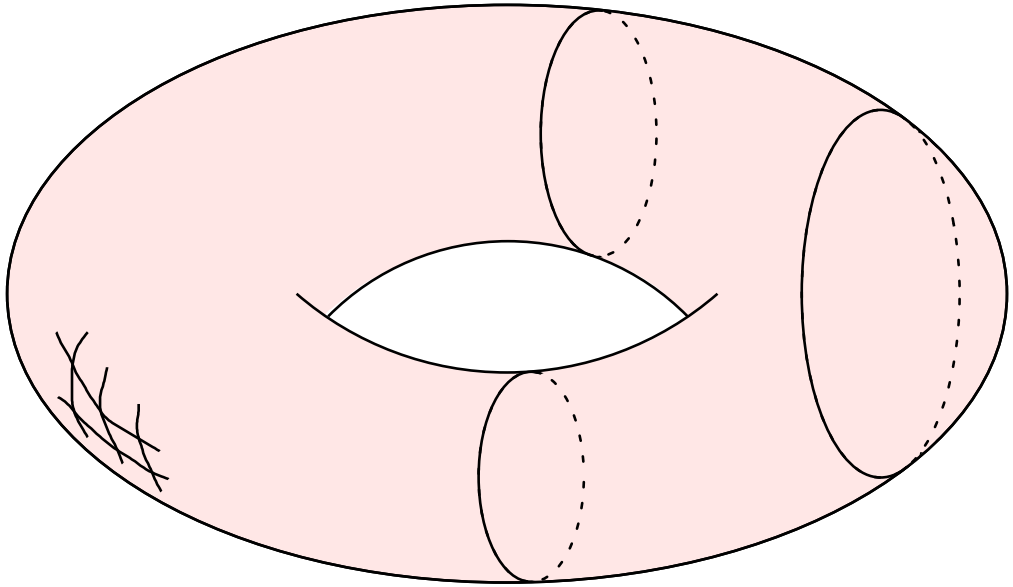


Topology studies shapes

Homology ($\mathbb{Z}/2\mathbb{Z}$)



Two cycles are equivalent if they differ by a boundary.

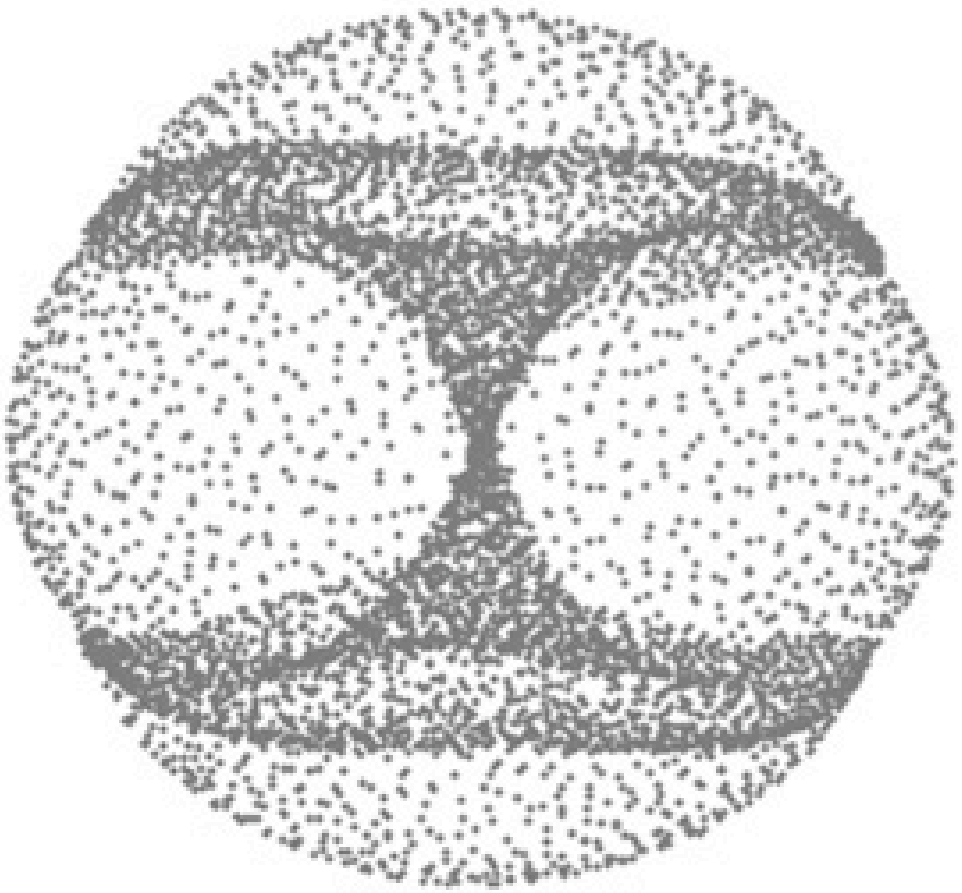
H_k measures equivalence classes of k-cycles.

“Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone …”

-Aleksandr Solzhenitsyn, *The First Circle*

Topology applied to data: Persistent homology

What shape is this?



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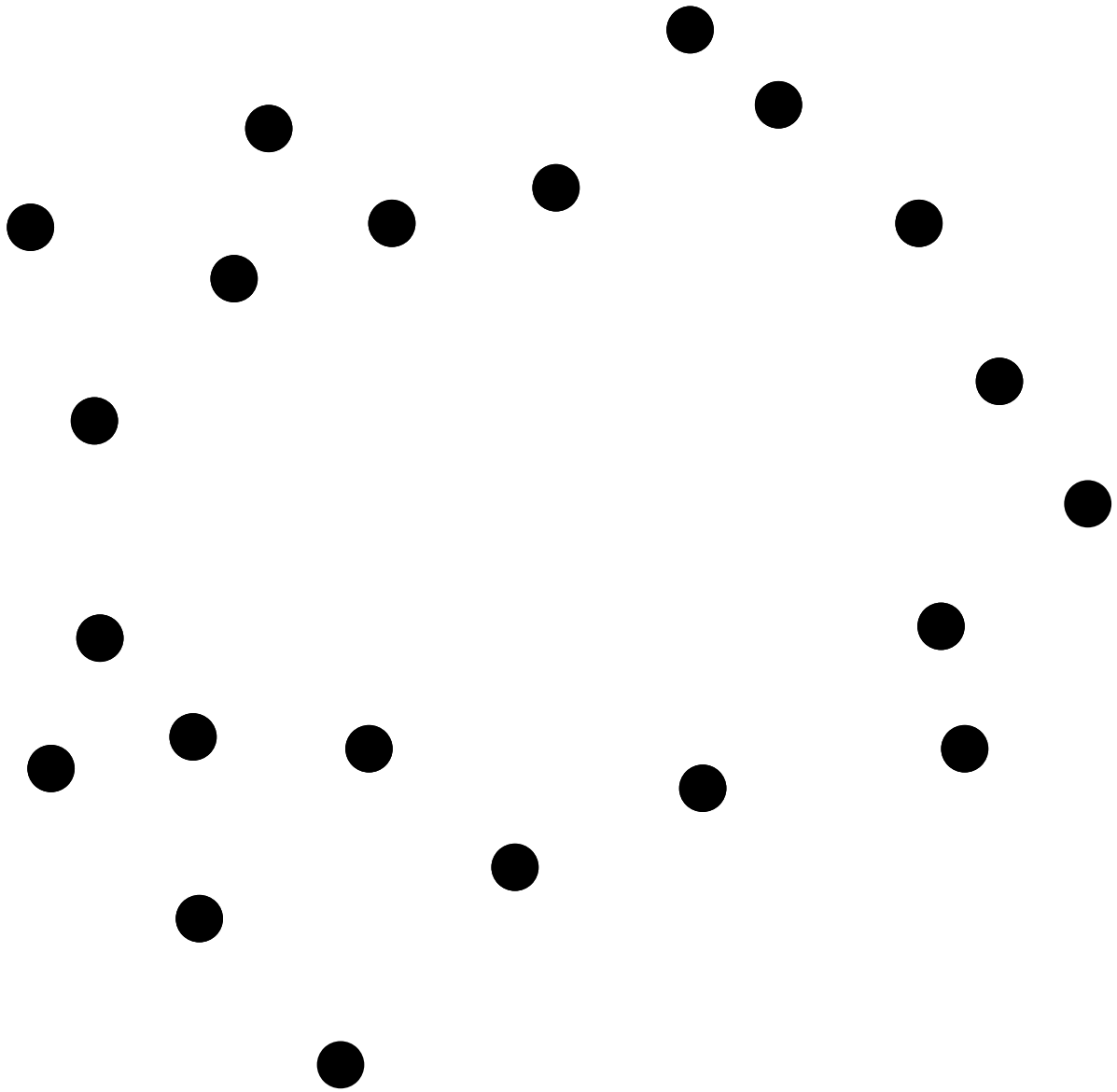

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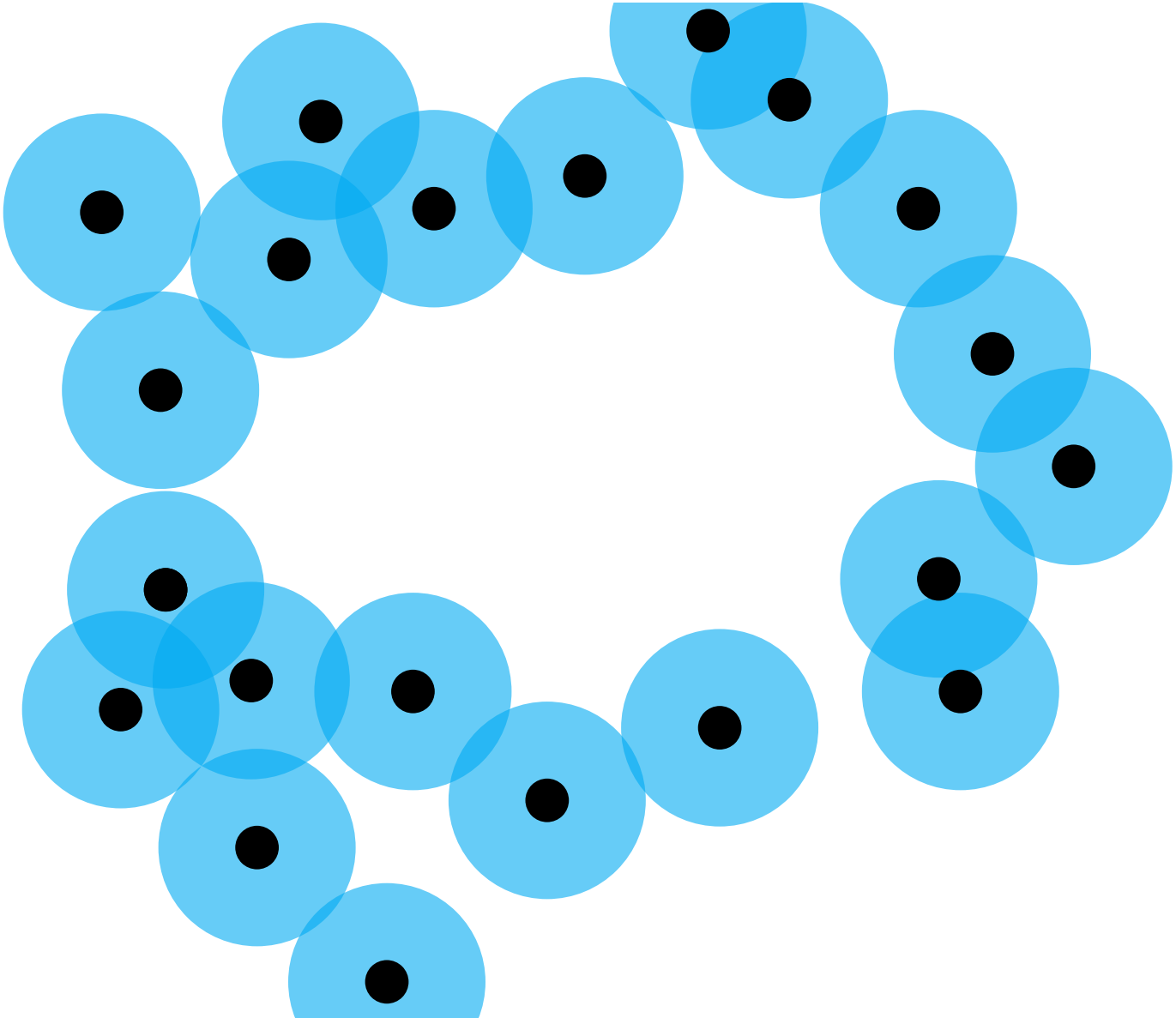

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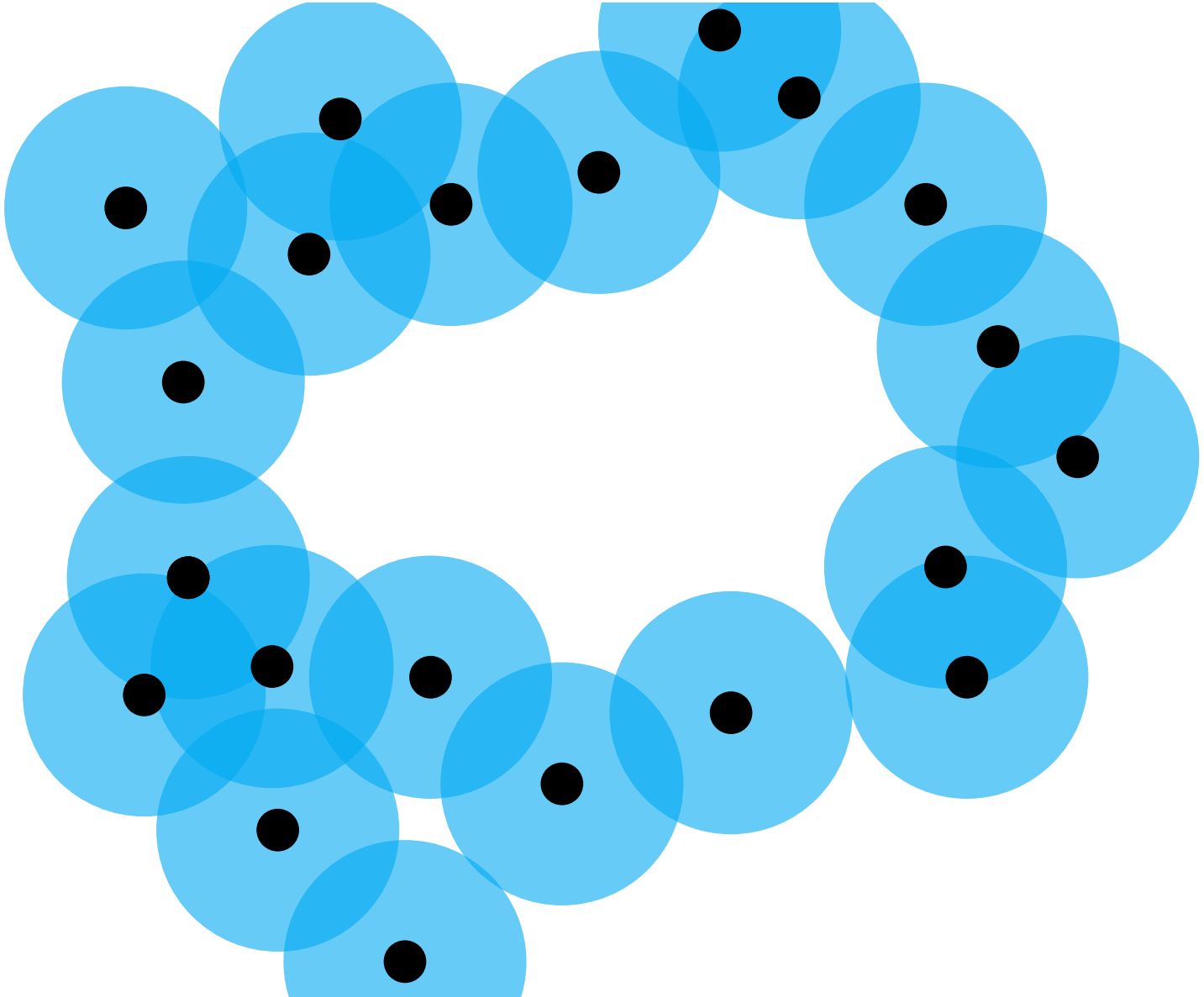

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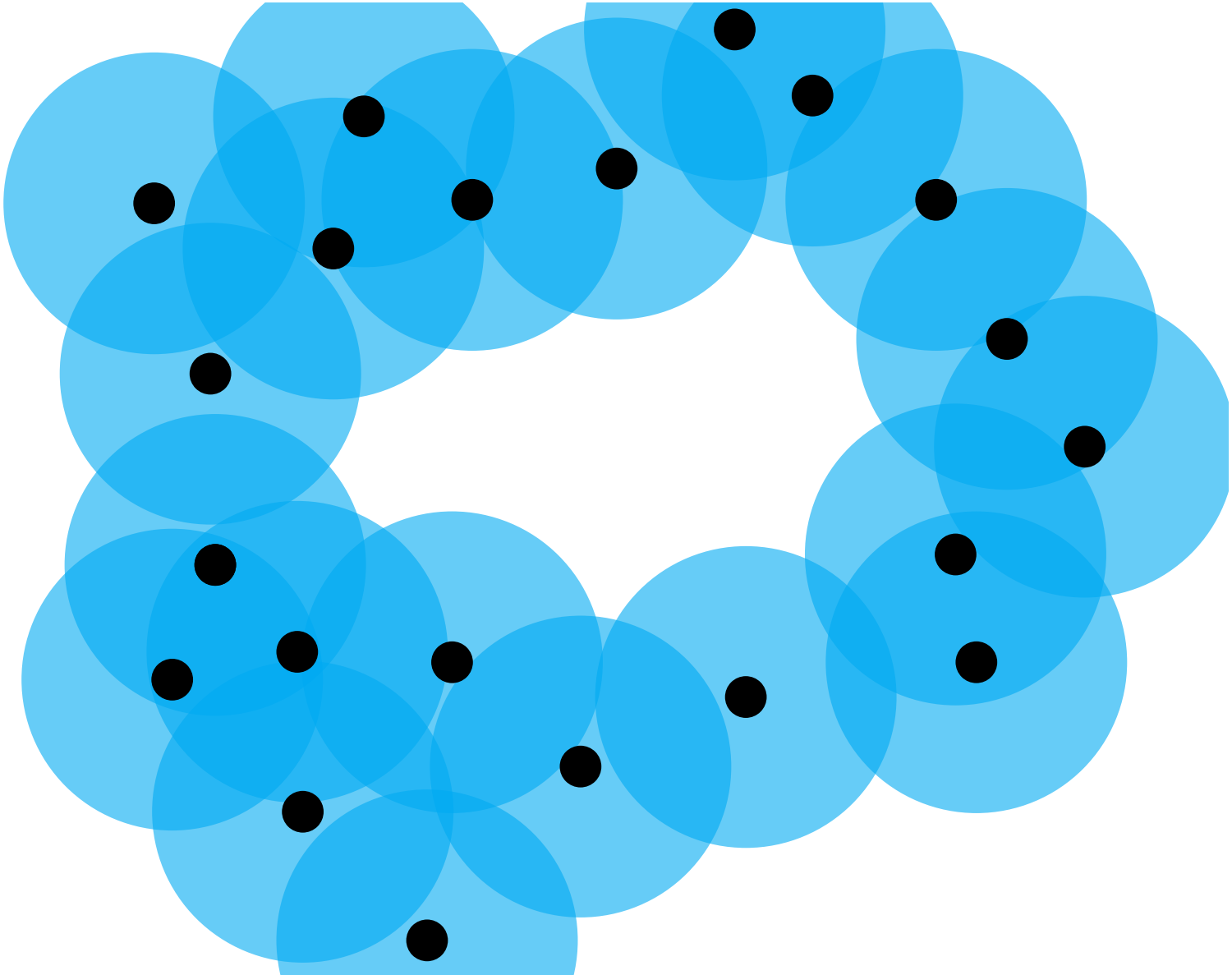

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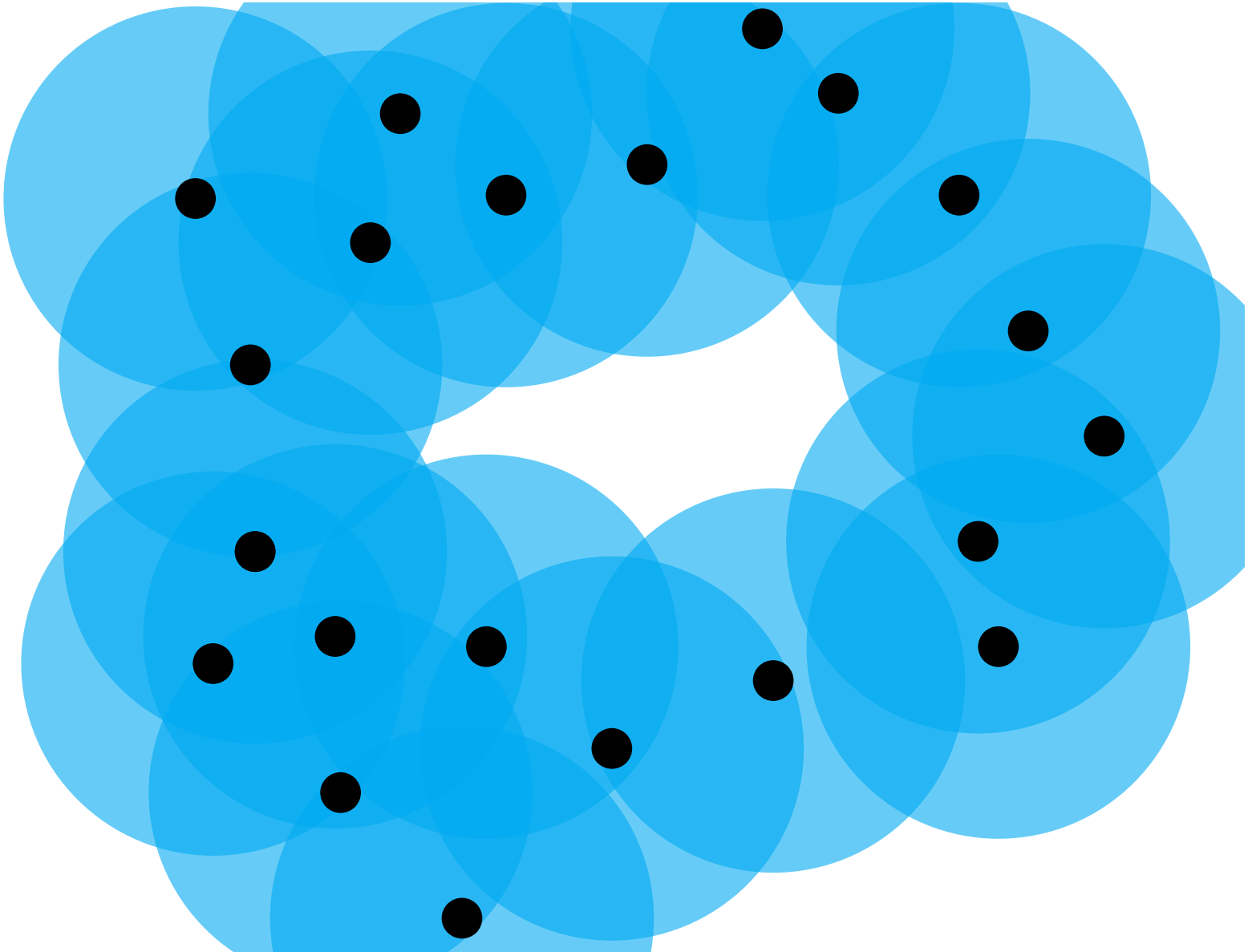

Topology applied to data: Persistent homology

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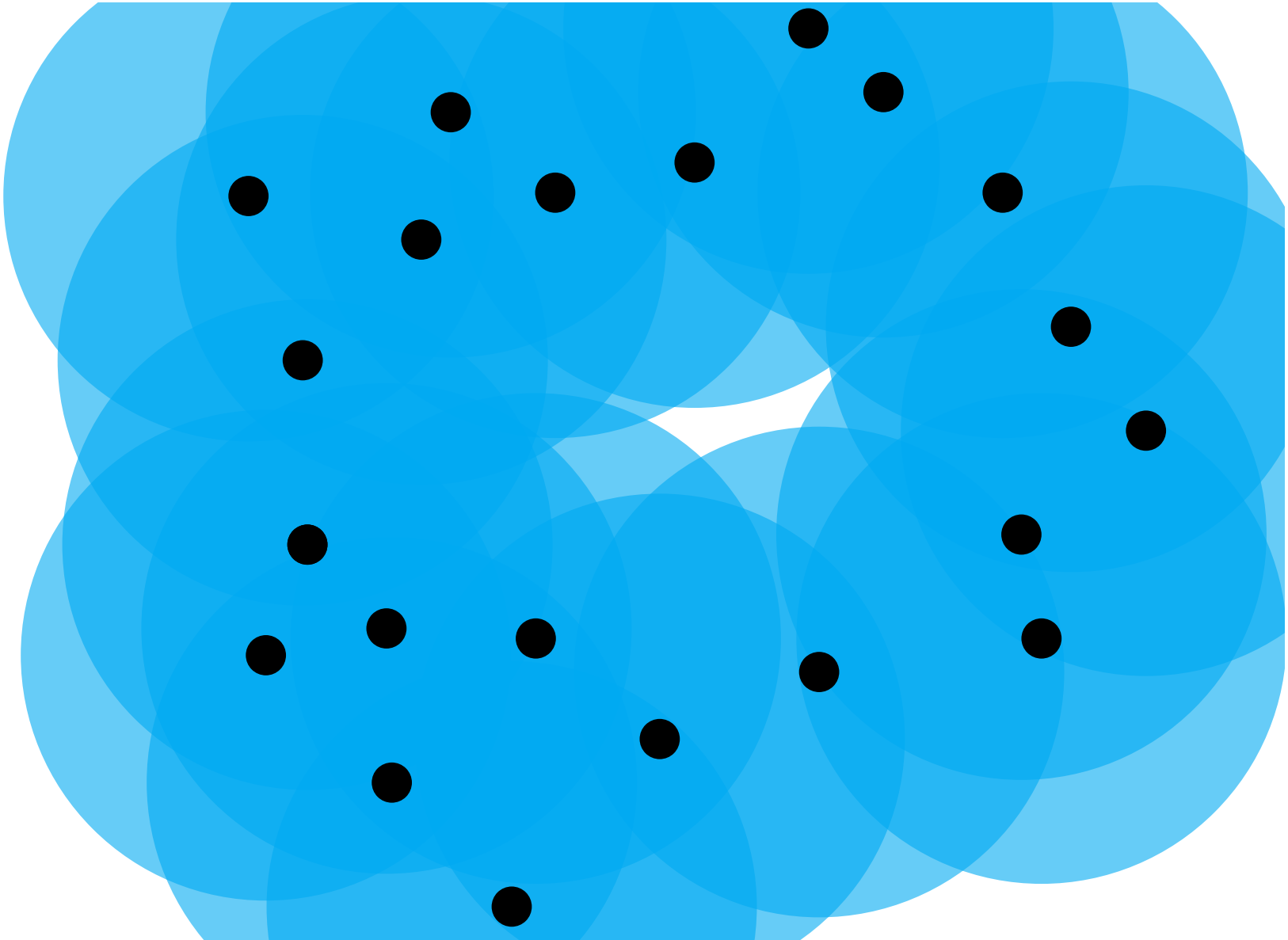

Topology applied to data: Persistent homology

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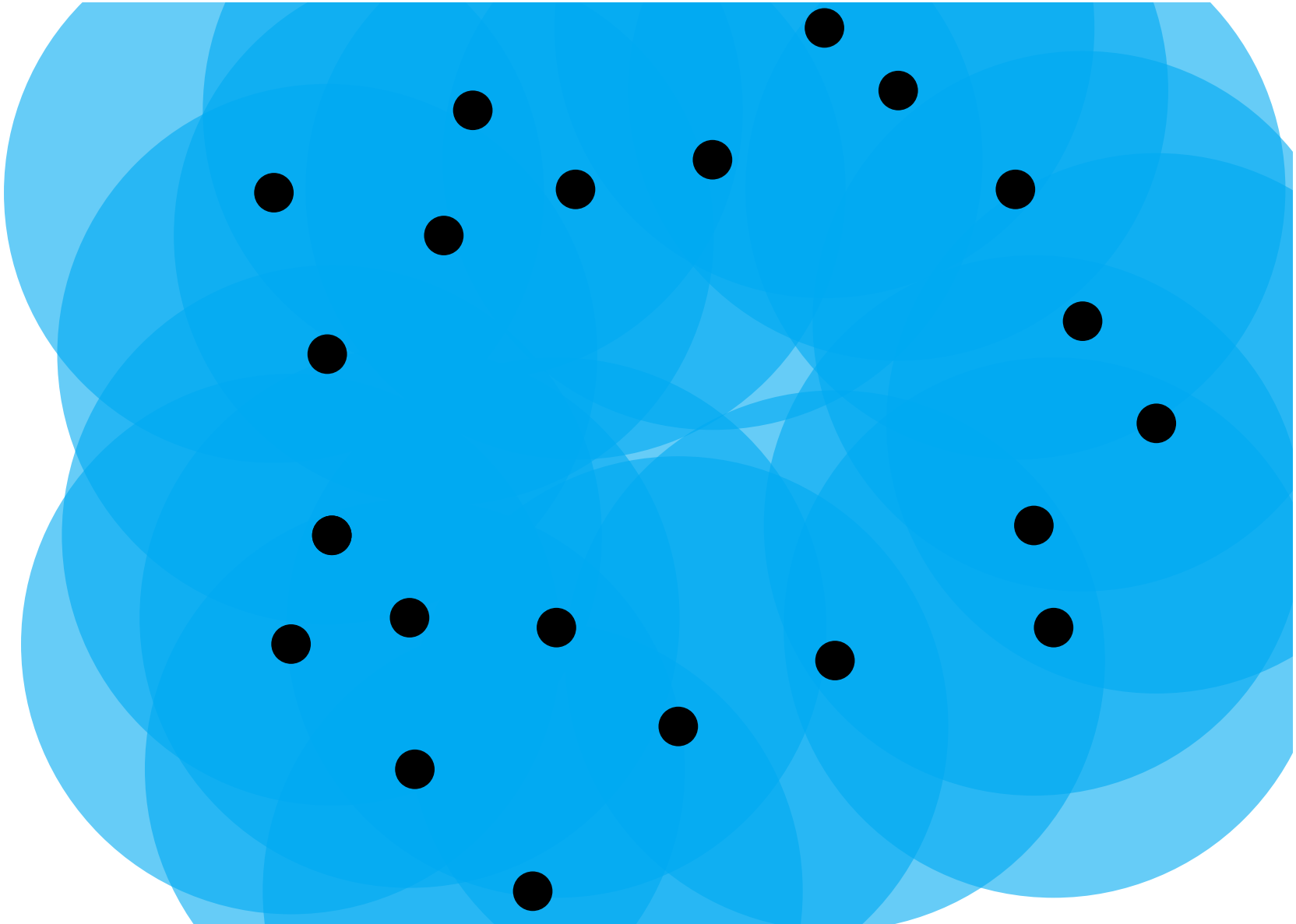
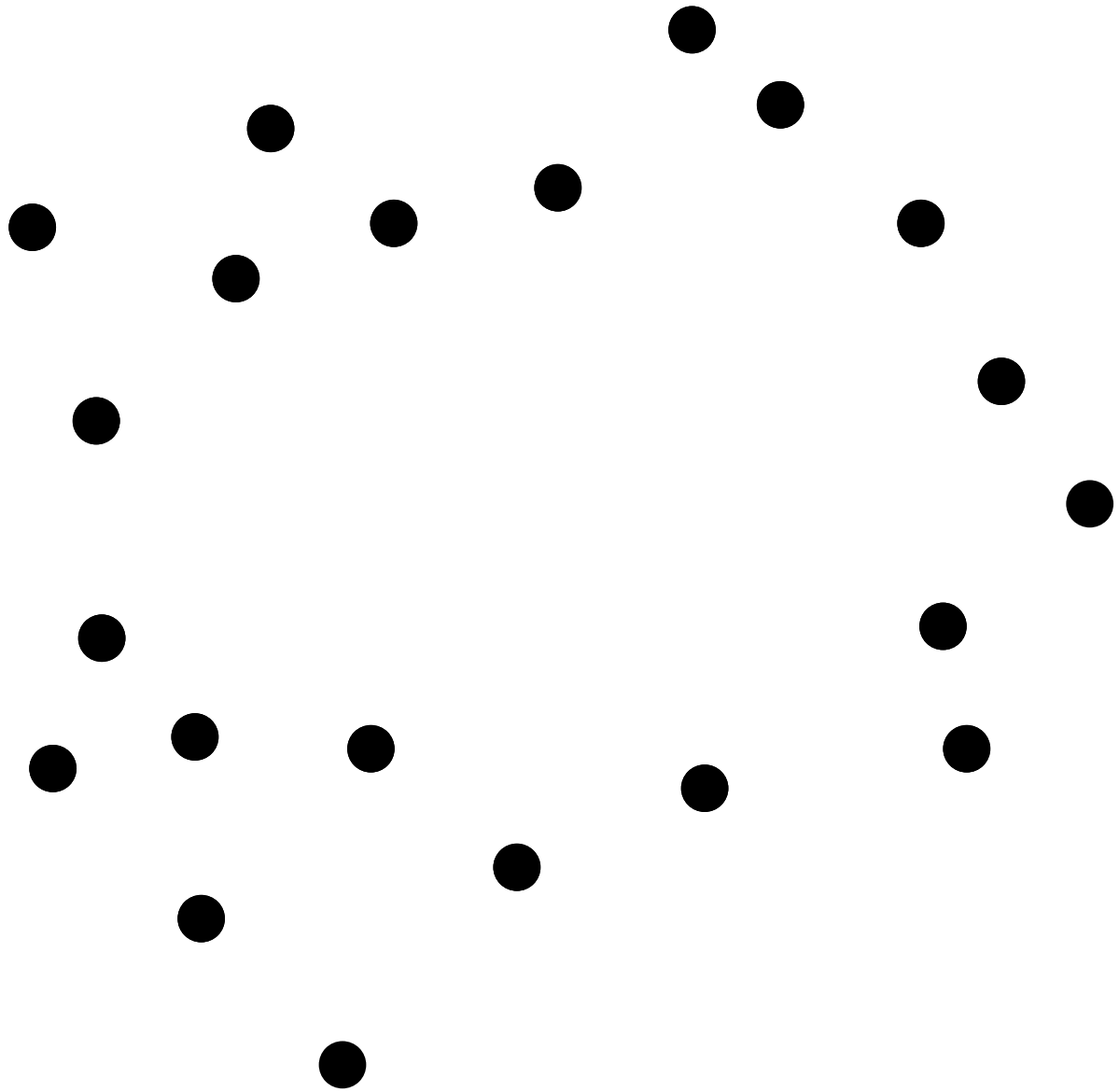


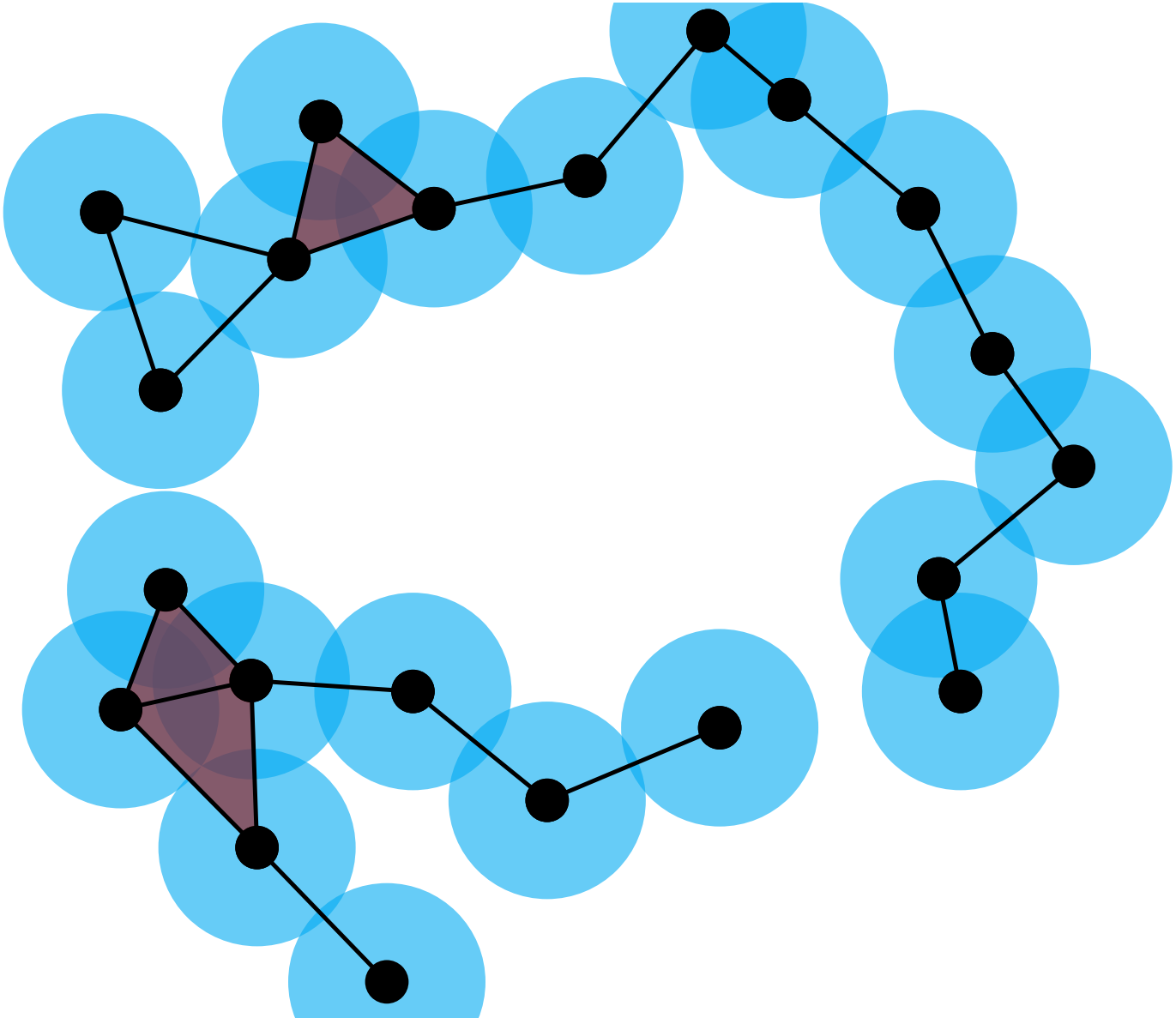
Topology applied to data: Persistent homology

Čech simplicial complex ~ union of balls



Topology applied to data: Persistent homology

Čech simplicial complex ~ union of balls



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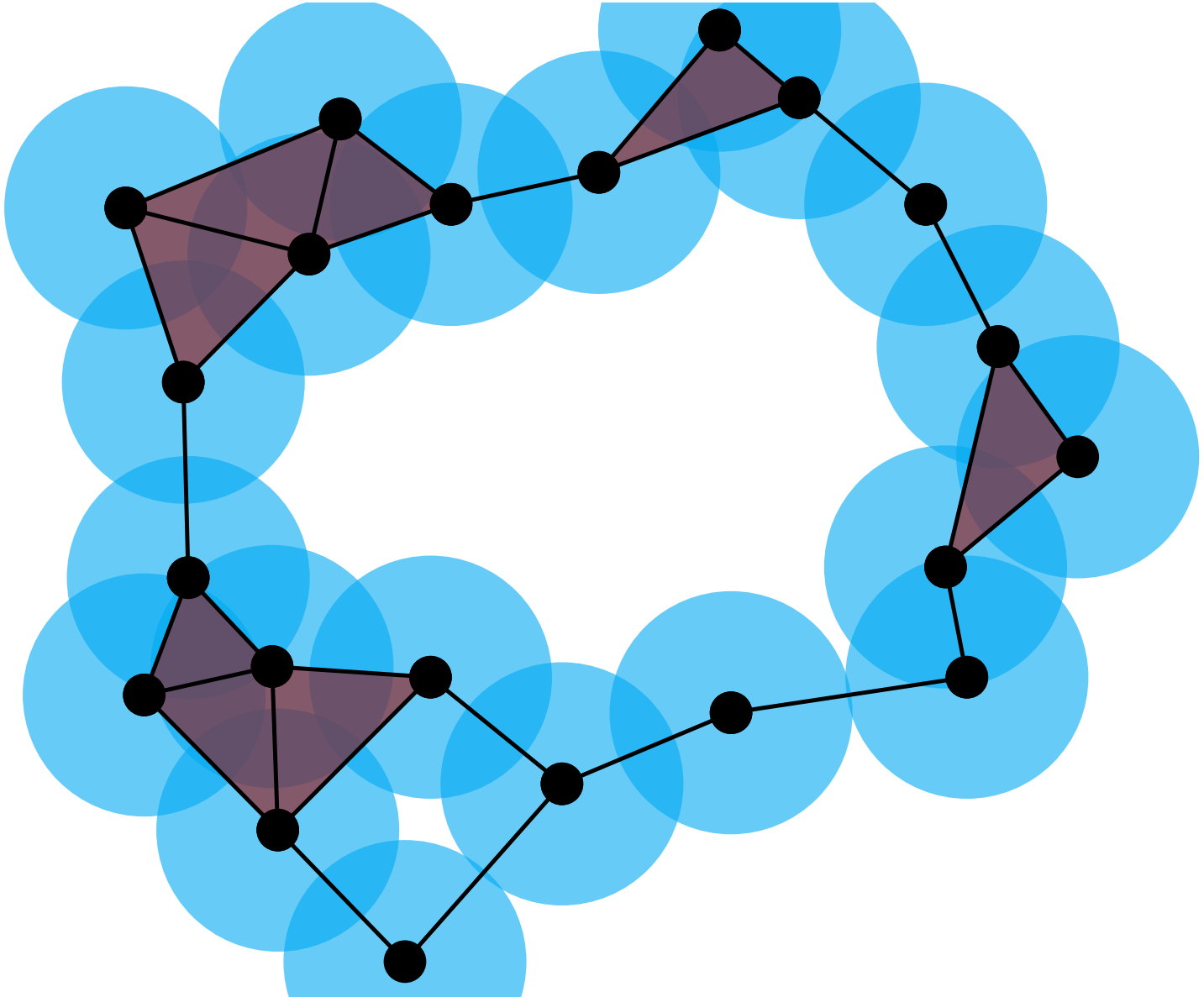

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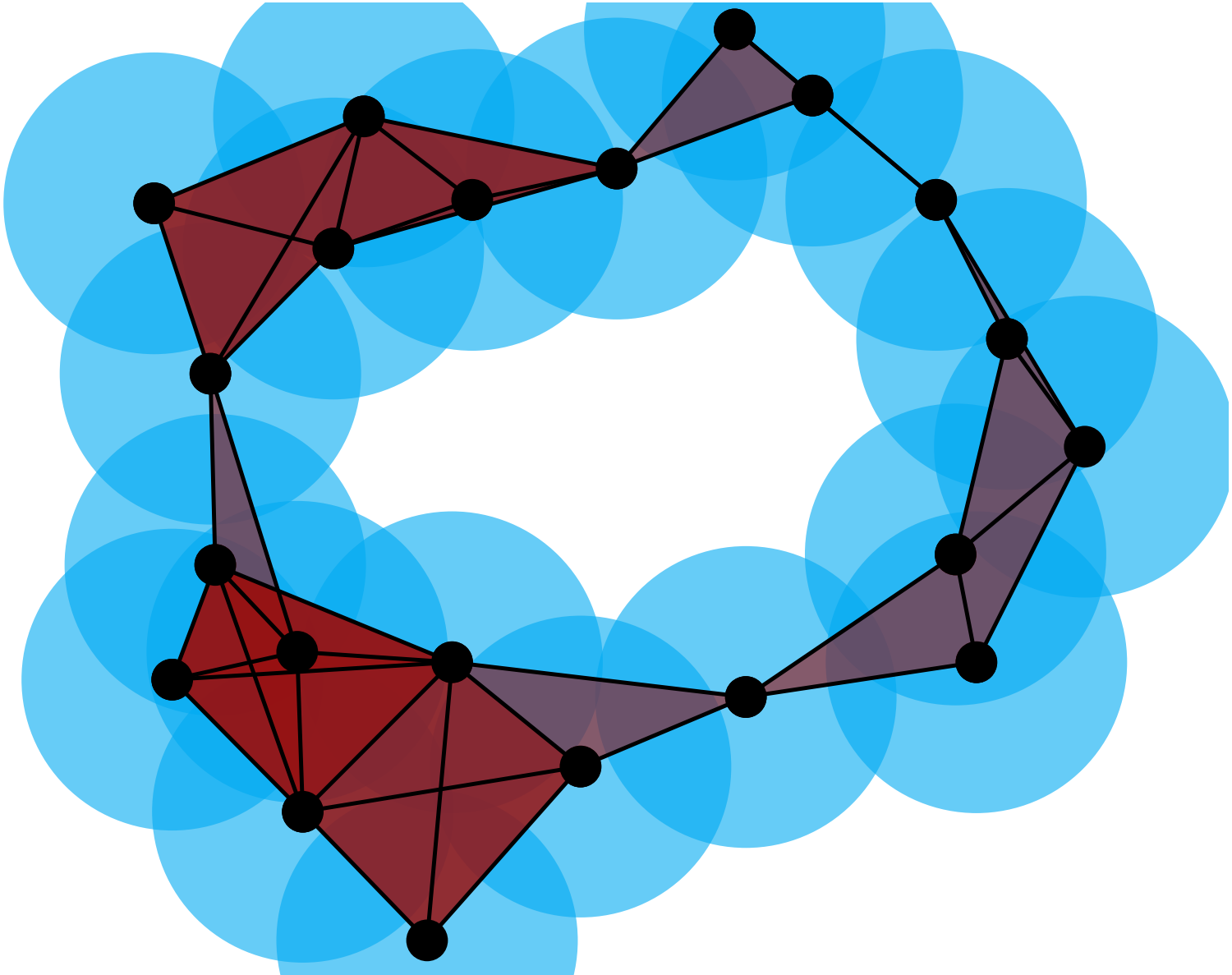

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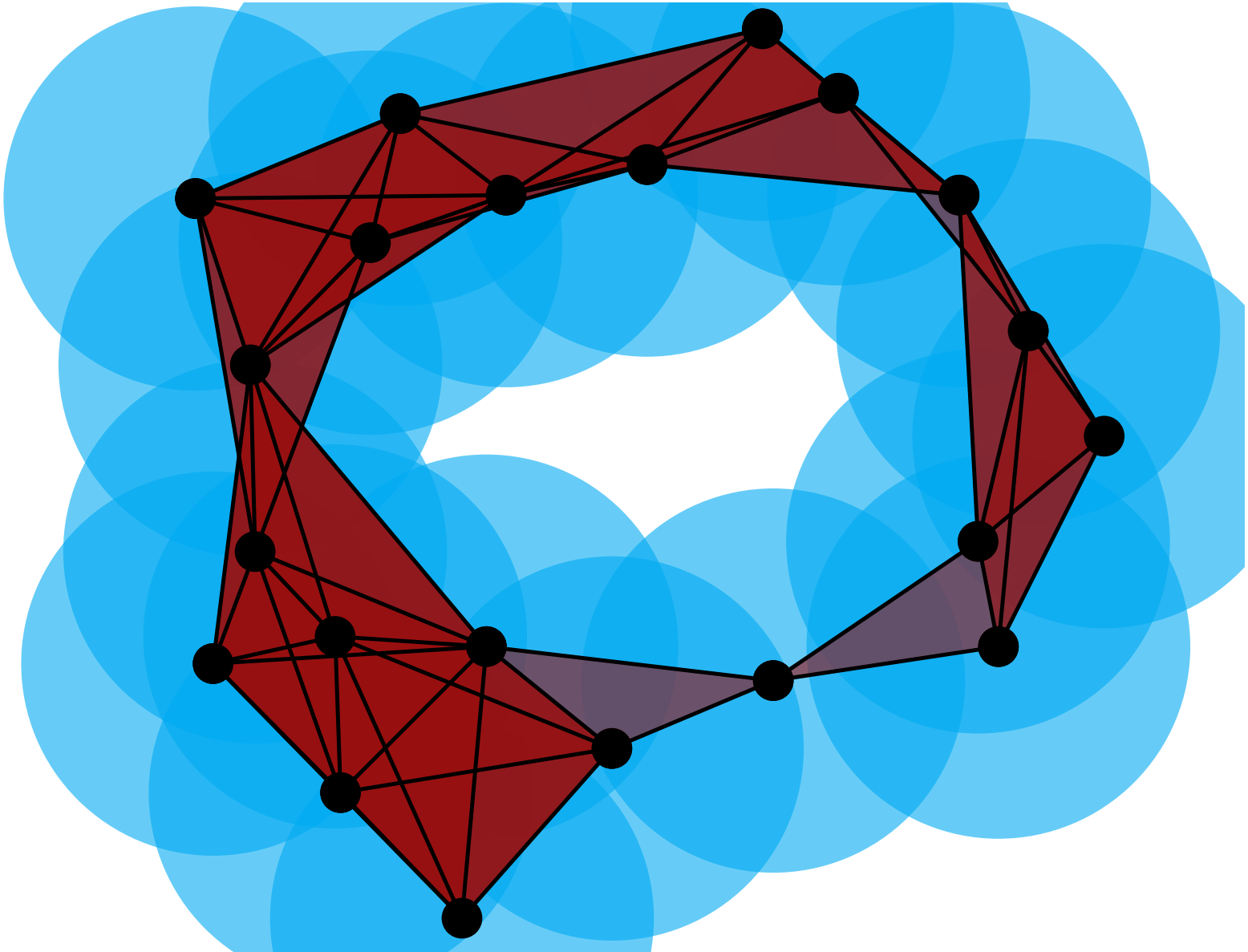

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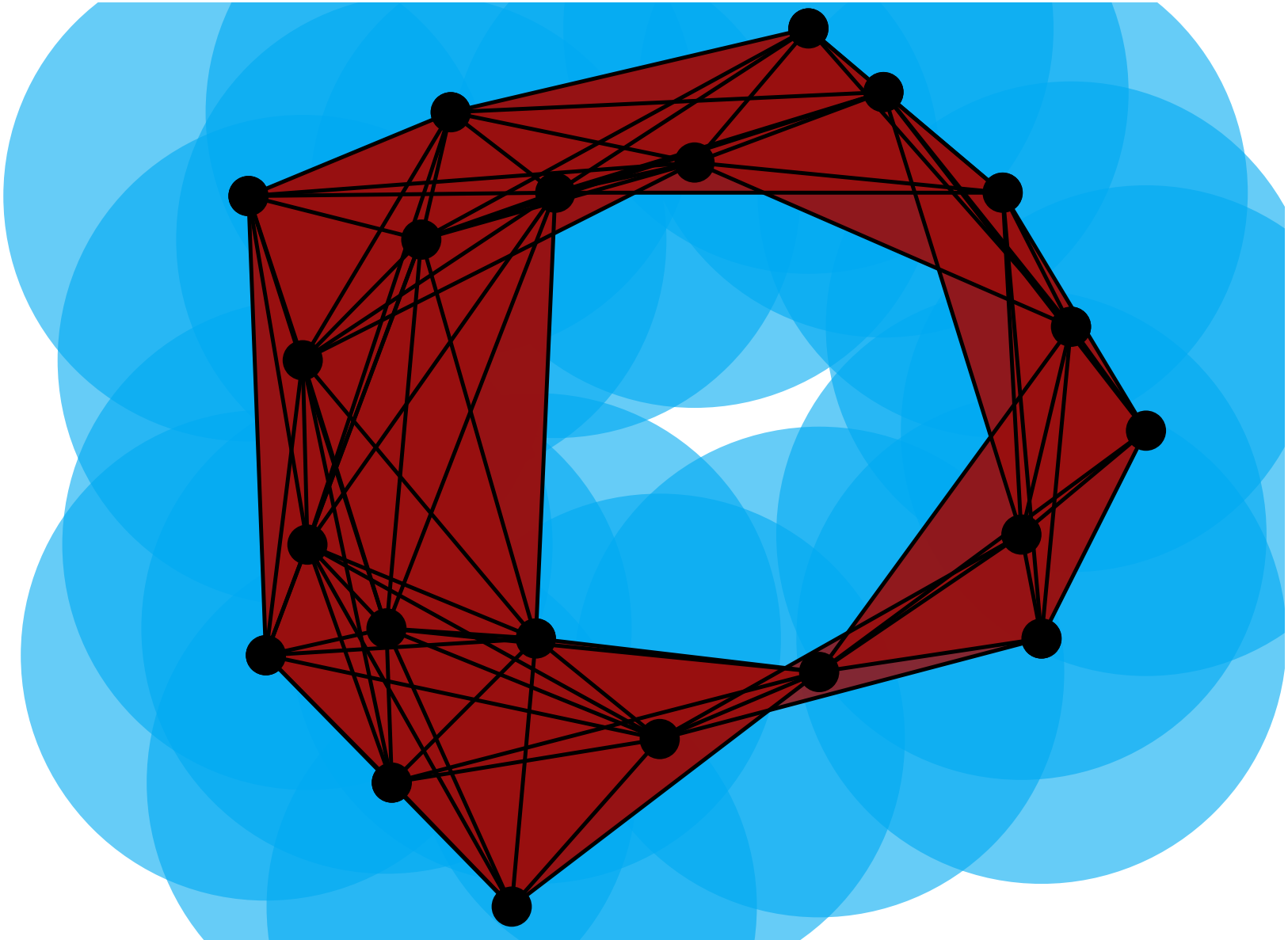

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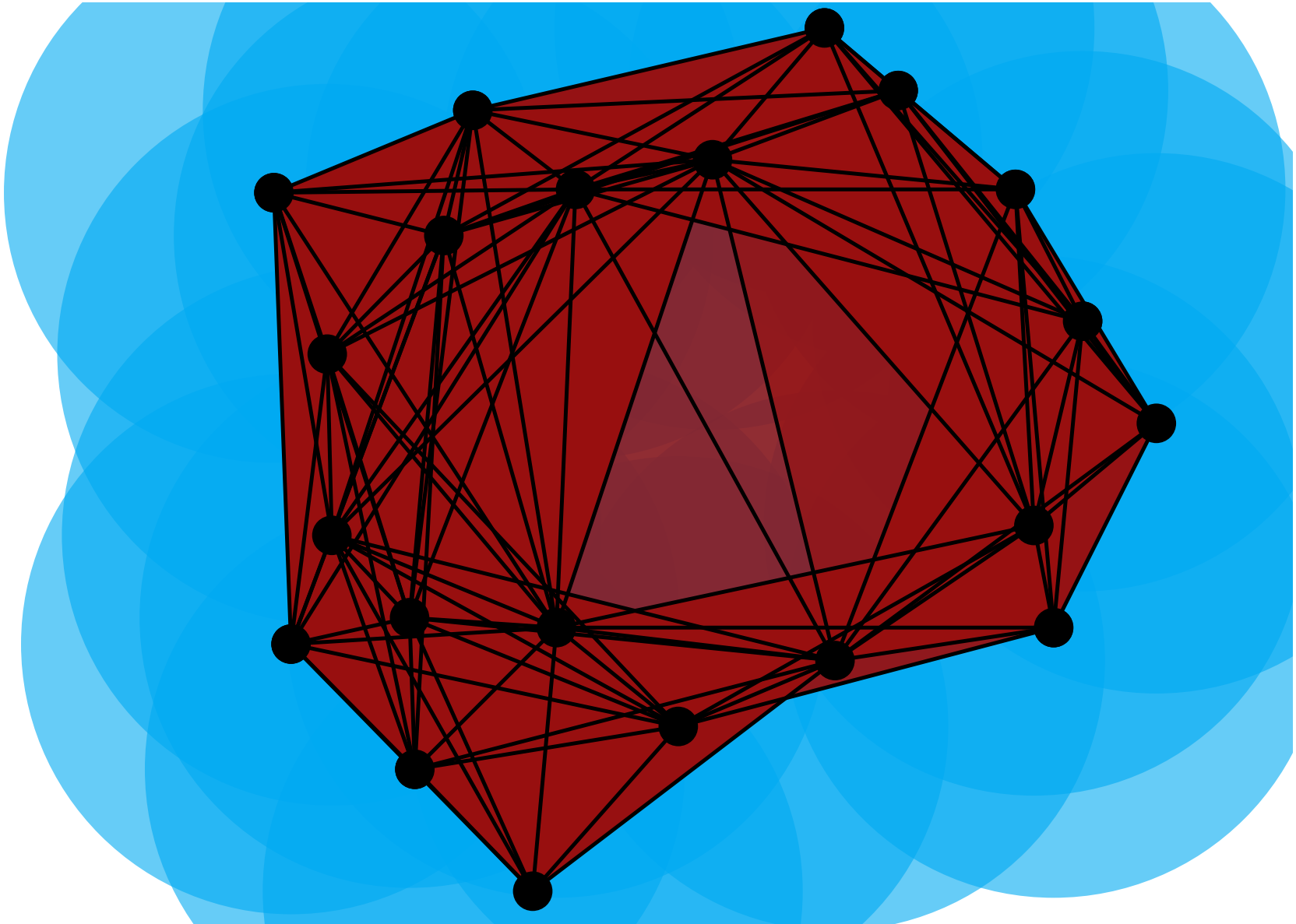

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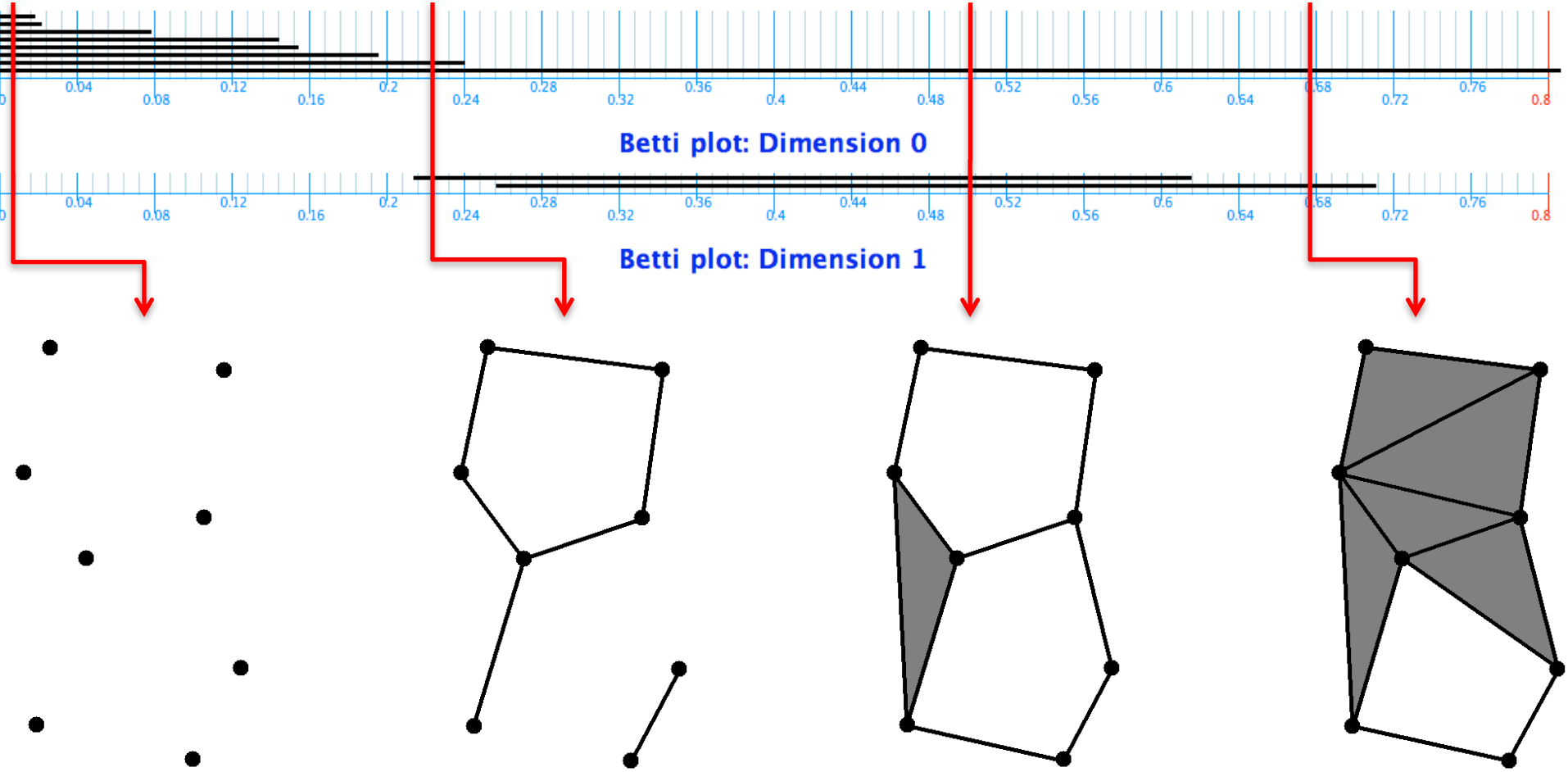
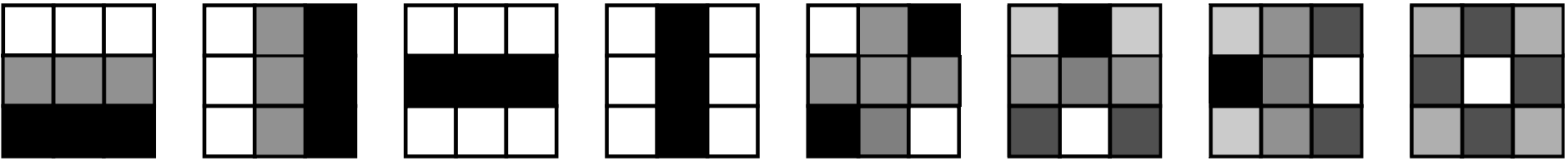


Significant features persist.

Topology applied to data: Persistent homology

Example: 3x3 high-contrast patches from images

Points in 9-dimensional space



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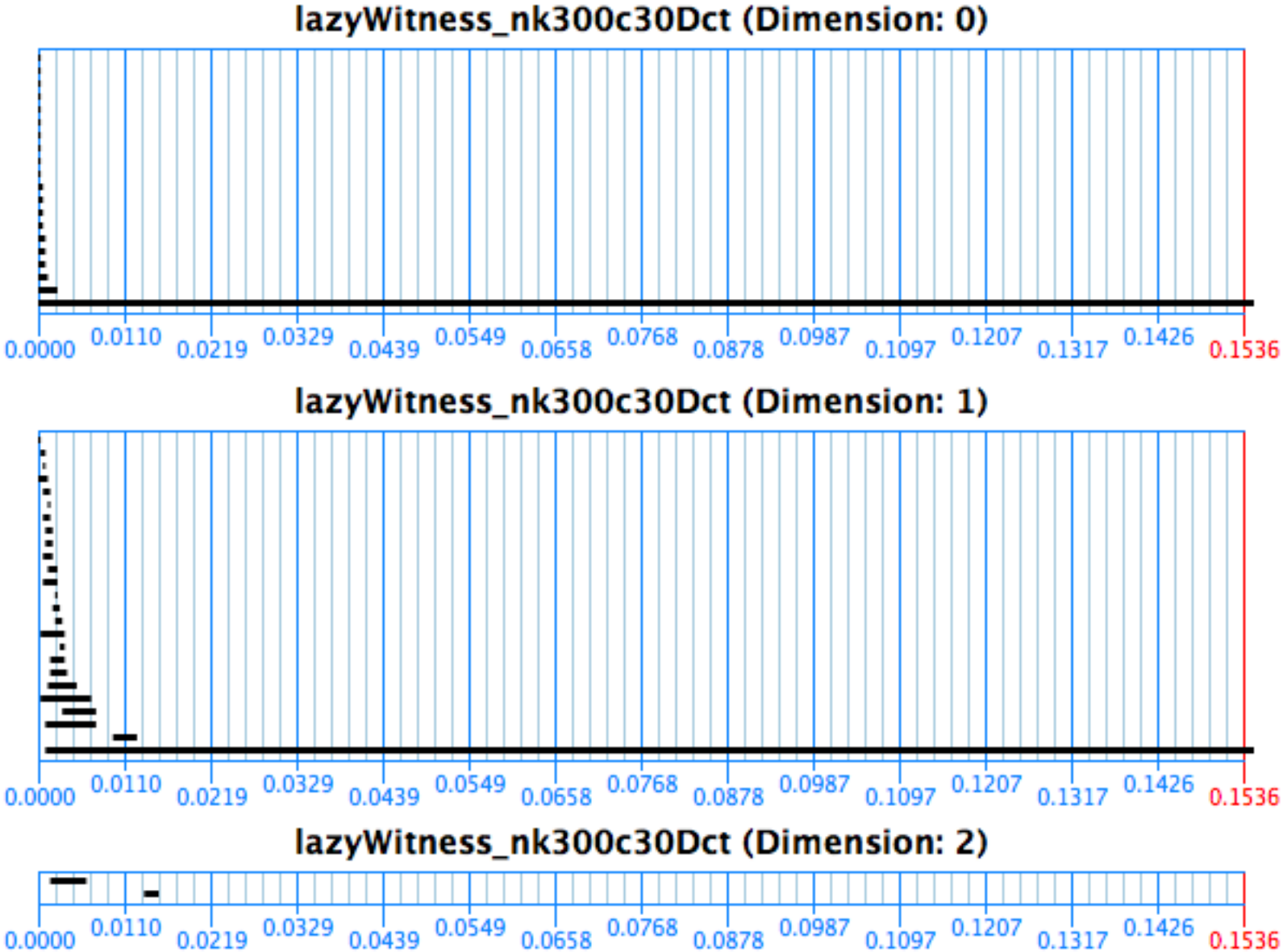
Topology applied to data: Persistent homology

1st densest group of patches



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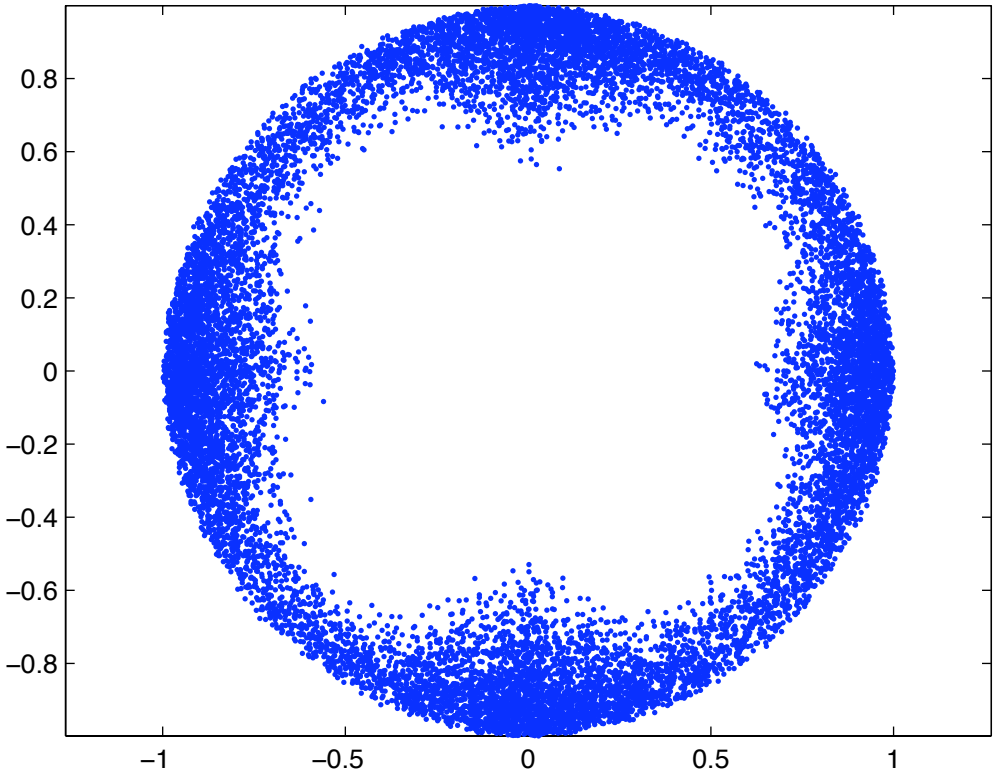
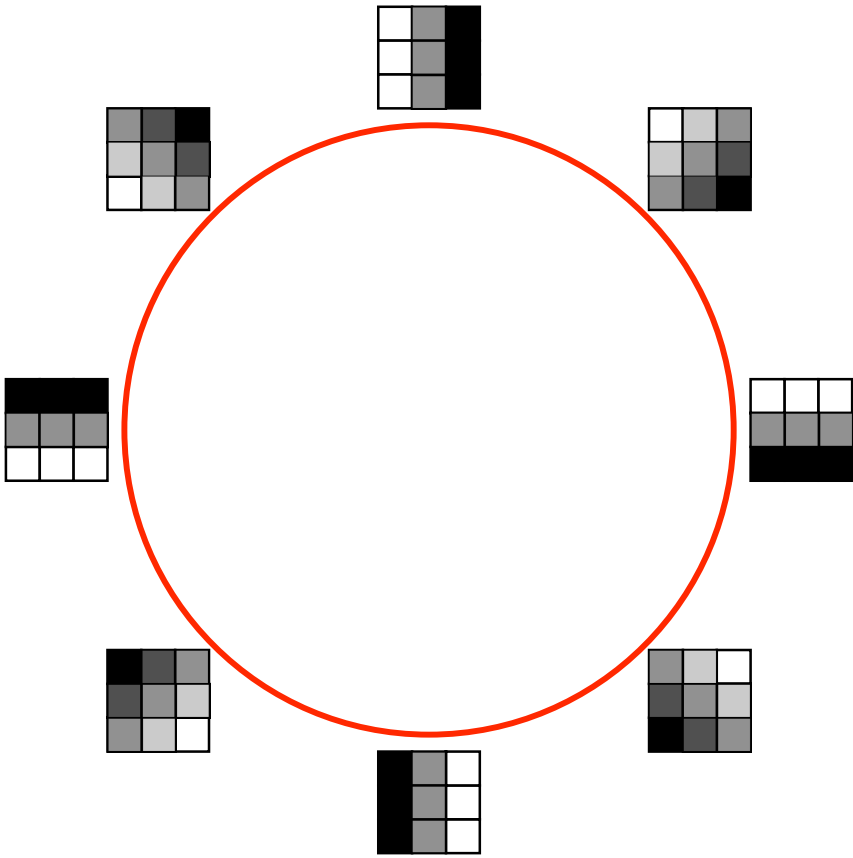
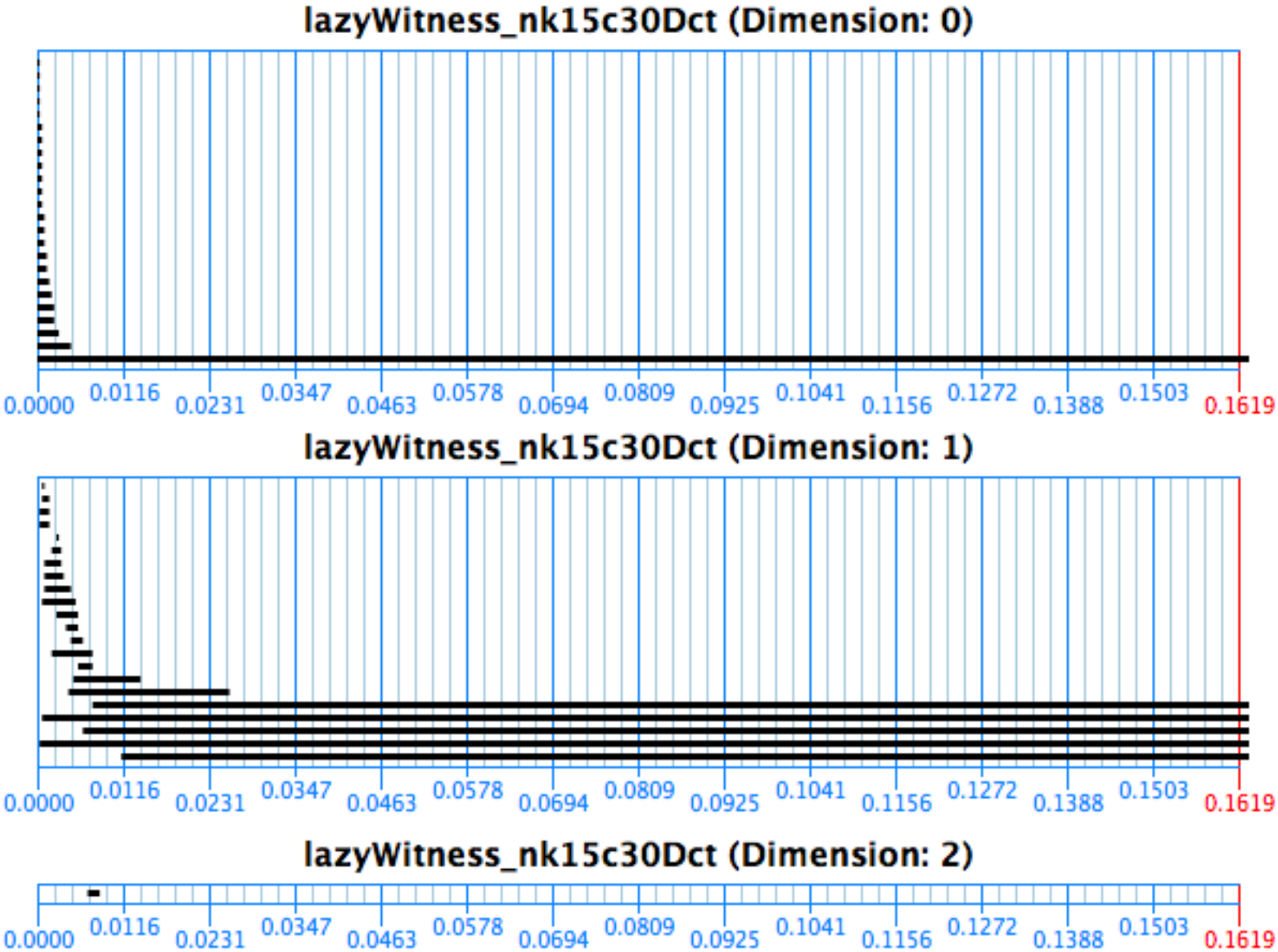




Interpretation: nature prefers linearity

Topology applied to data: Persistent homology

2nd densest group of patches



Topology applied to data: Persistent homology

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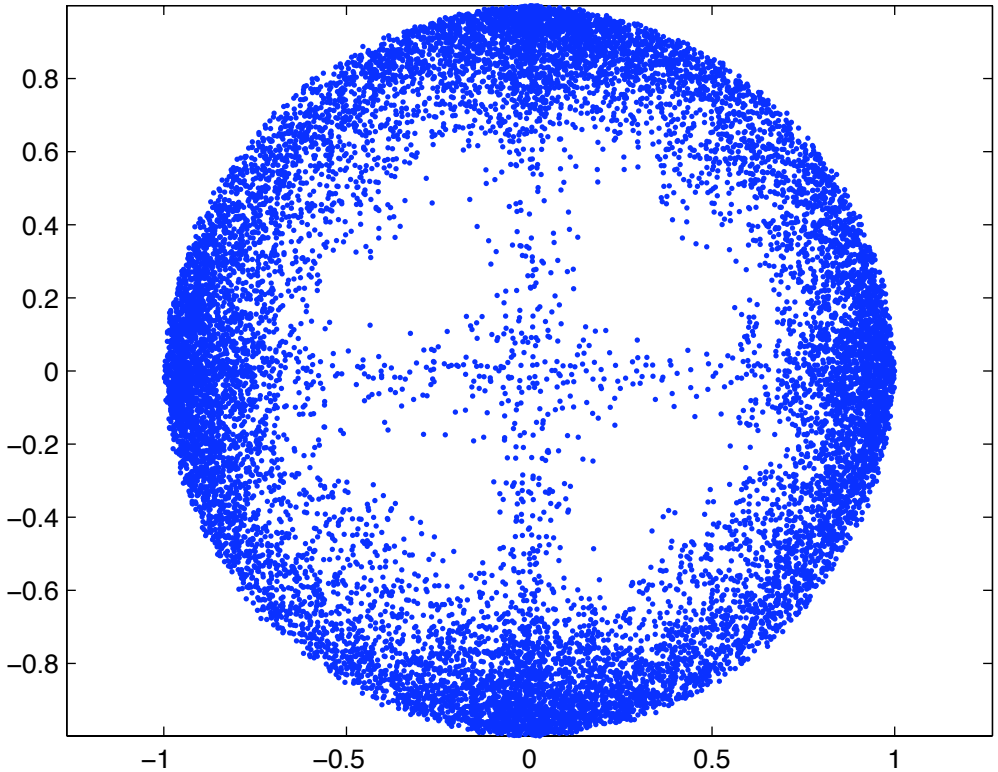
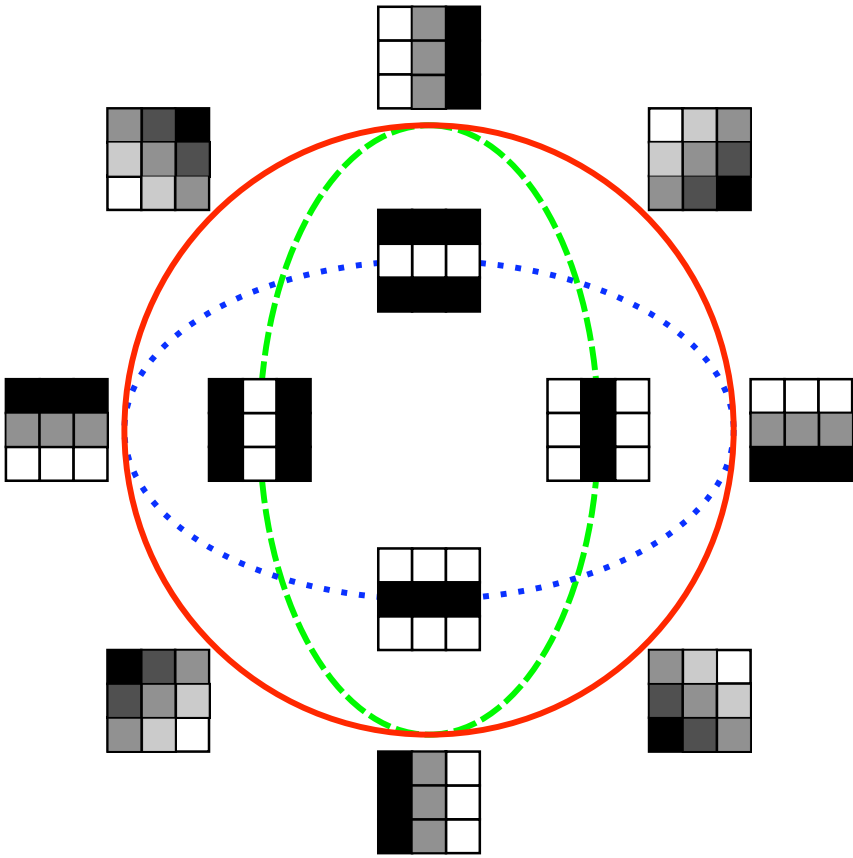
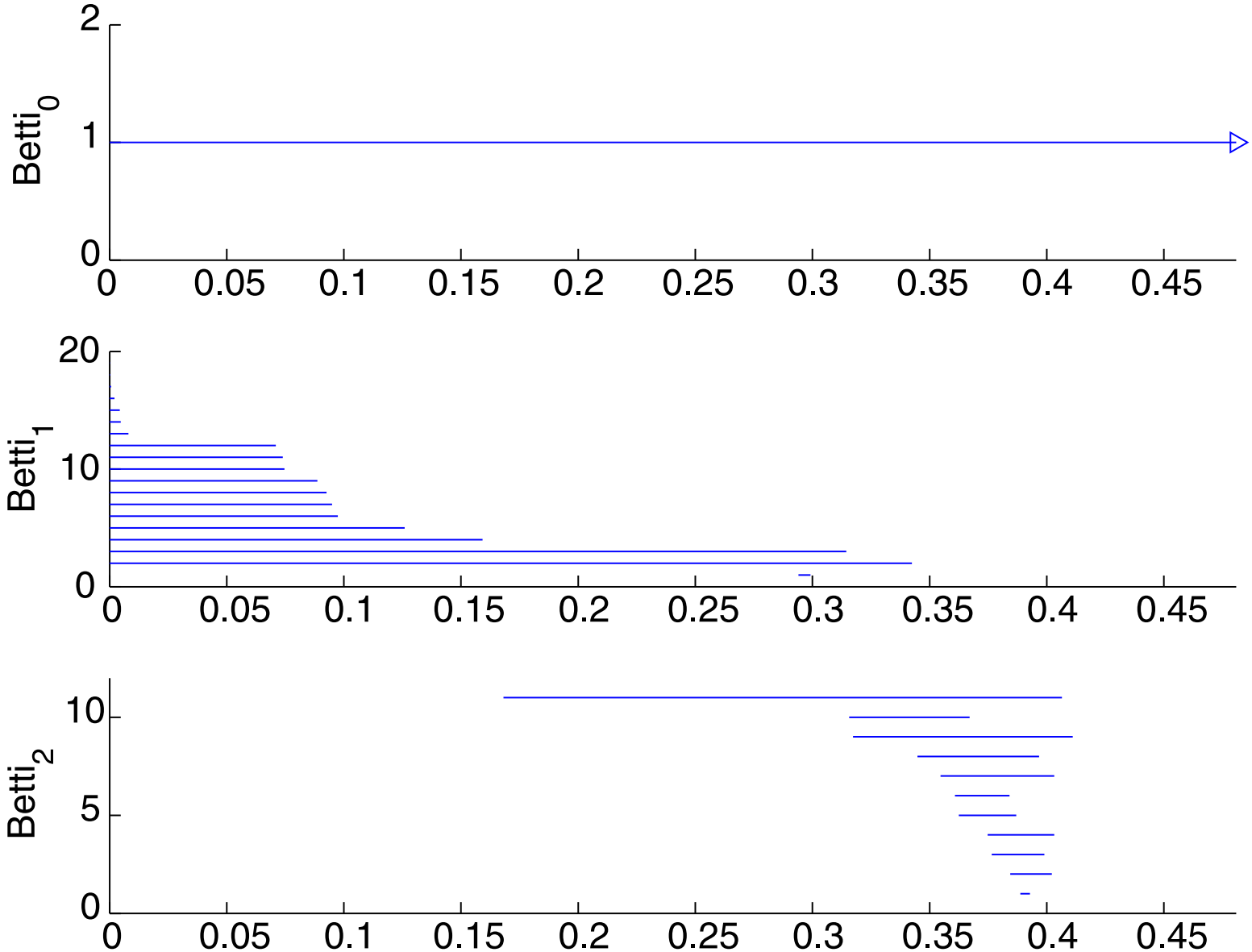




Interpretation: nature prefers horizontal and vertical directions

Topology applied to data: Persistent homology

3rd densest group of patches



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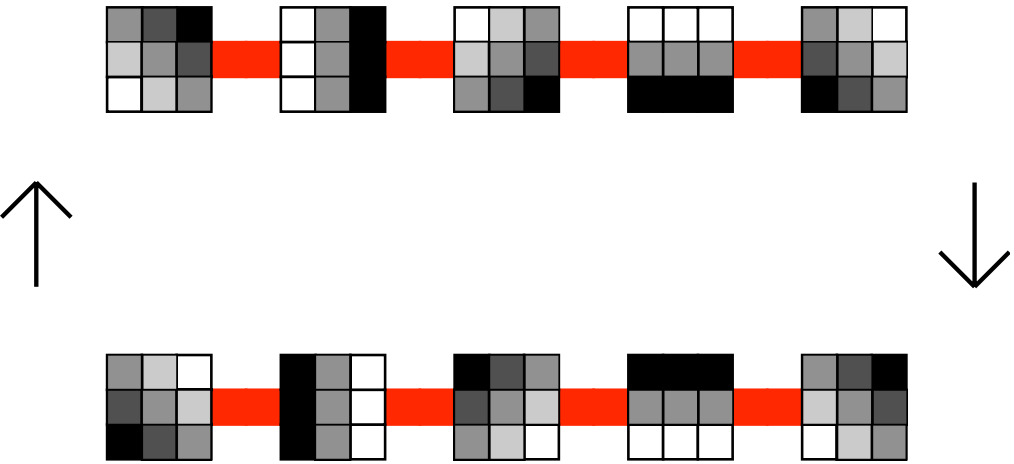

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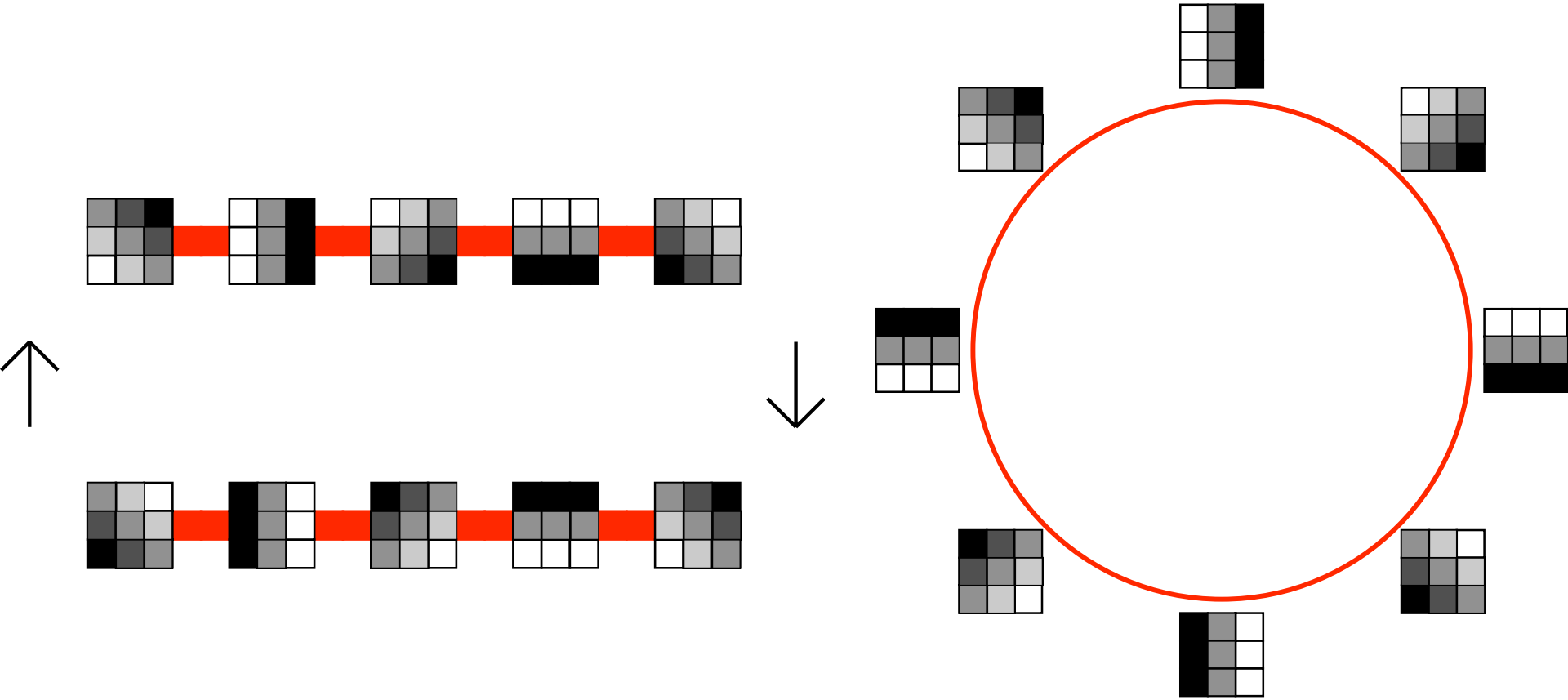

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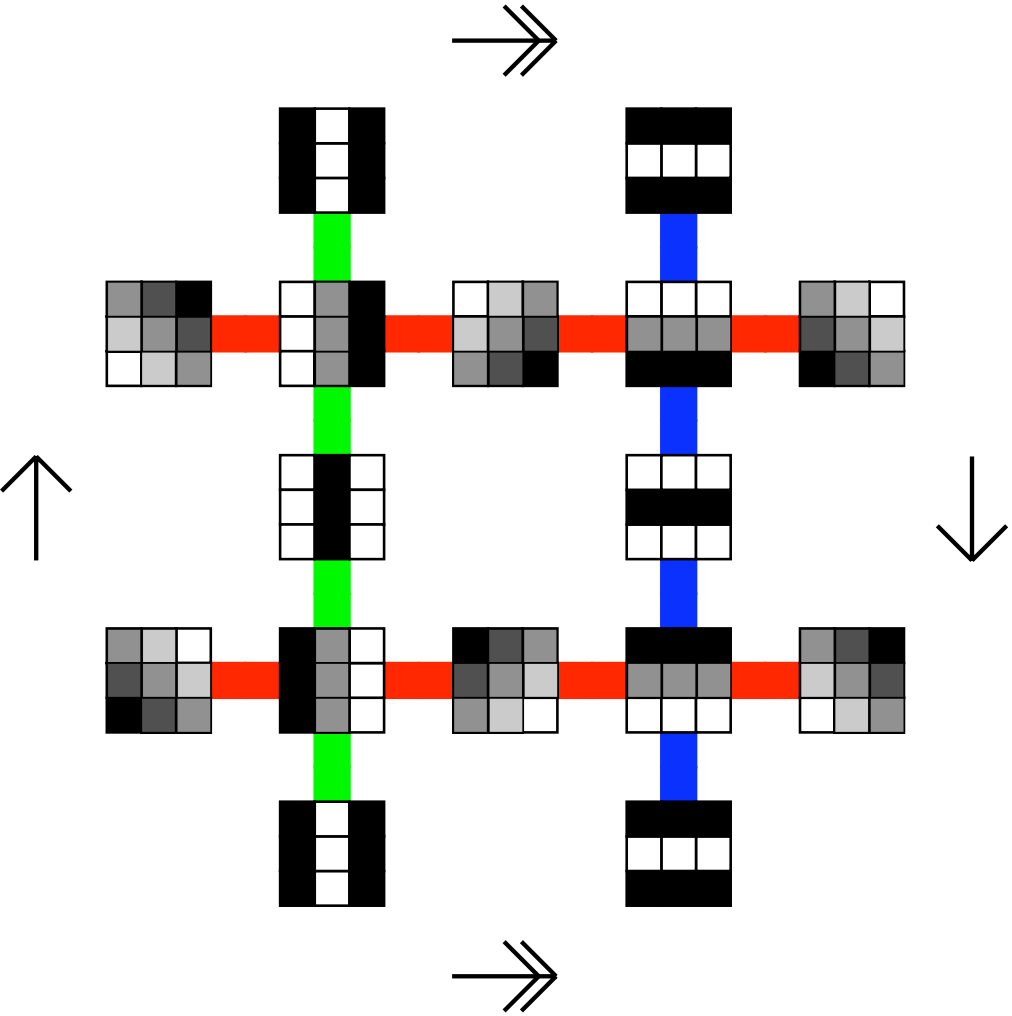

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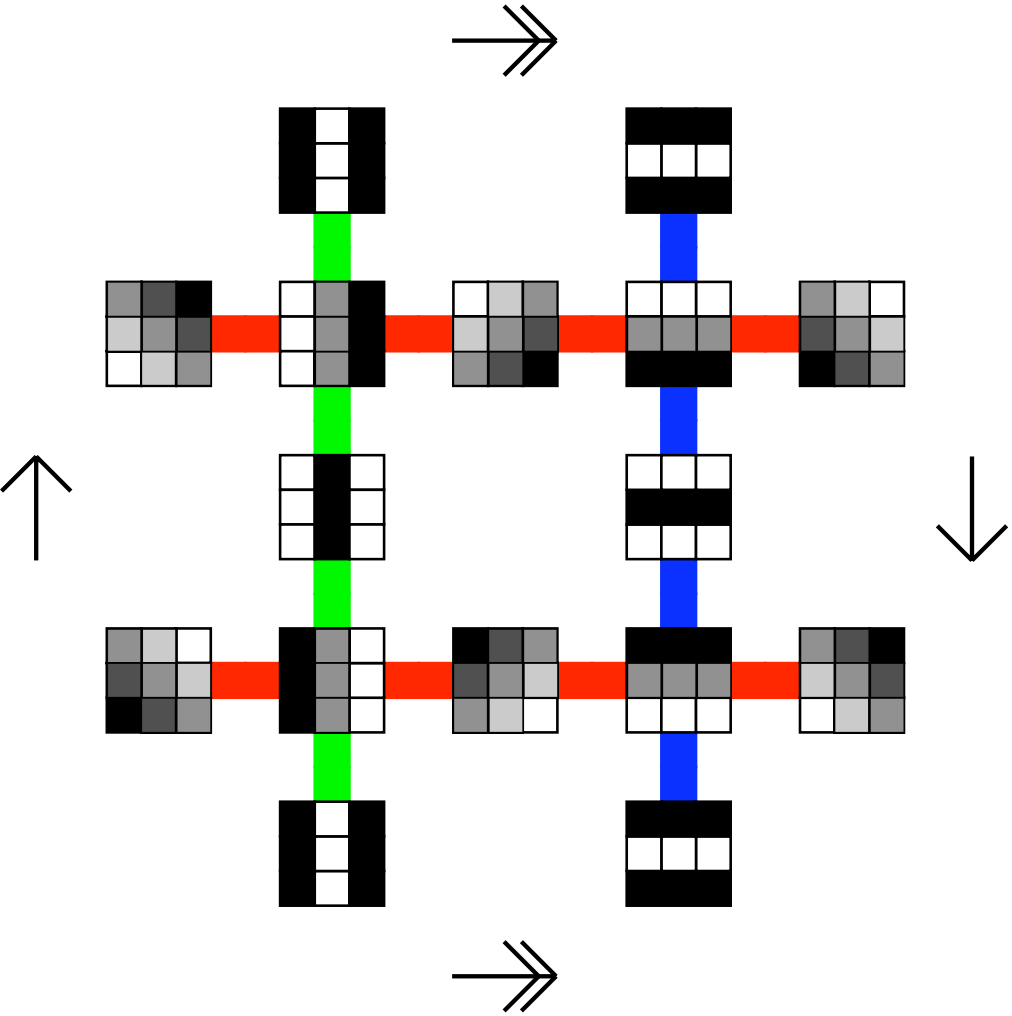

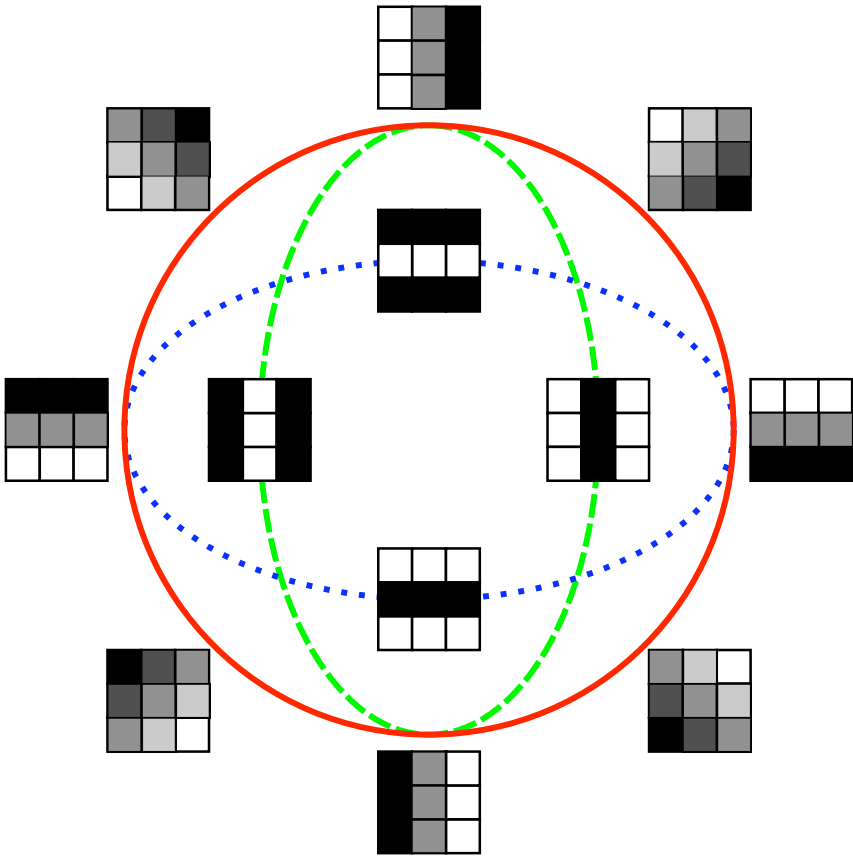

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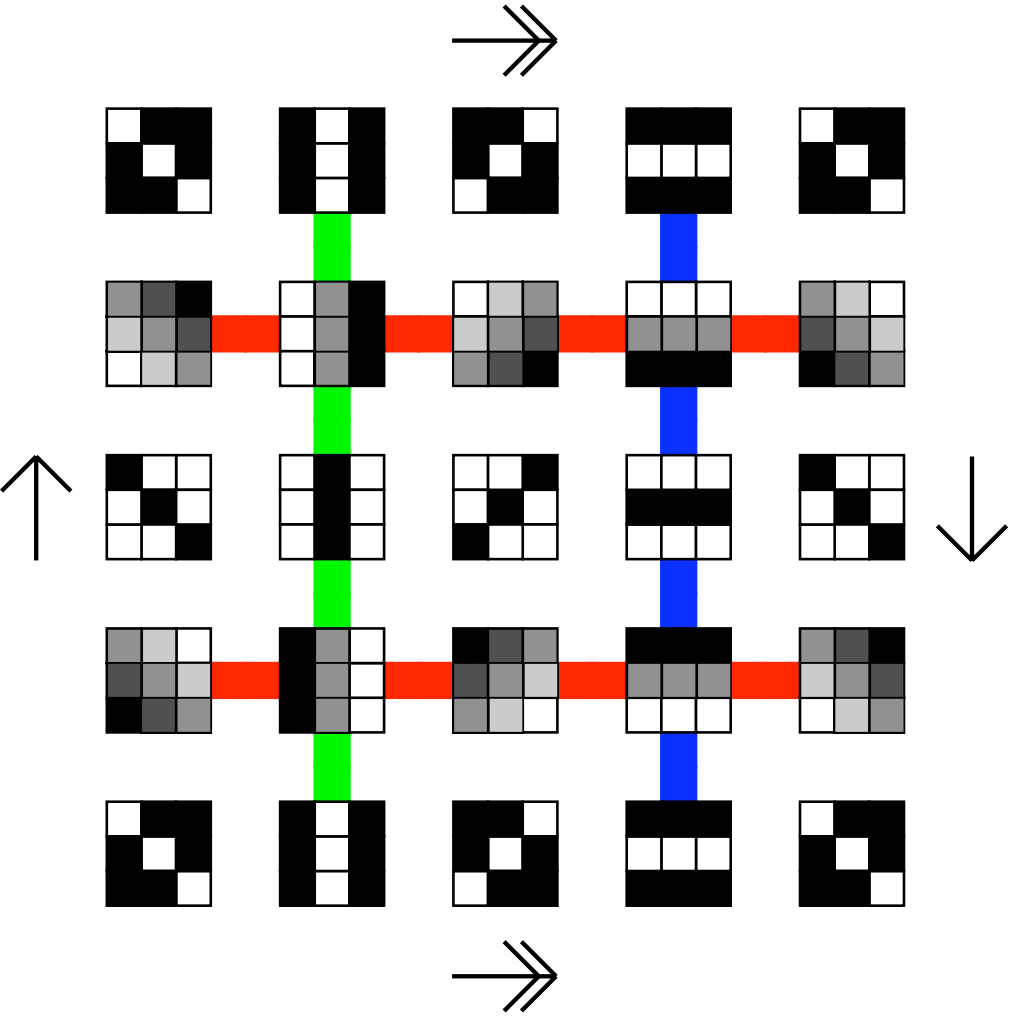

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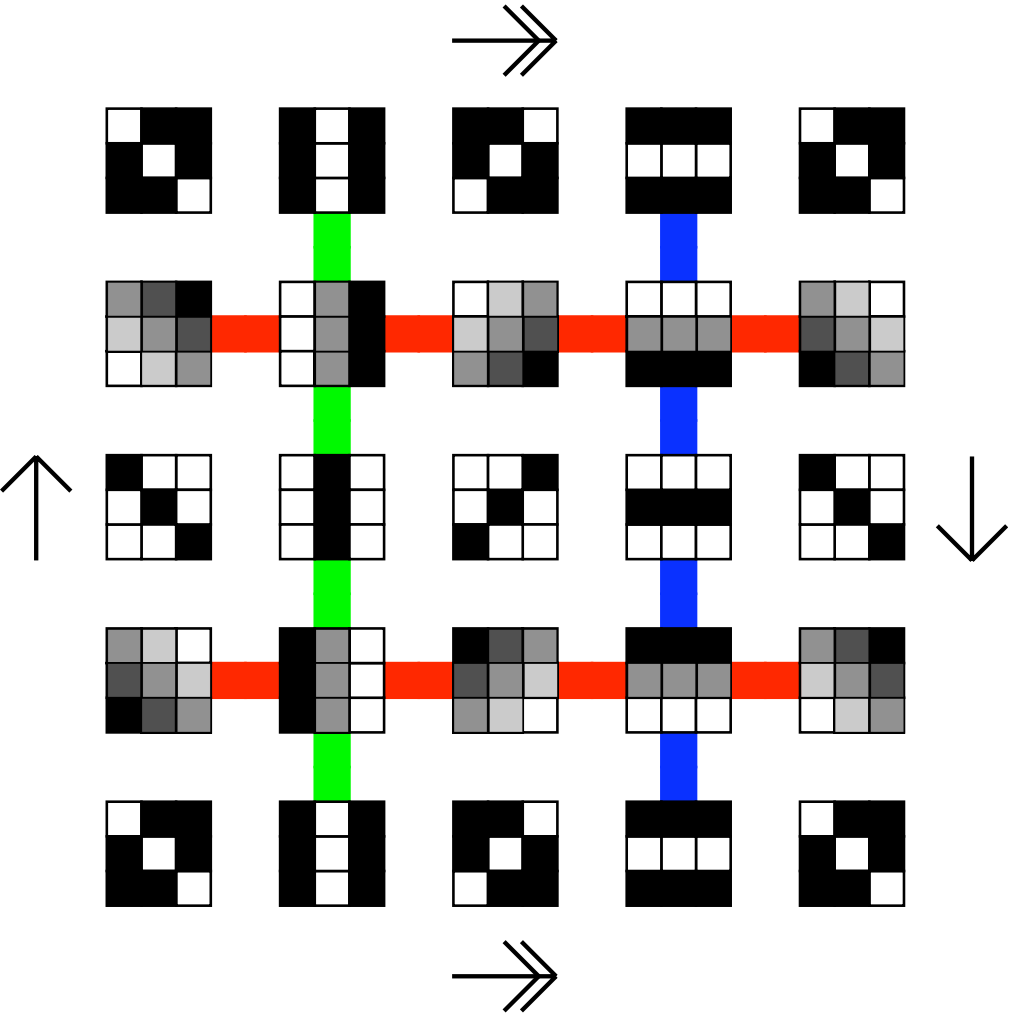

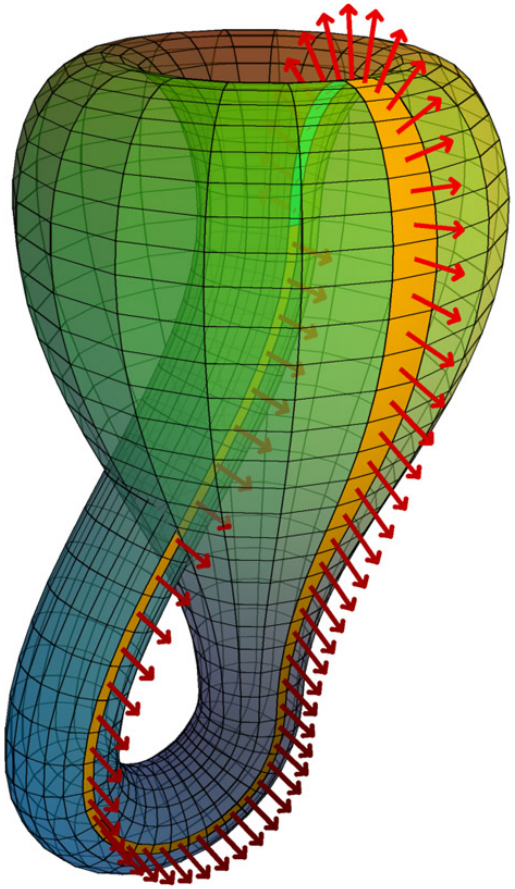

https://plus.maths.org/content/imaging-maths-inside-klein-bottle

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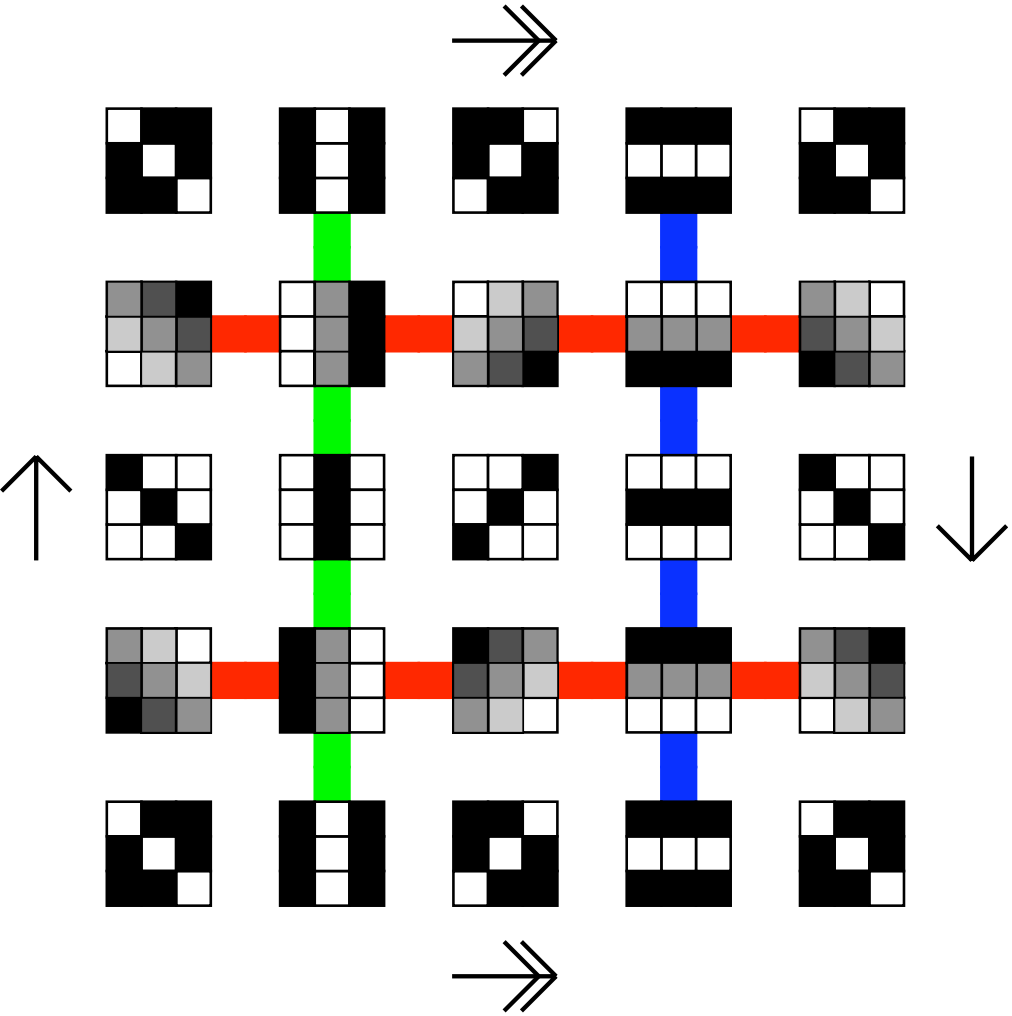
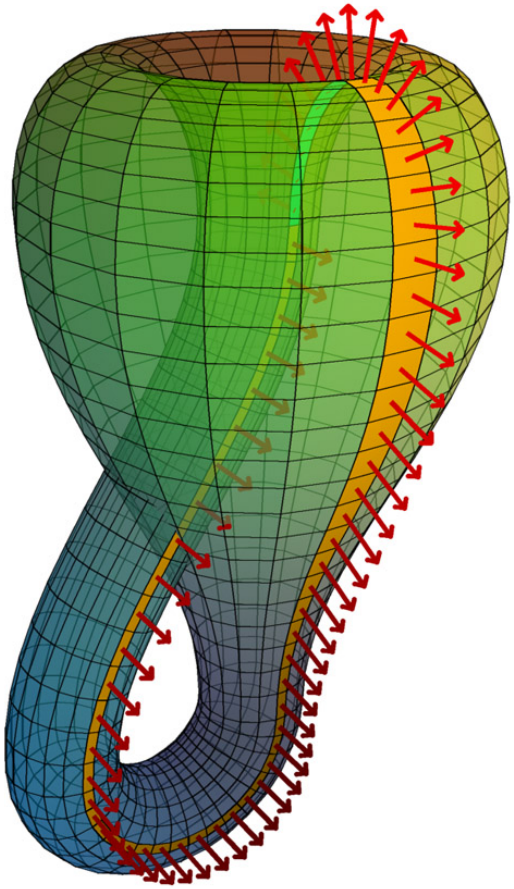


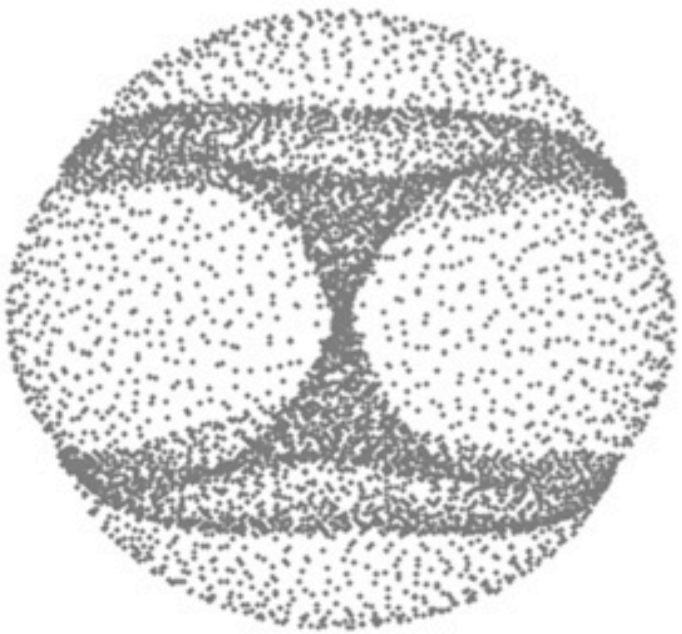


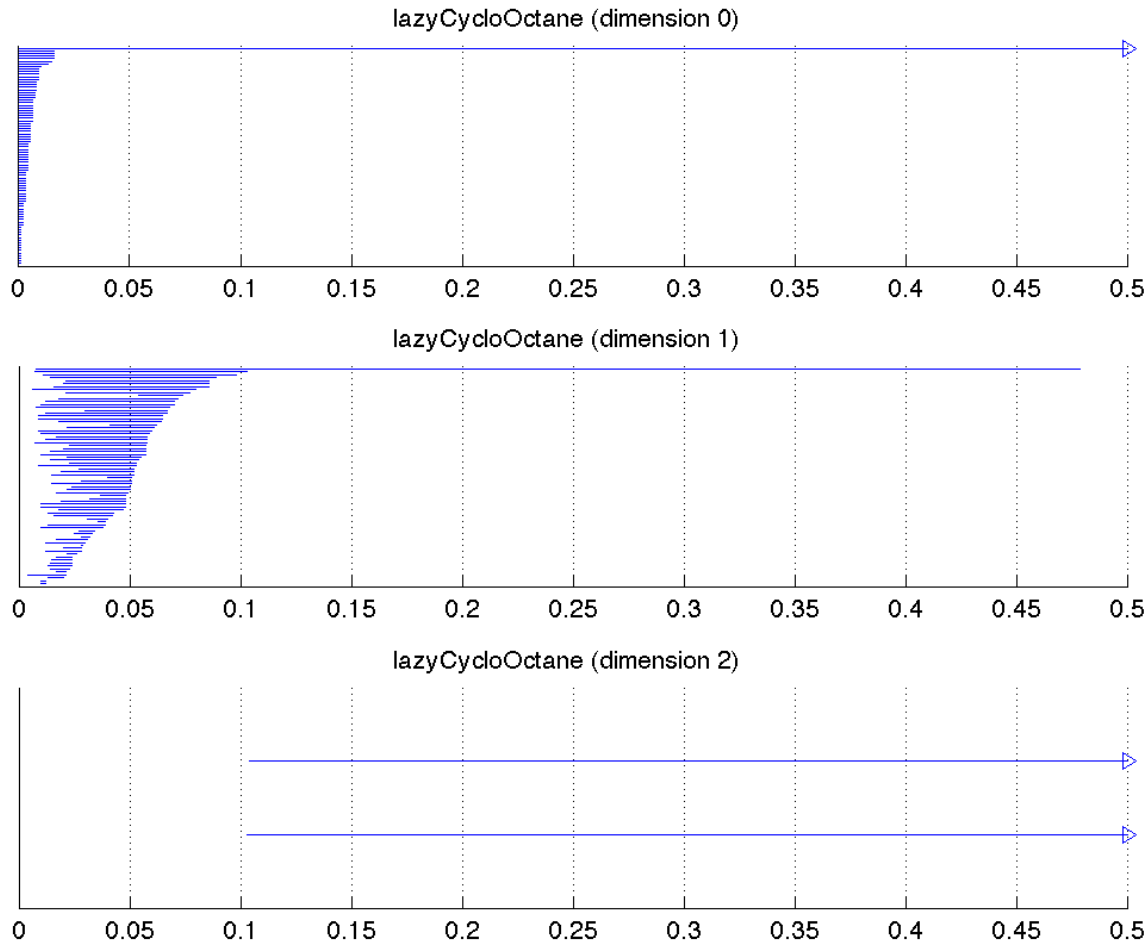
Interpretation: nature prefers linear and quadratic patches at all angles

Topology applied to data: Persistent homology

Example: Cyclo-Octane (C_8H_{16}) data

1,031,644 points in 72-dimensional space





Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data by Shawn Martin and Jean-Paul Watson, 2010.

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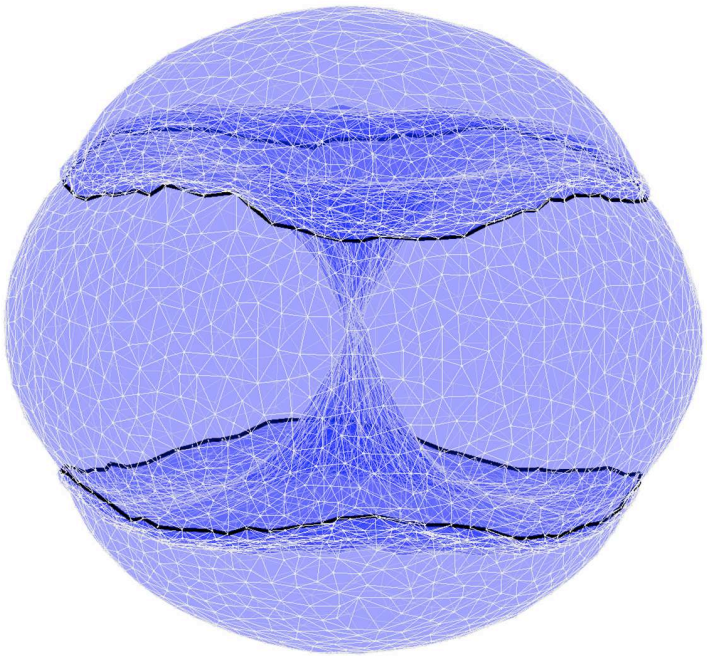

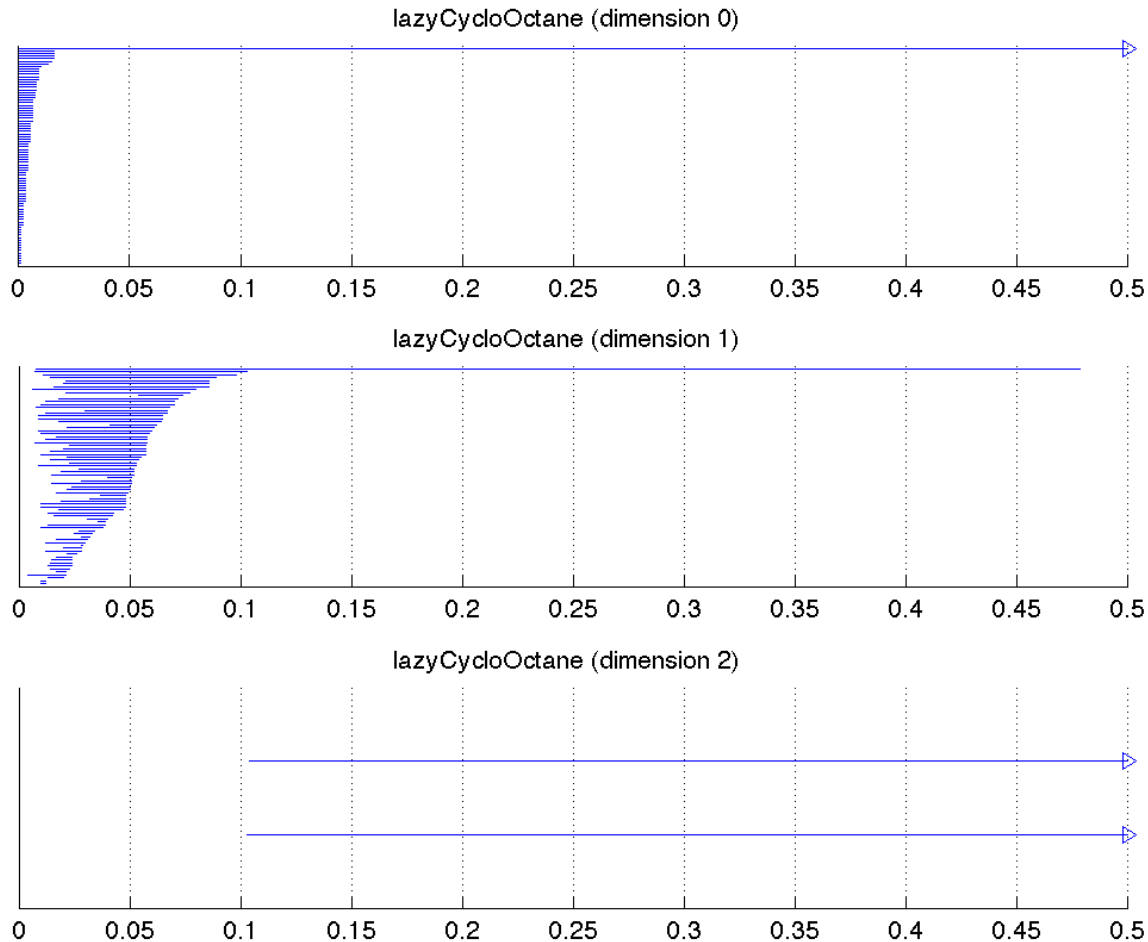

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References

- *An Attempt to Define the Nature of Chemical Diabetes Using a Multidimensional Analysis* by G. M. Reaven and R. G. Miller, 1979.
- *Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data* by Shawn Martin and Jean-Paul Watson, 2010.
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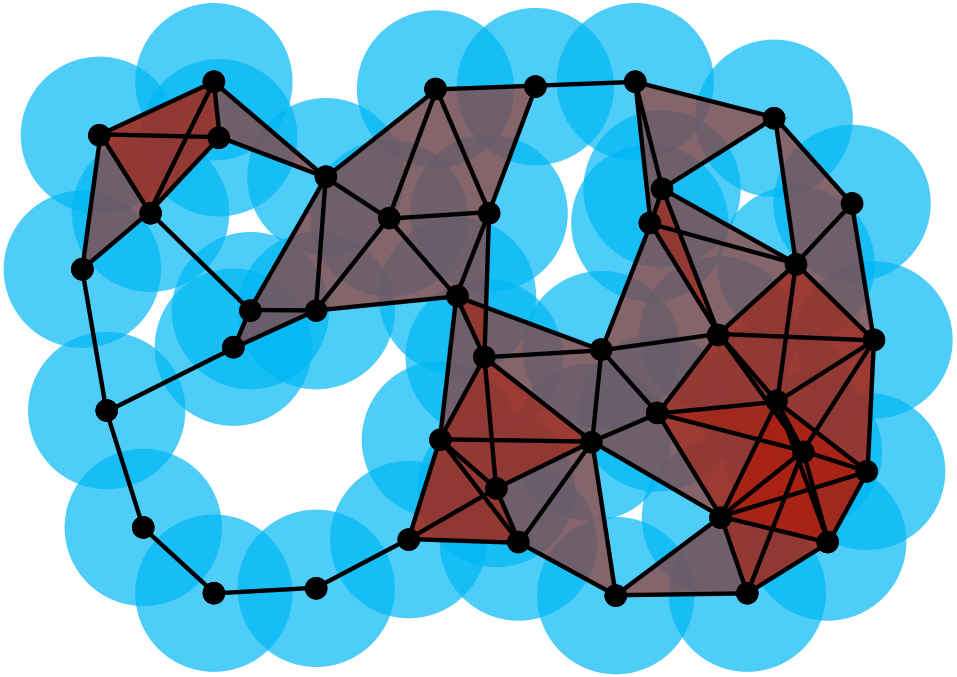
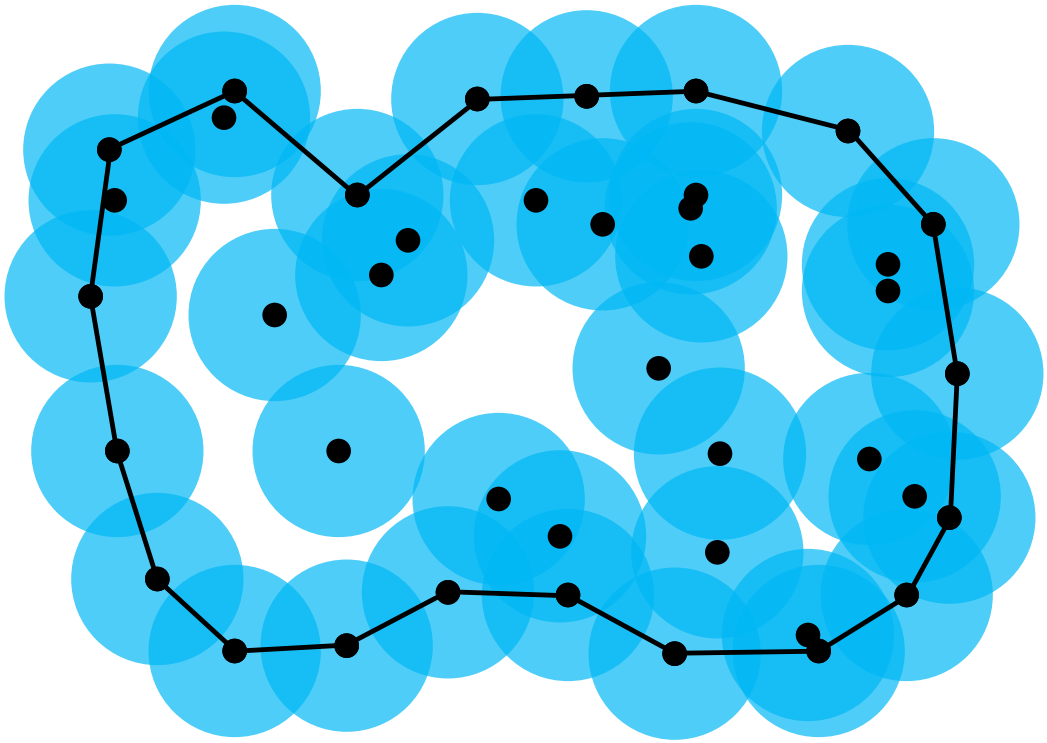
Evasion Paths in Mobile Sensor Networks



Henry Adams and Gunnar Carlsson, *Evasion paths in mobile sensor networks*, International Journal of Robotics Research 34 (2015), 90-104.

Evasion problem

- Sensors move in a ball-shaped domain $B \subset \mathbb{R}^d$ over time interval I. Fixed sensors cover ∂B.
- Measure only the Čech complex.
- Is there an evasion path?



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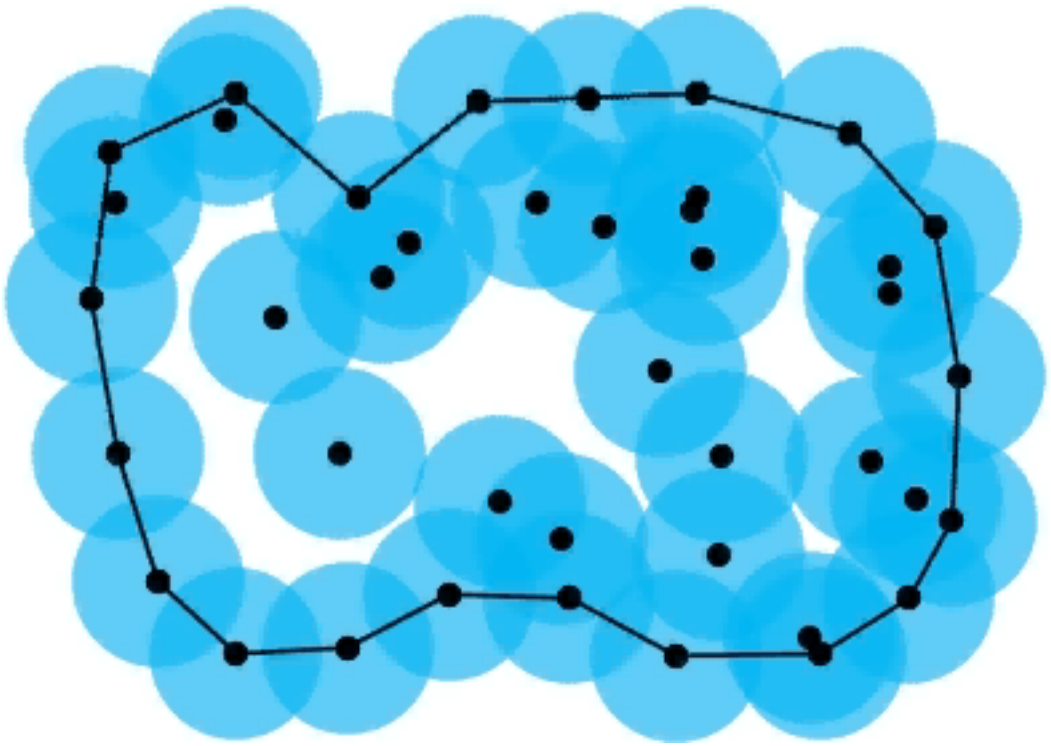

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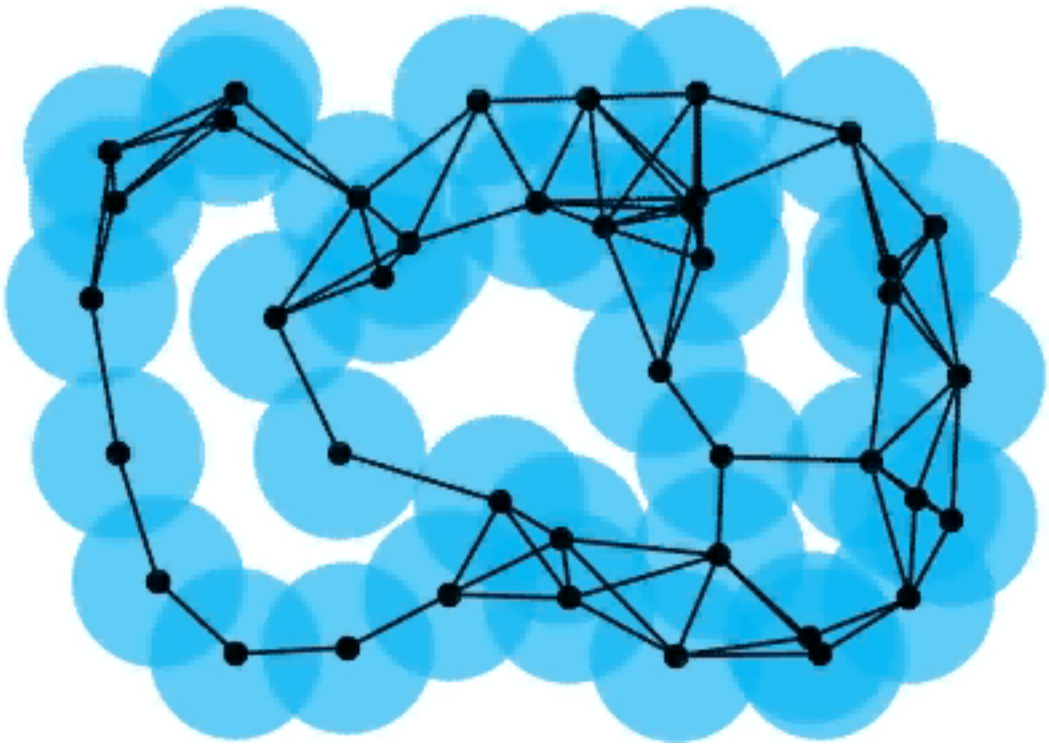

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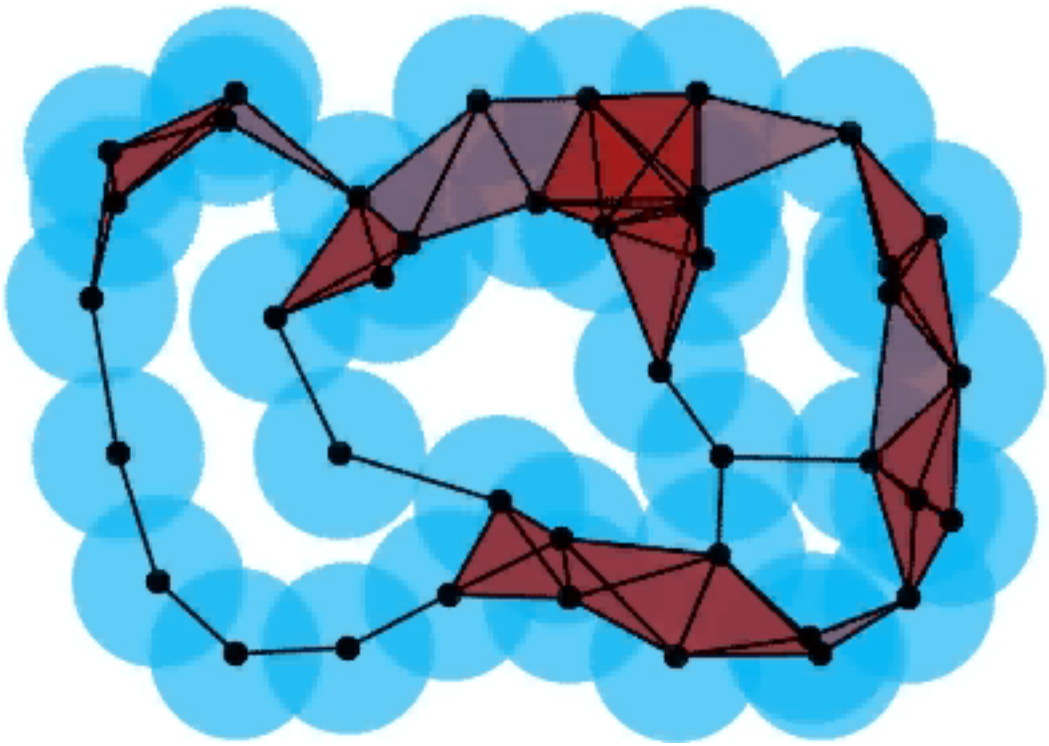

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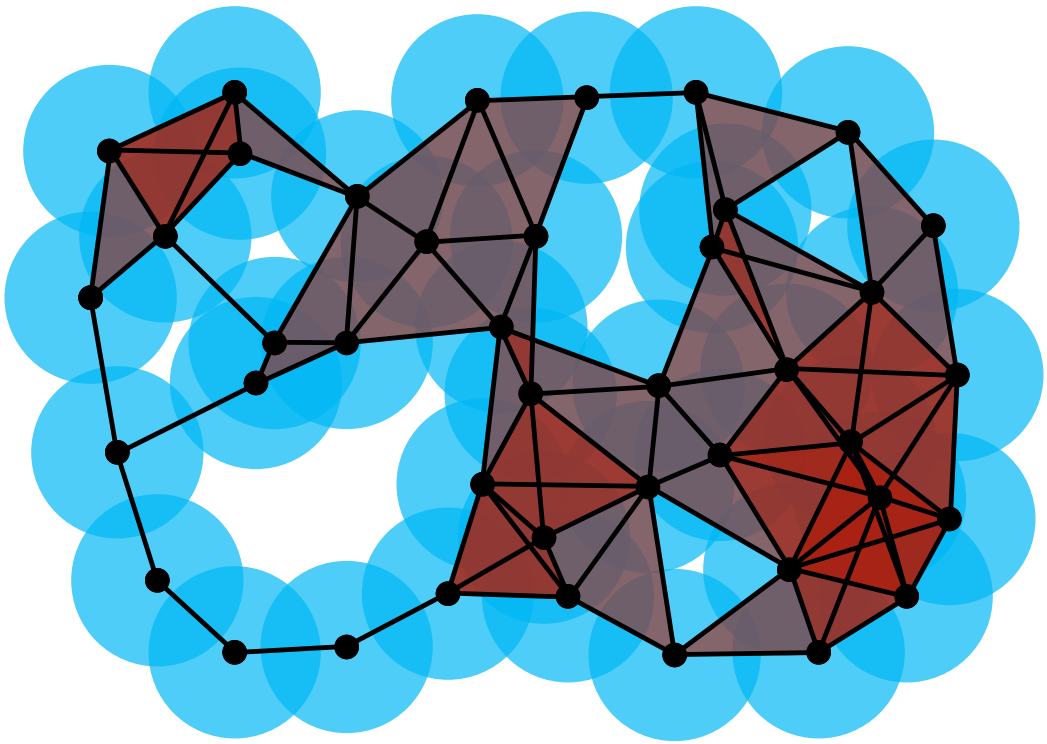

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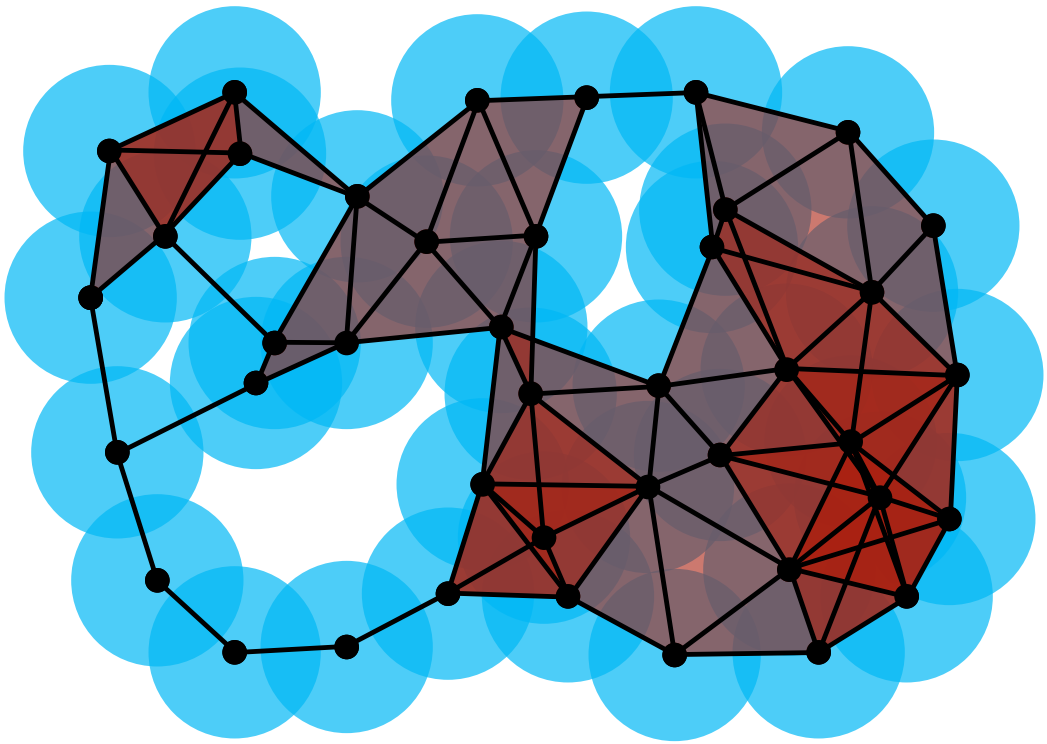

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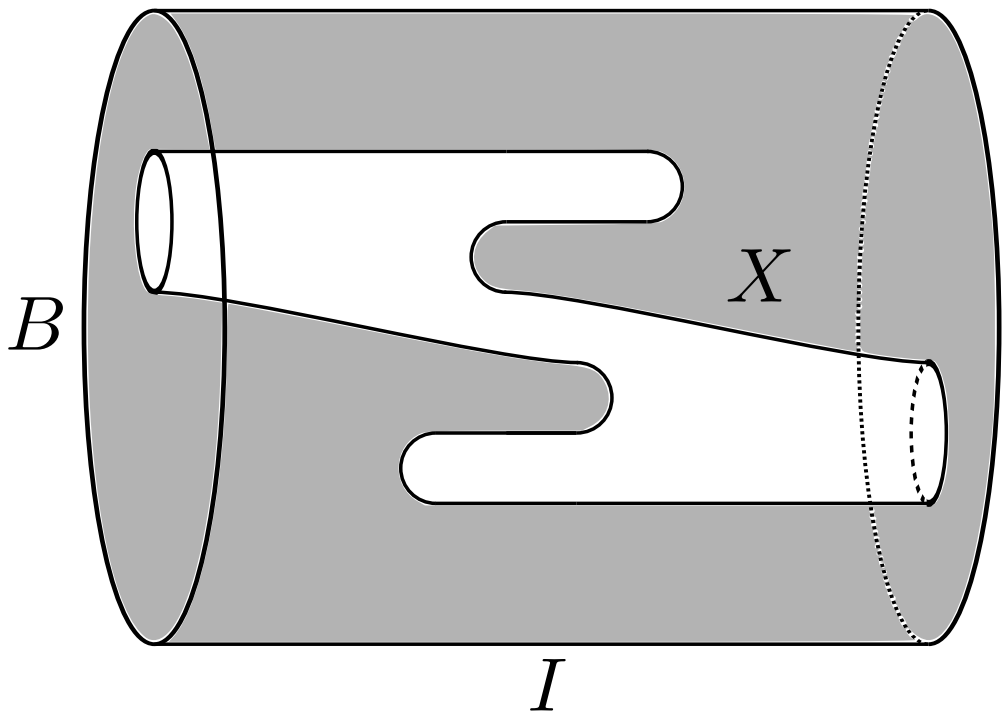

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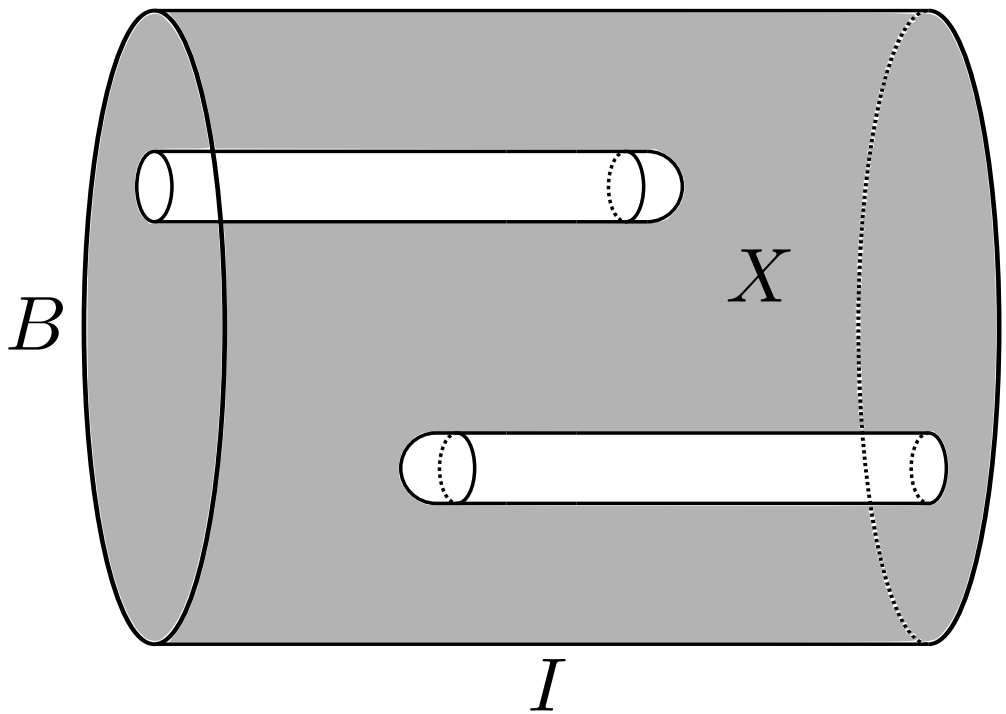

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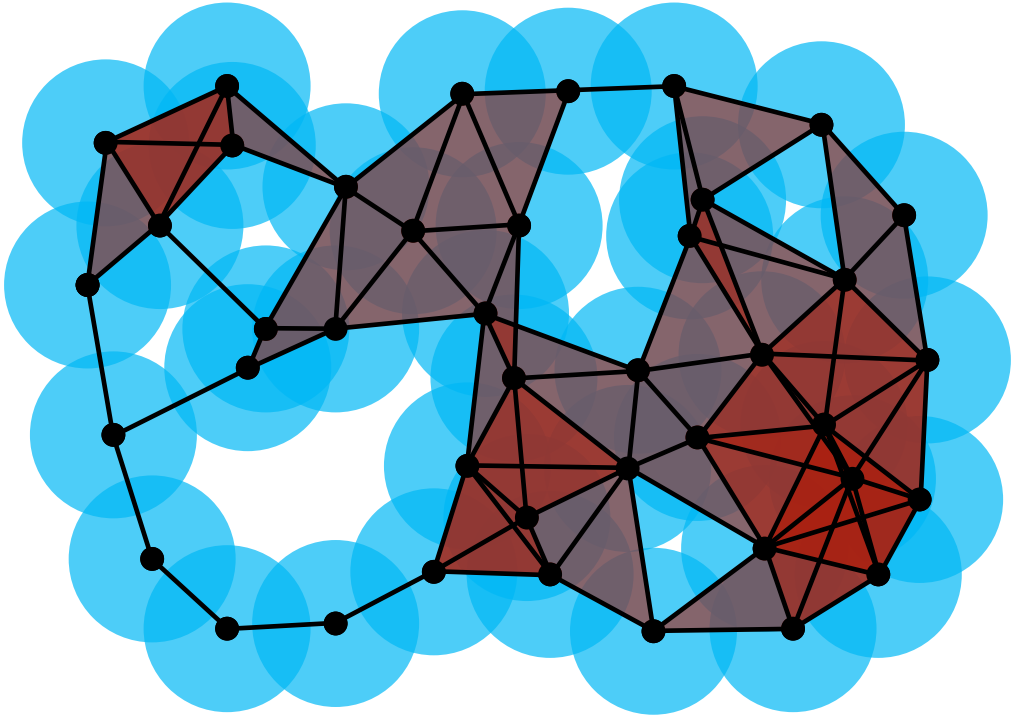

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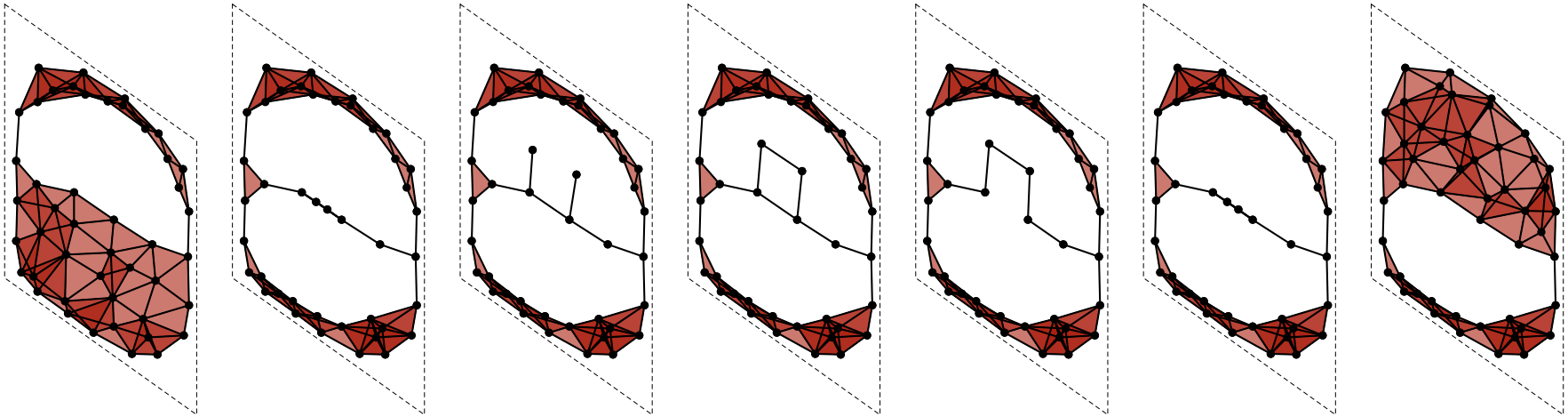

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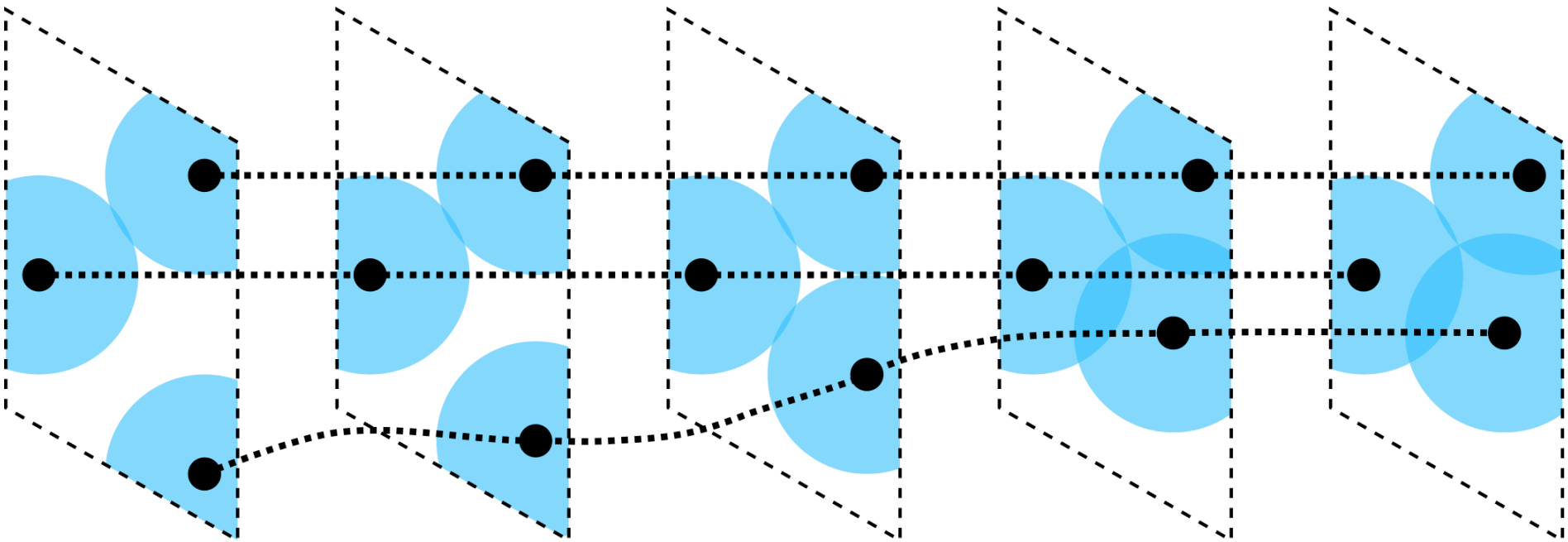

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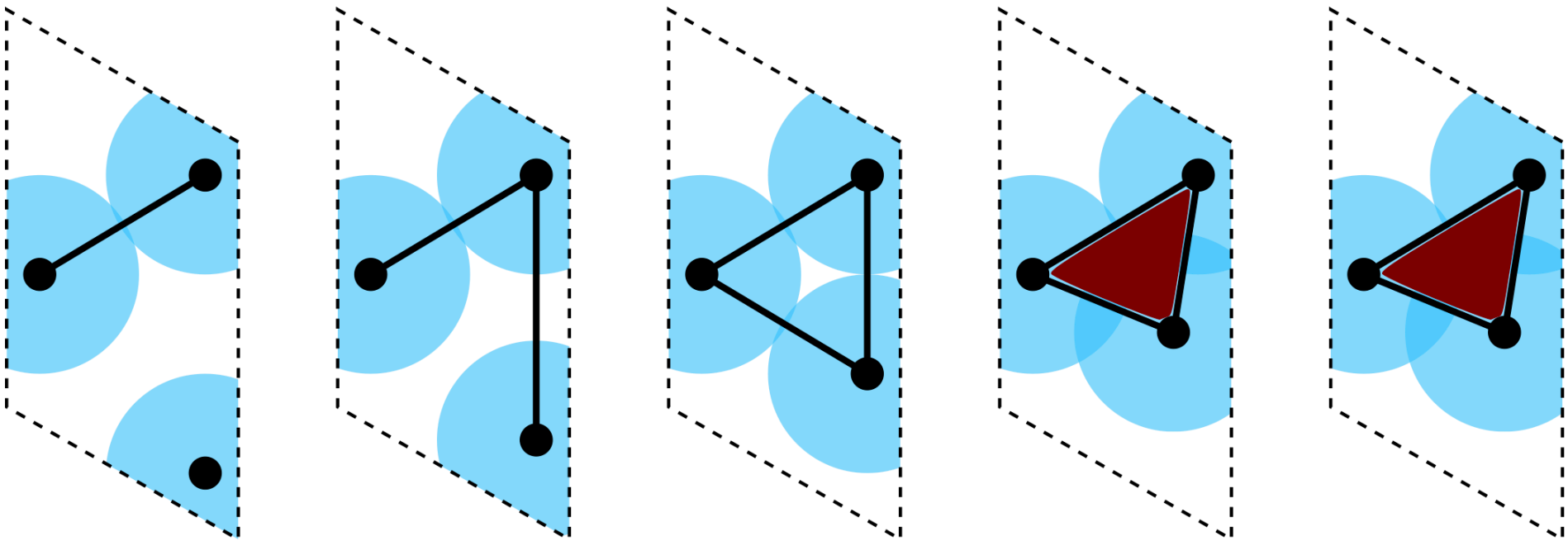

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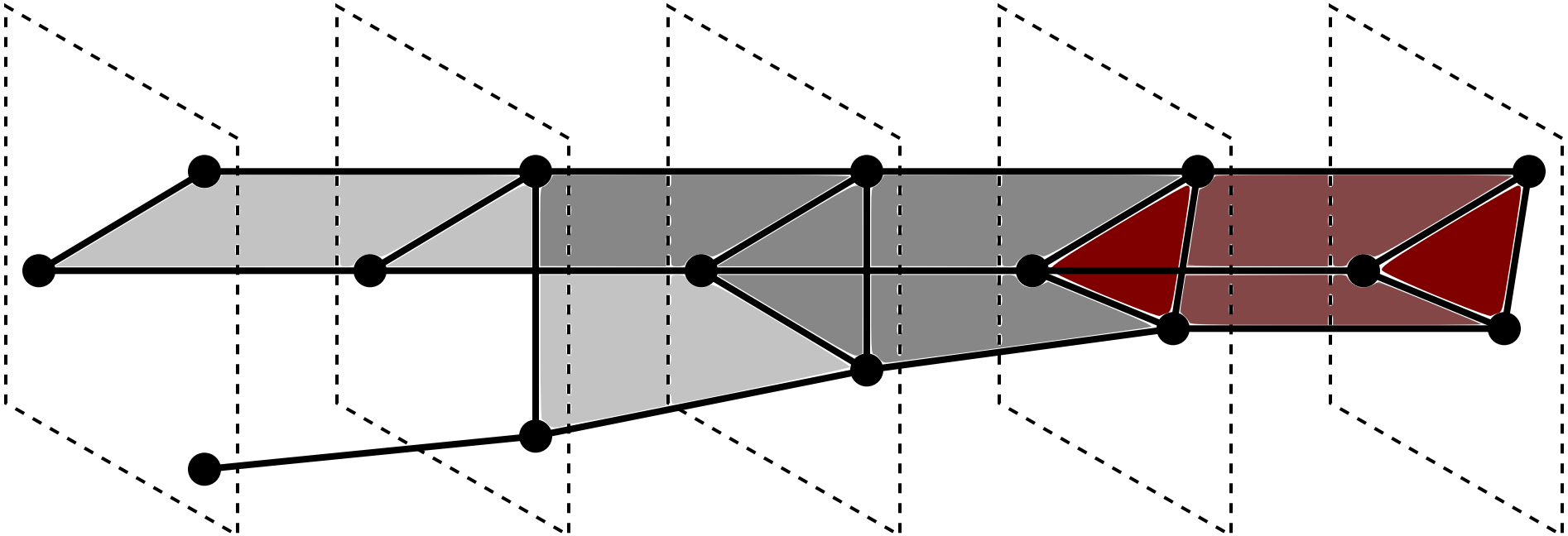

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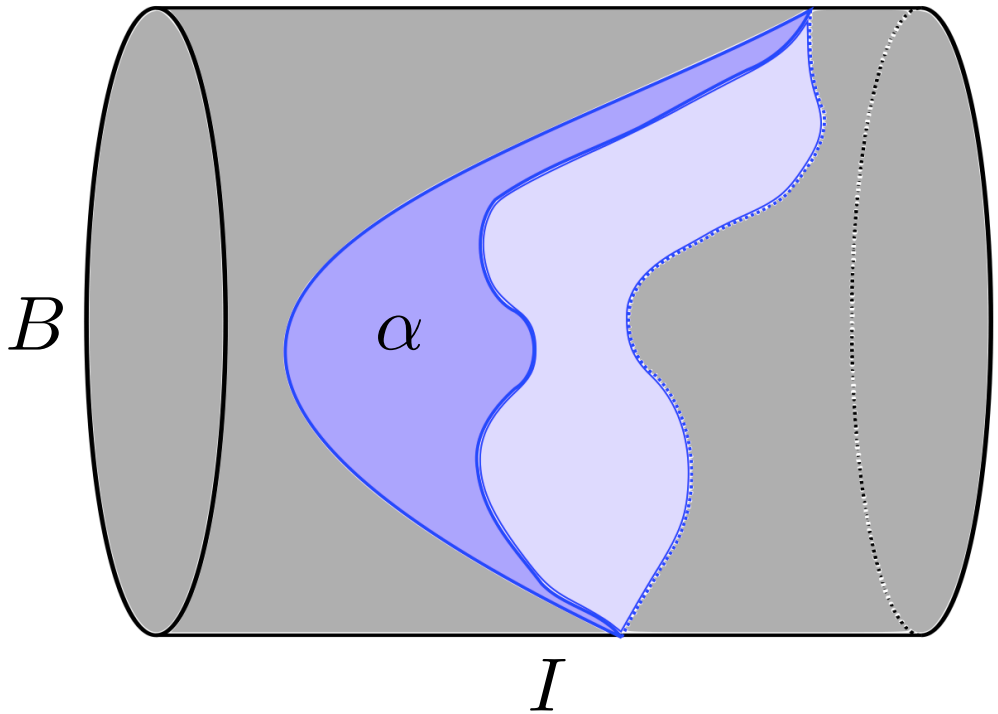

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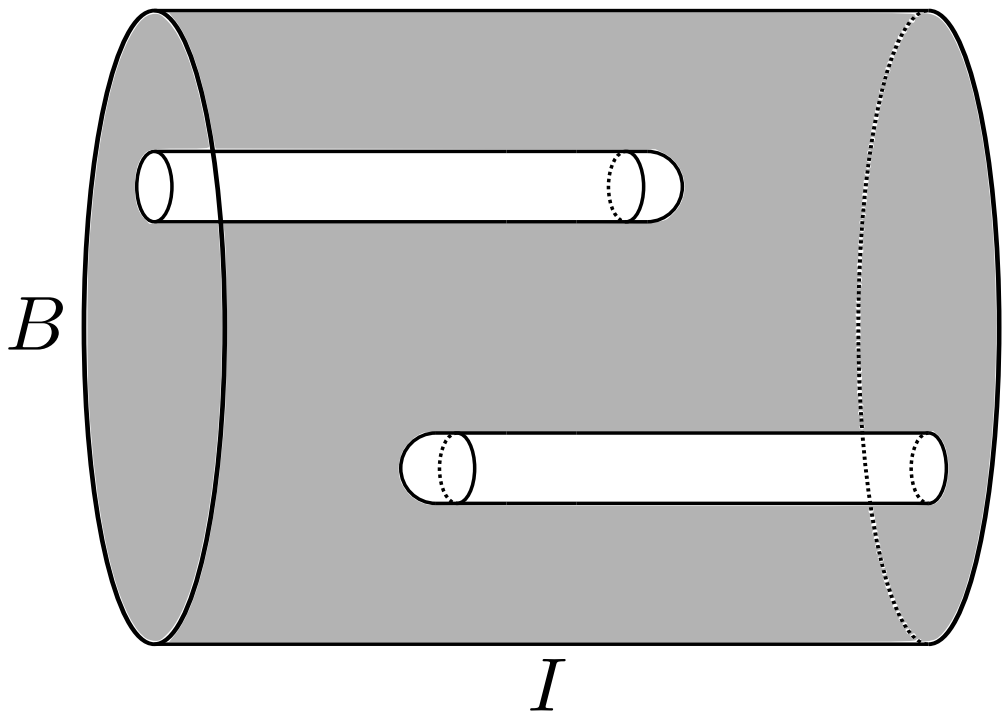

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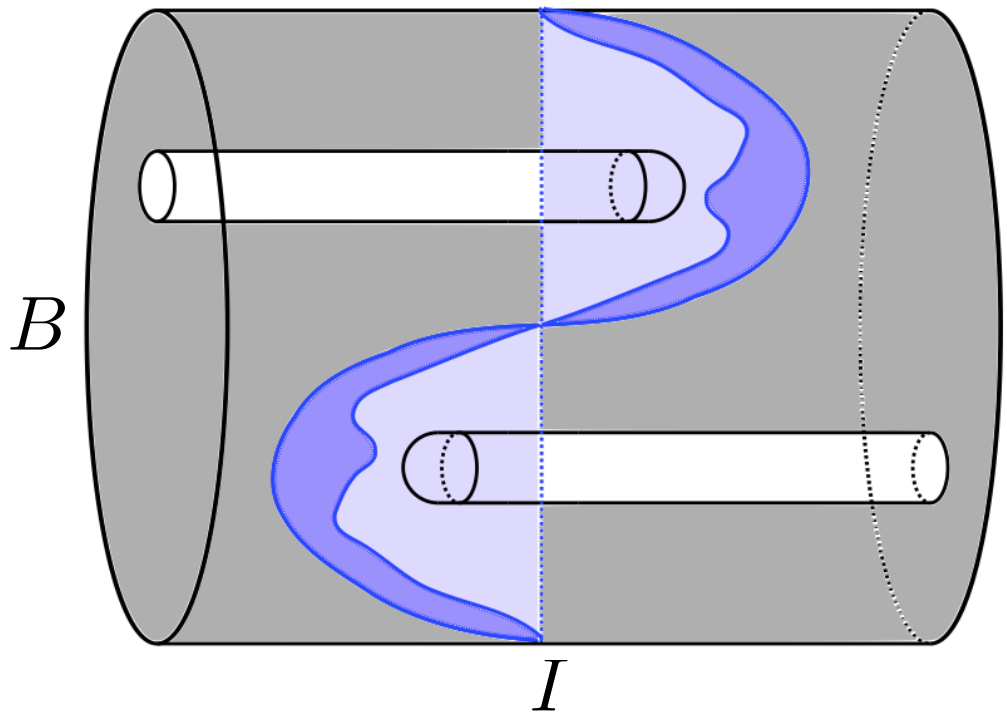

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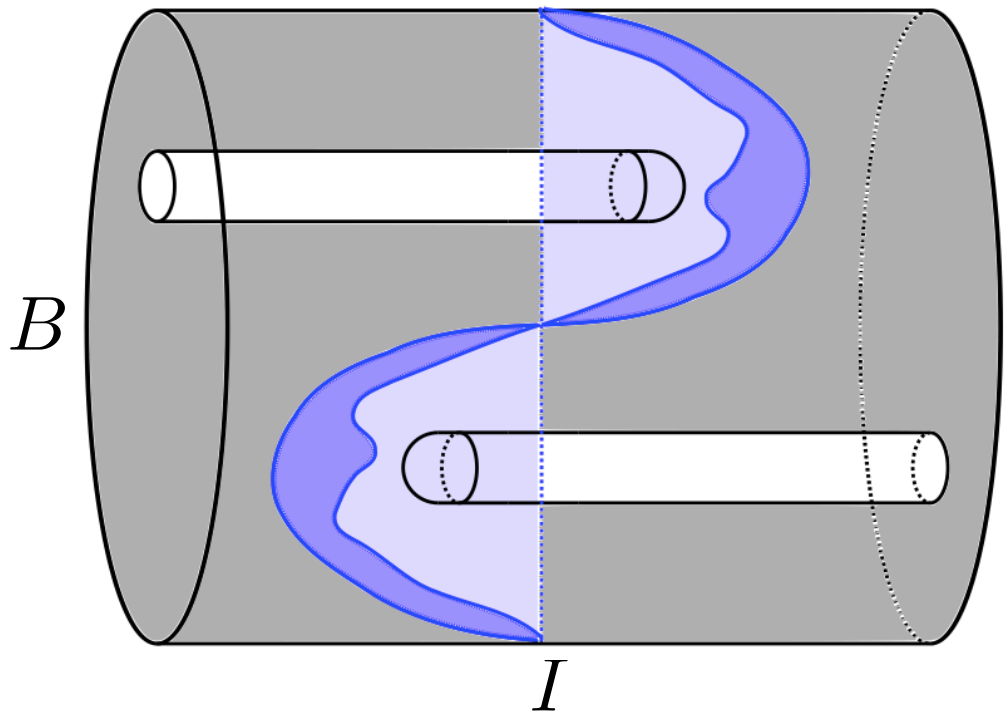

Coordinate-free Coverage in Sensor Networks with Controlled Boundaries via Homology by V. de Silva and R. Ghrist

Evasion problem

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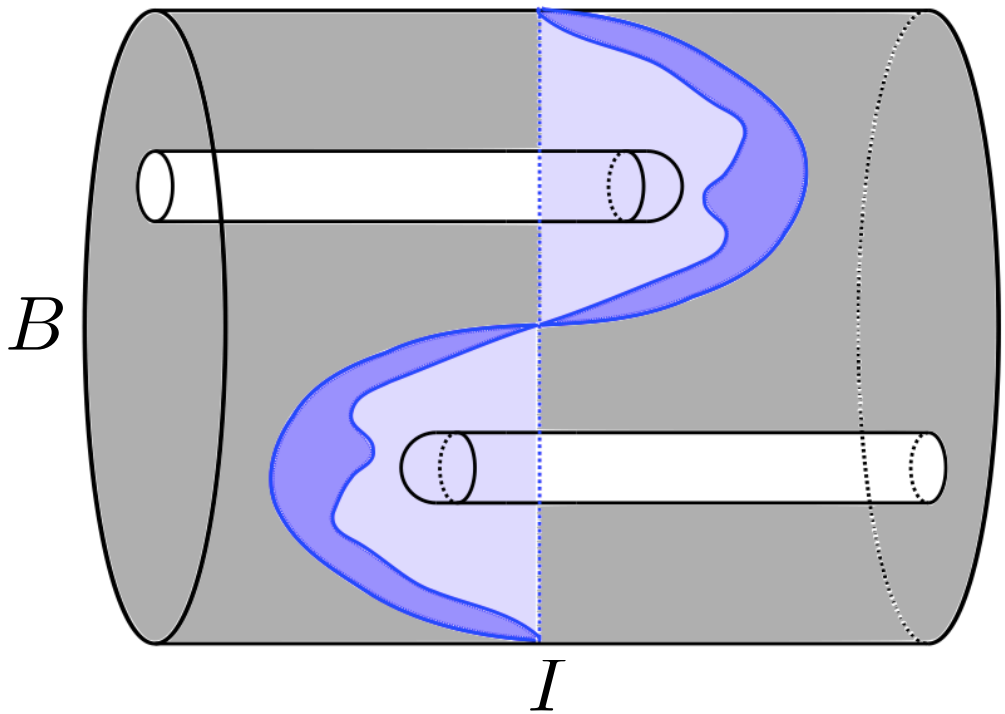

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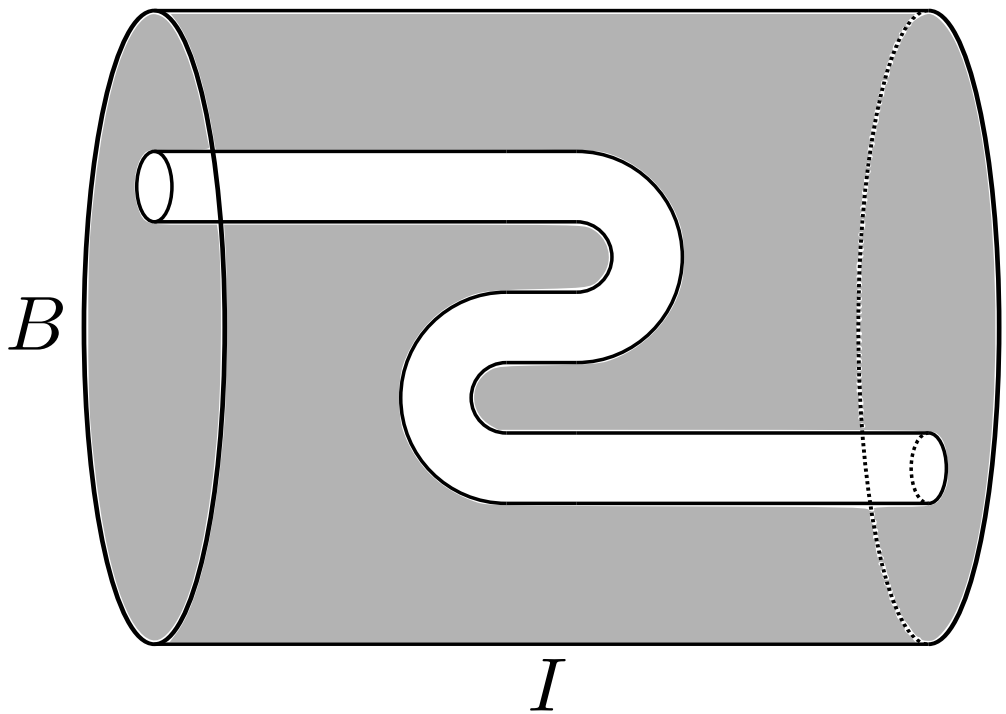

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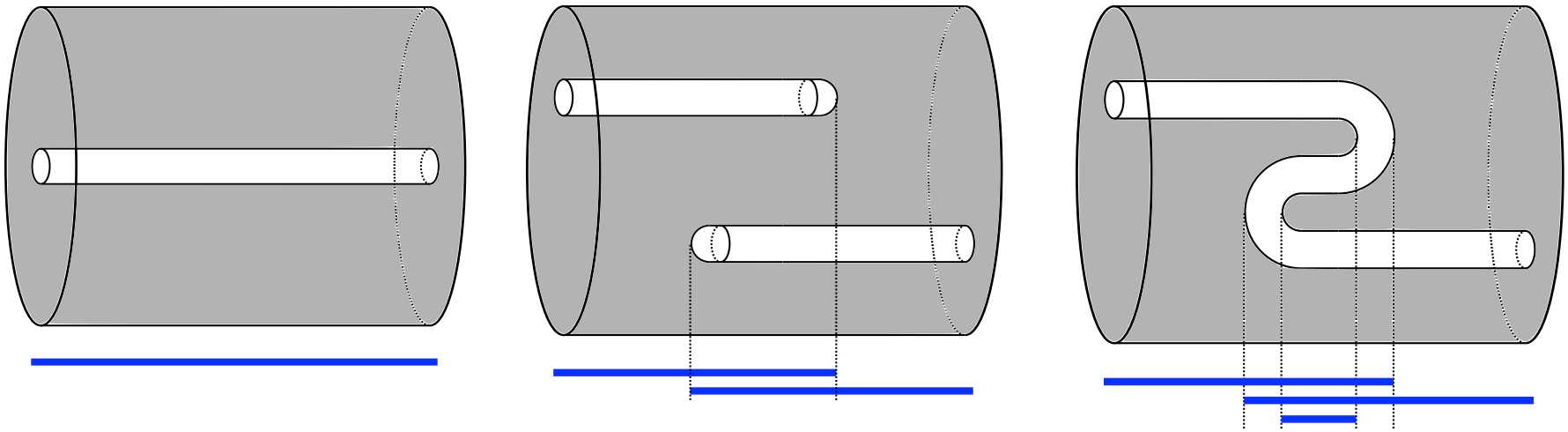
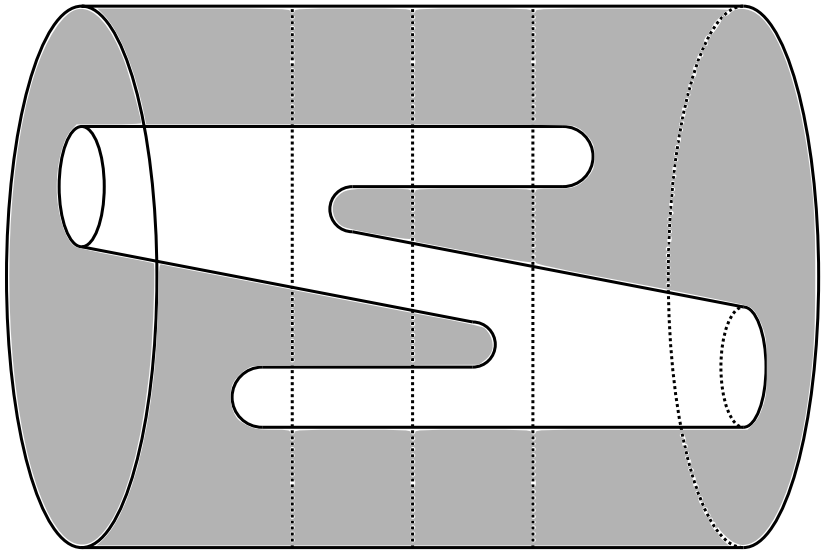
Zigzag persistence



Zigzag Persistence by G. Carlsson and V. de Silva

Zigzag persistence

- Form zigzag module for $X \to I$ with H_{d-1}.



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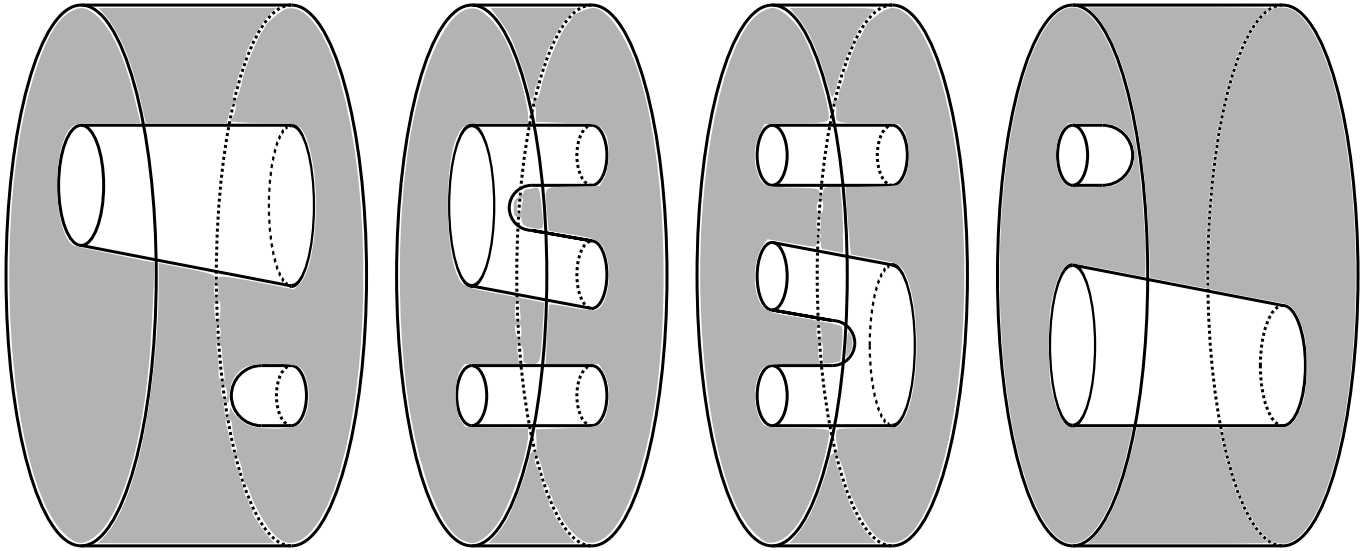

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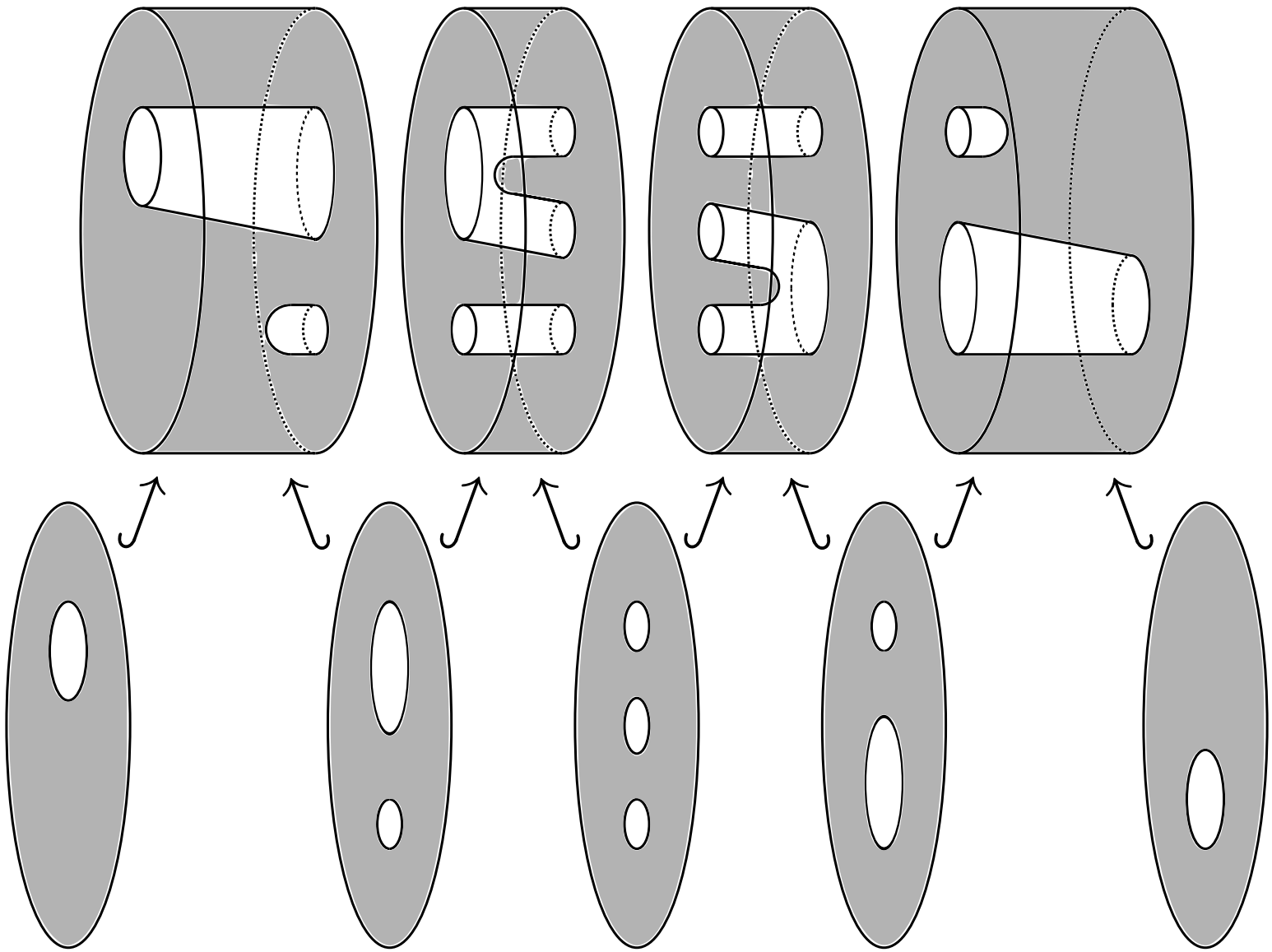

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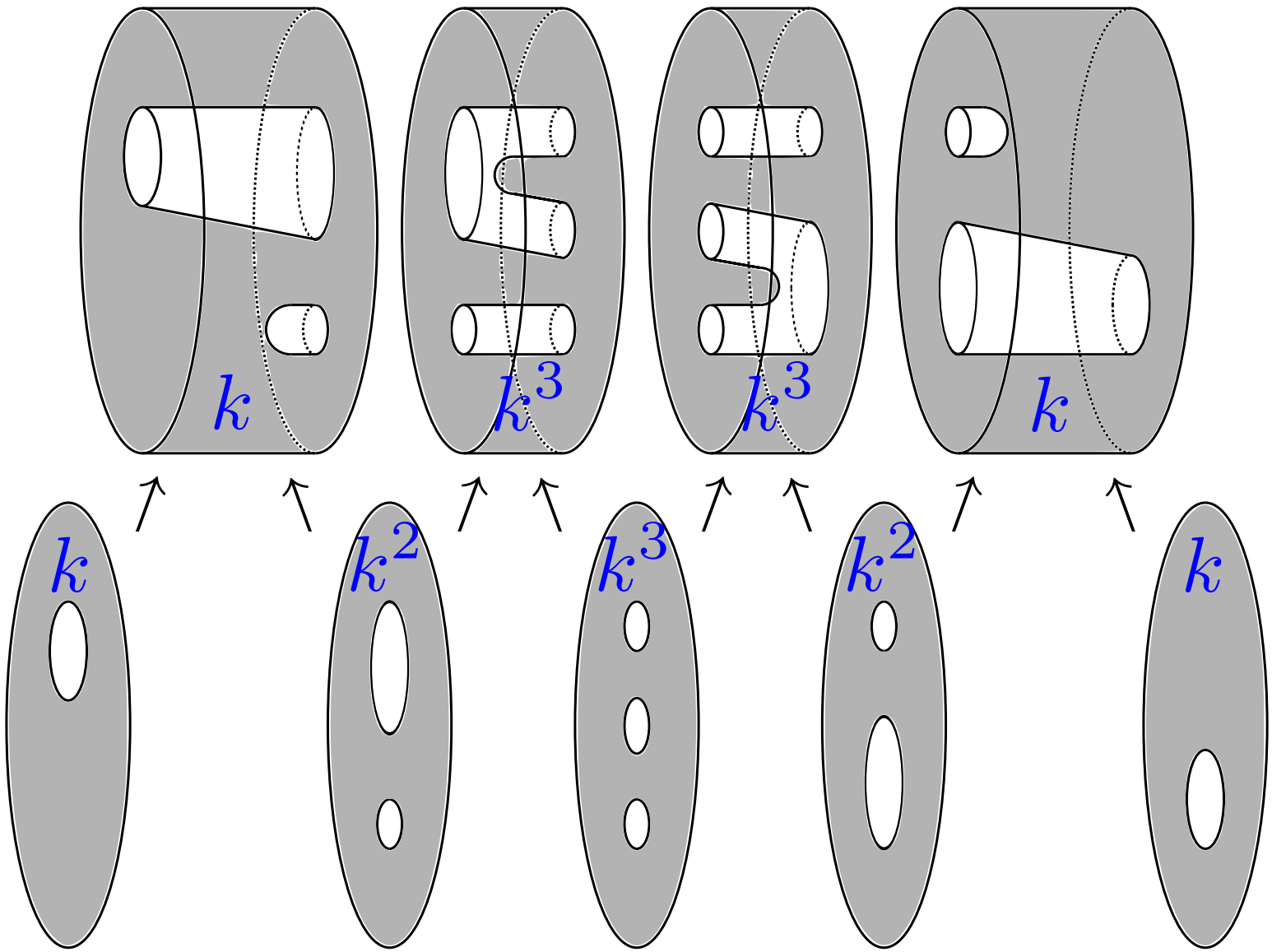

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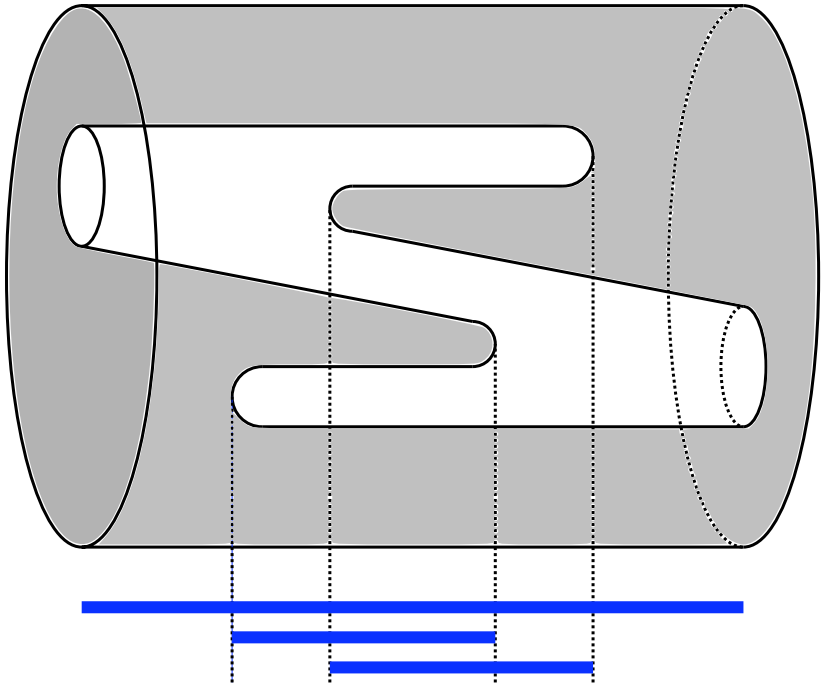

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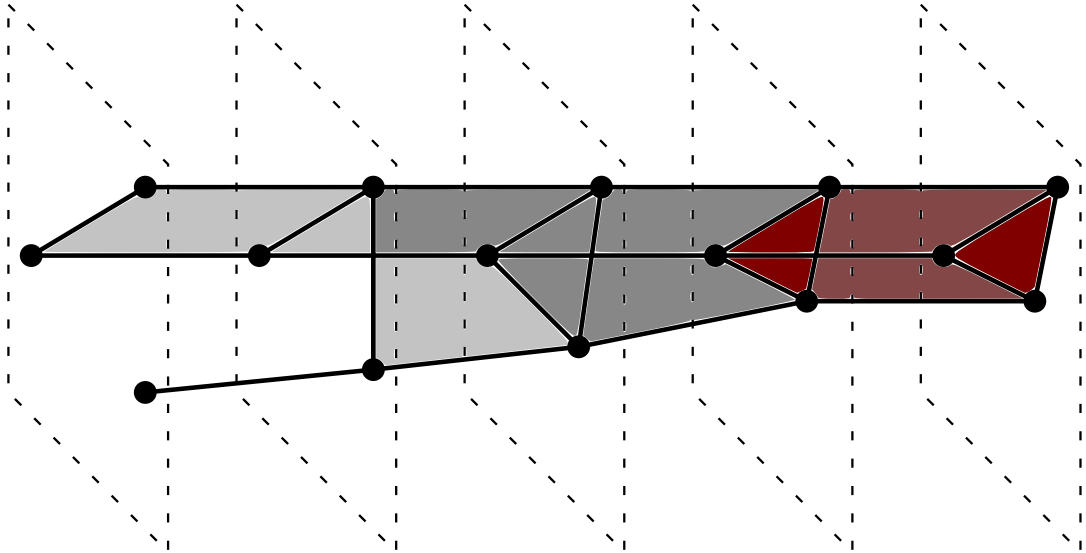

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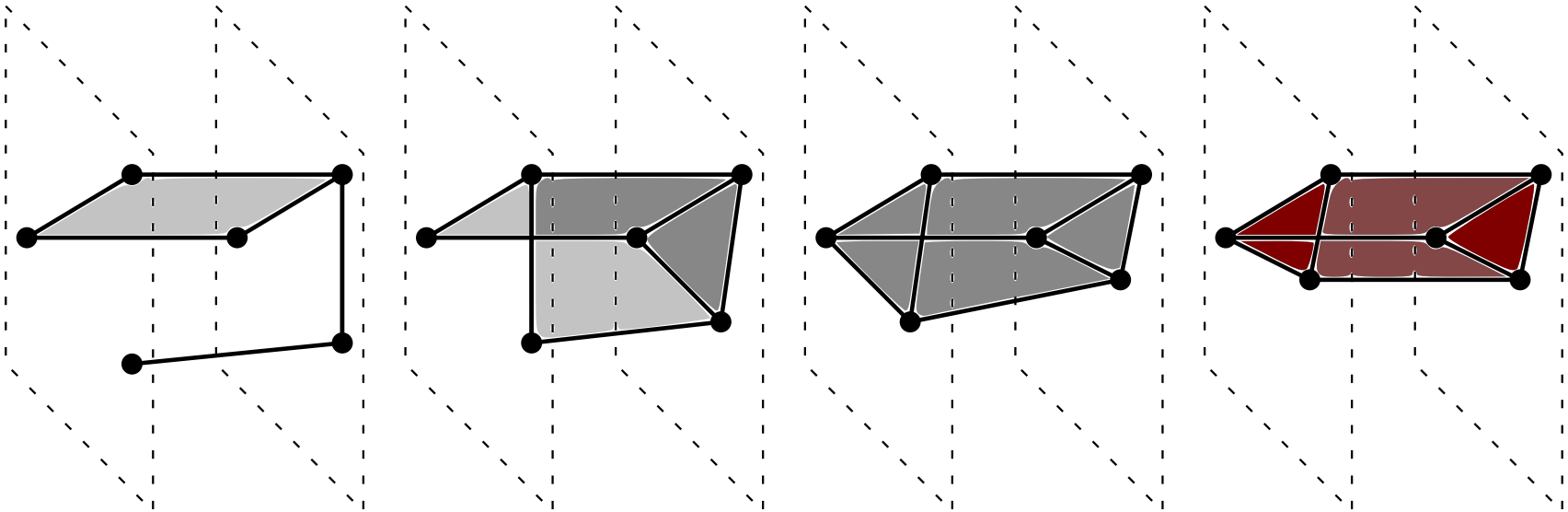

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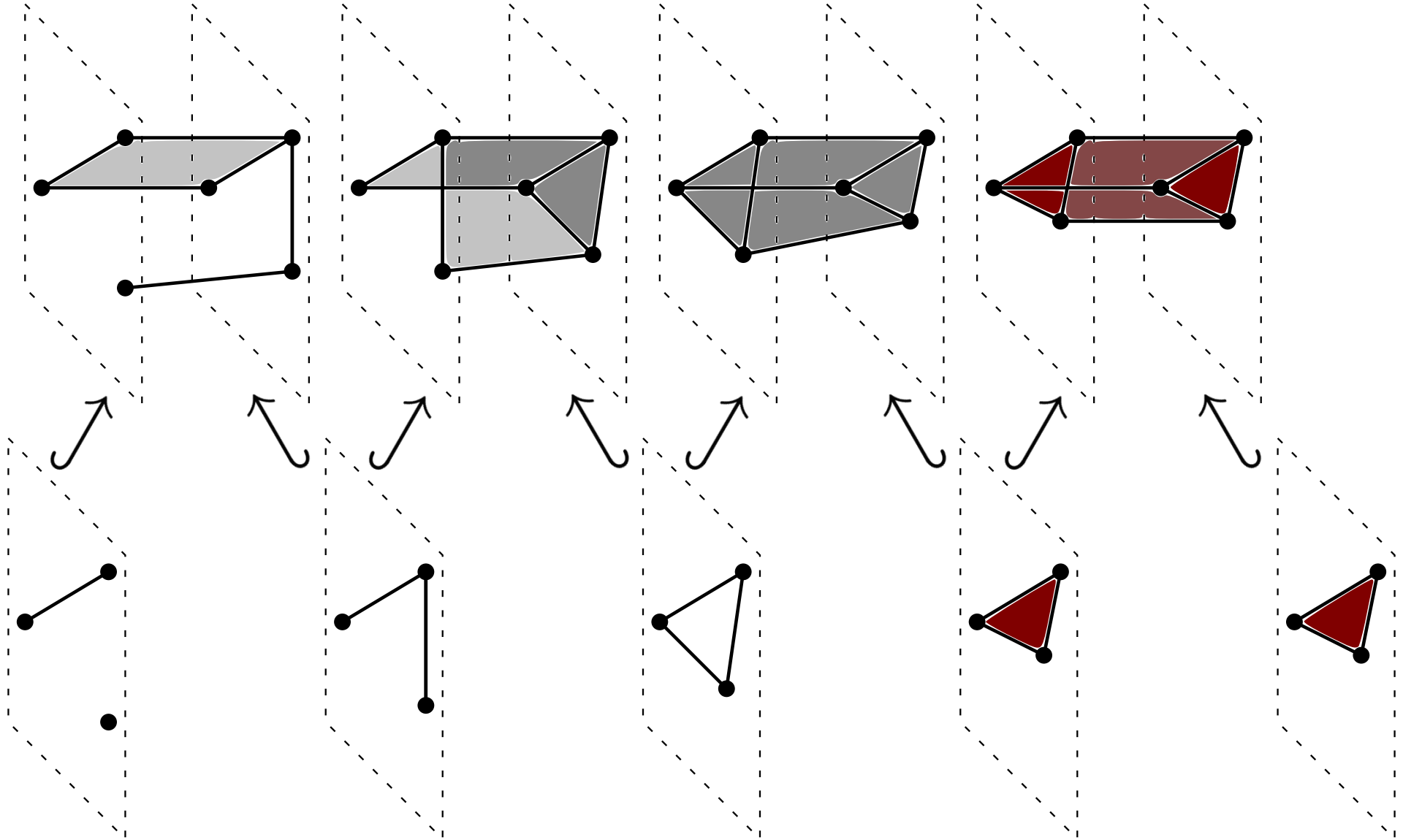

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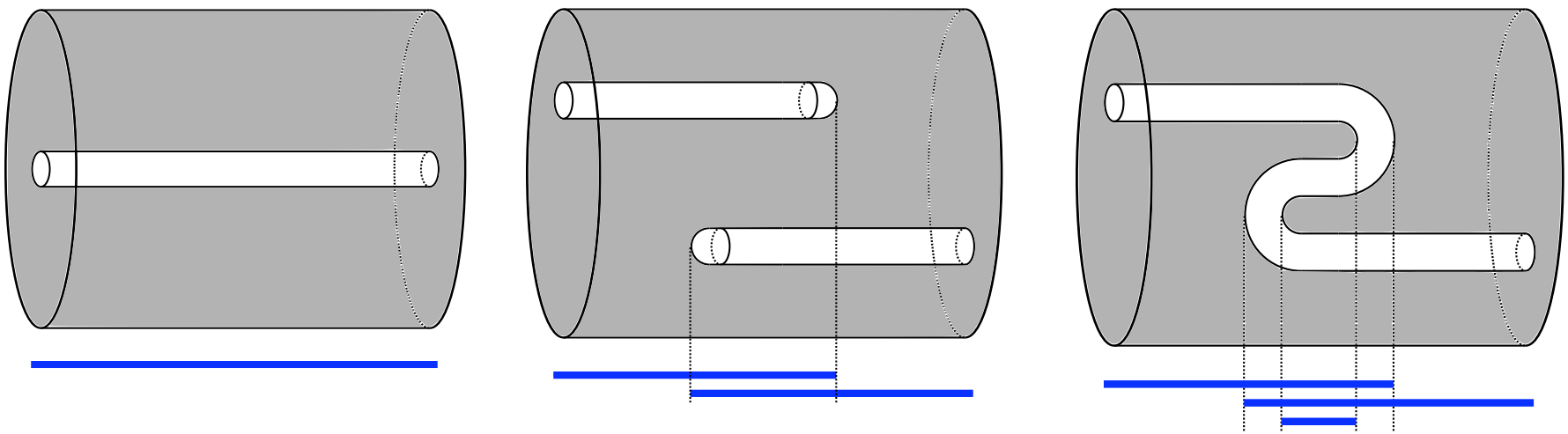

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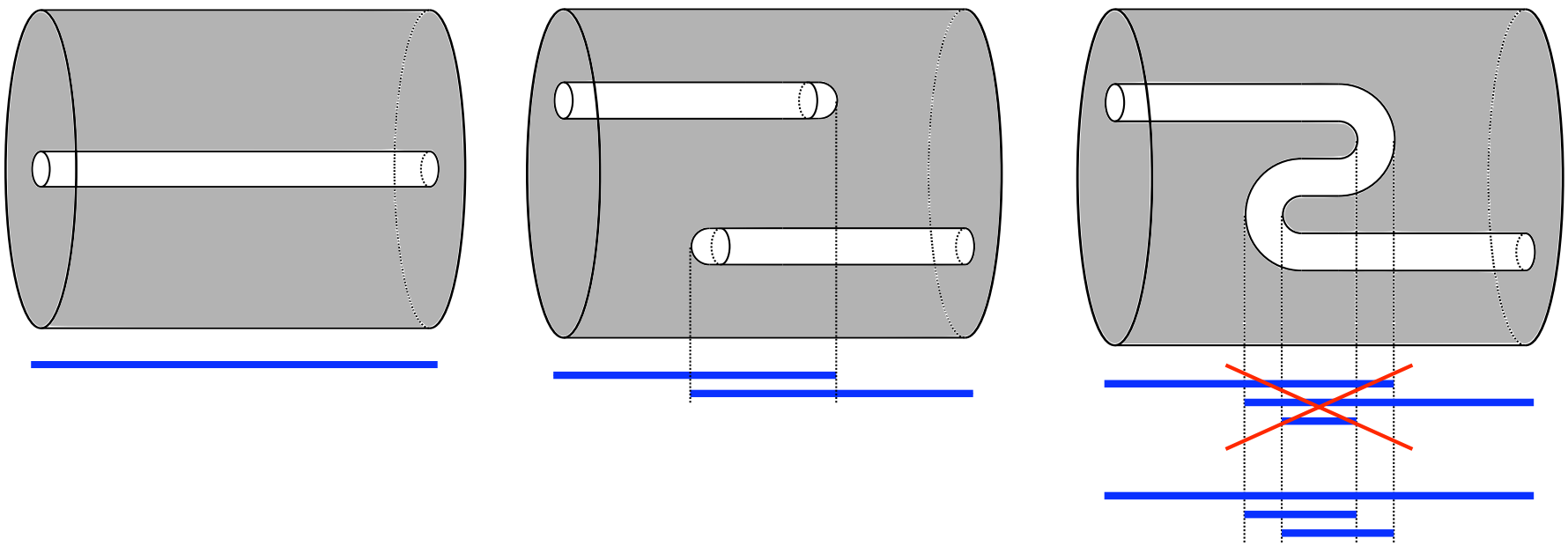
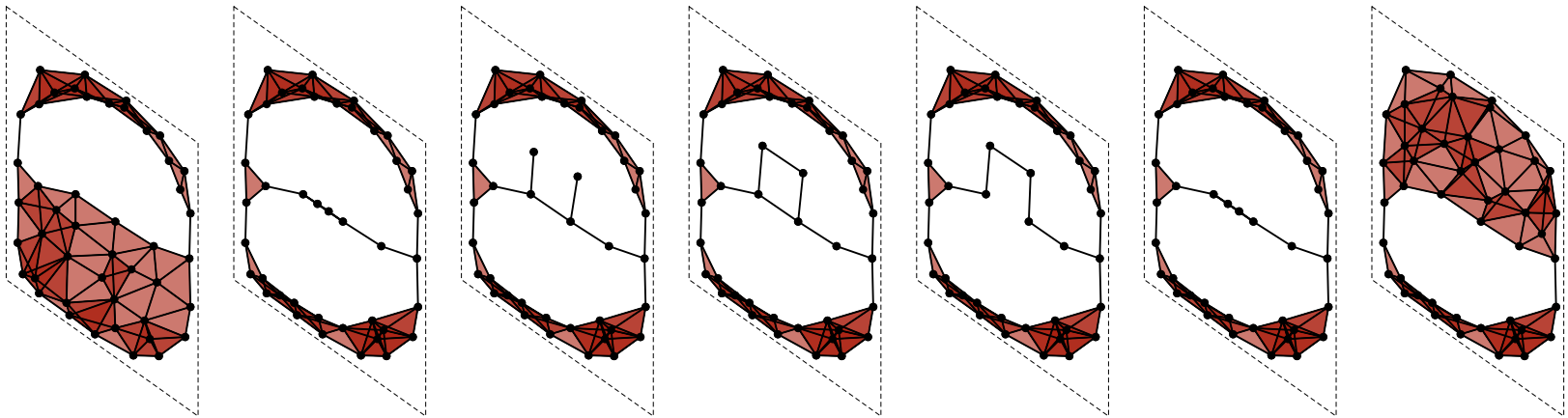


- <u>Hypothesis:</u> There is an evasion path $\Leftrightarrow$ there is a full-length bar.
- <u>Theorem:</u> If there is an evasion path then there is a full-length bar.
- Streaming computation.

Dependence on embedding $X \hookrightarrow B \times I$

- The time-varying Čech complex alone does not determine if an evasion path exists.



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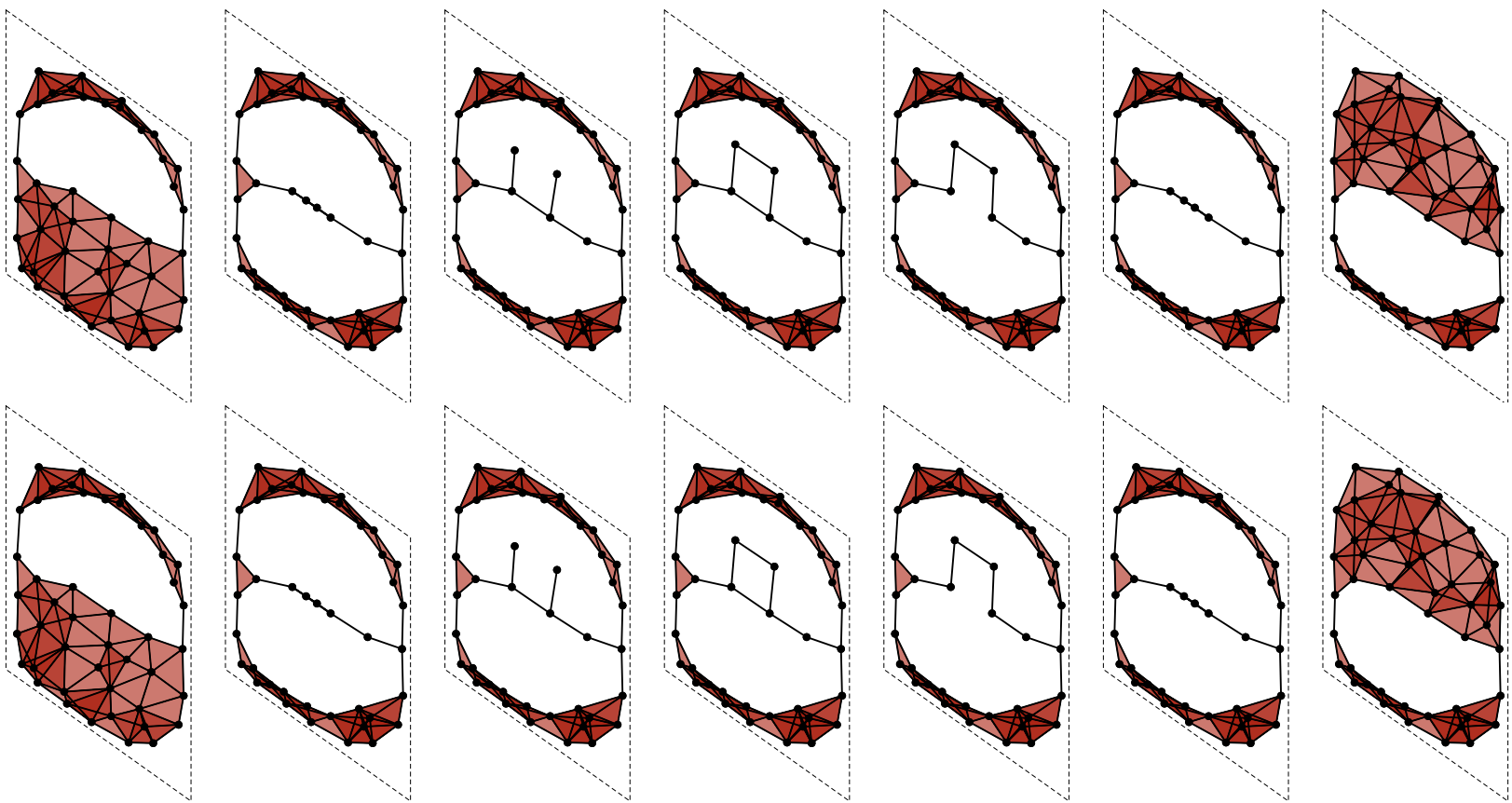

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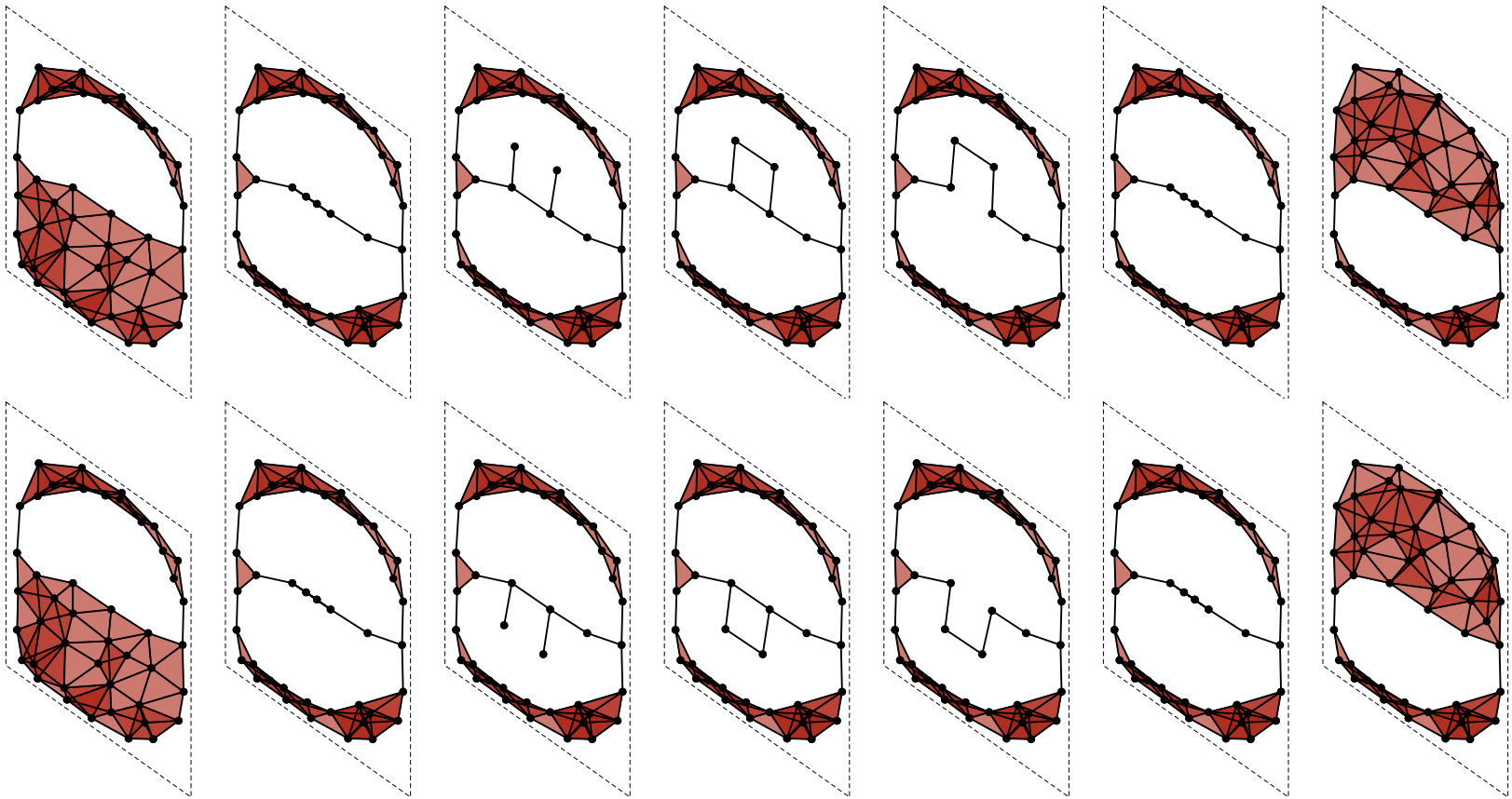

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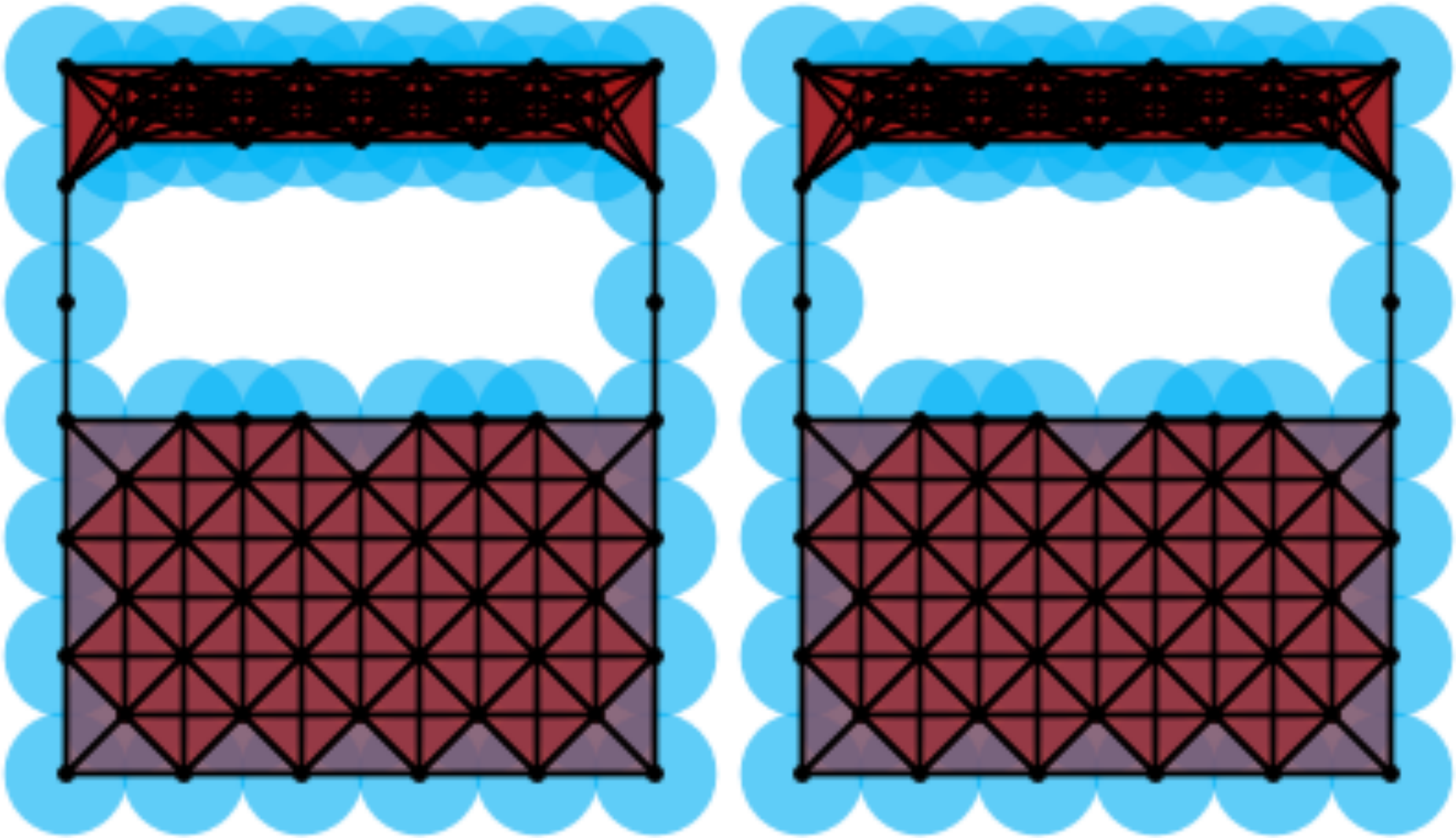

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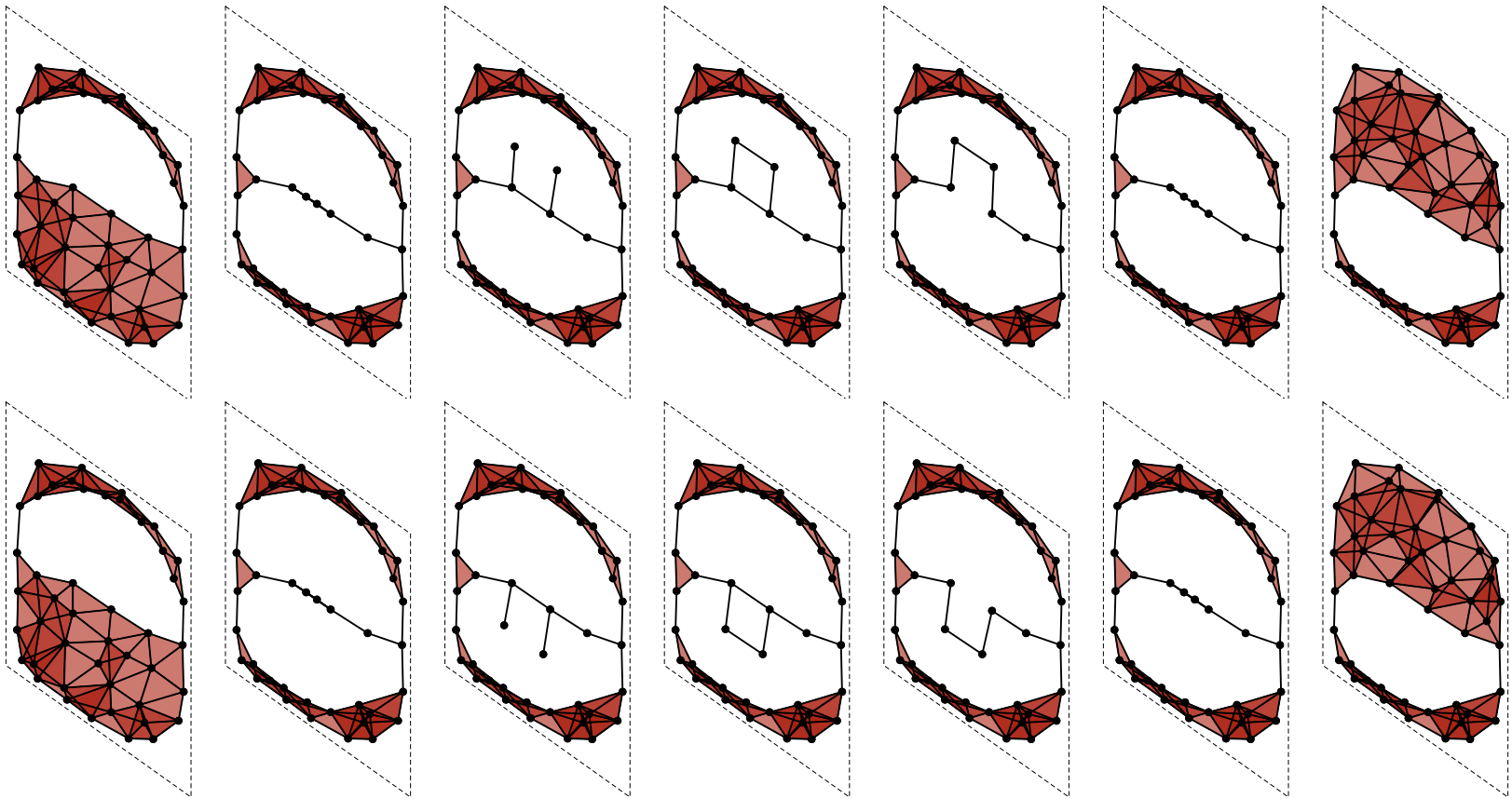

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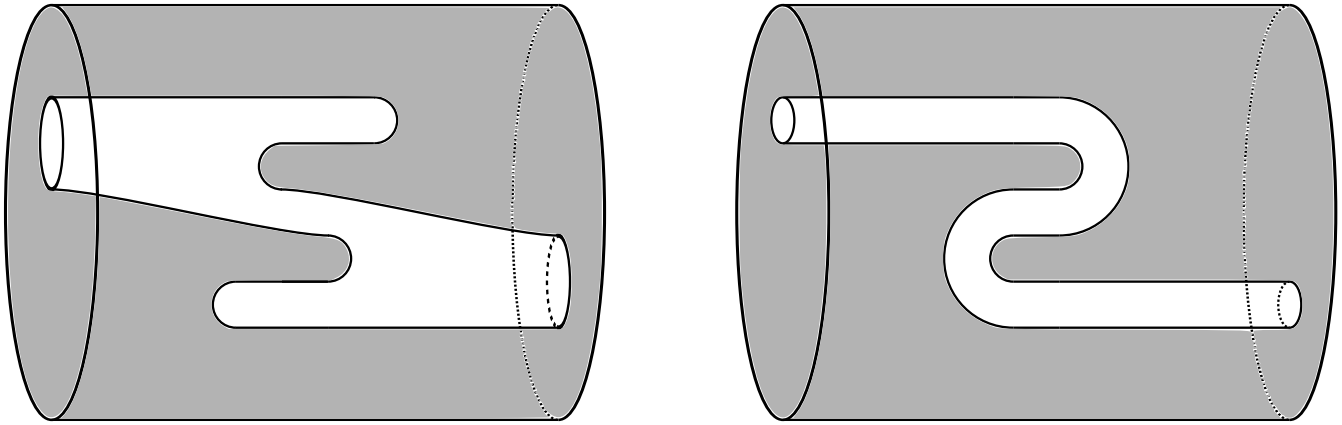

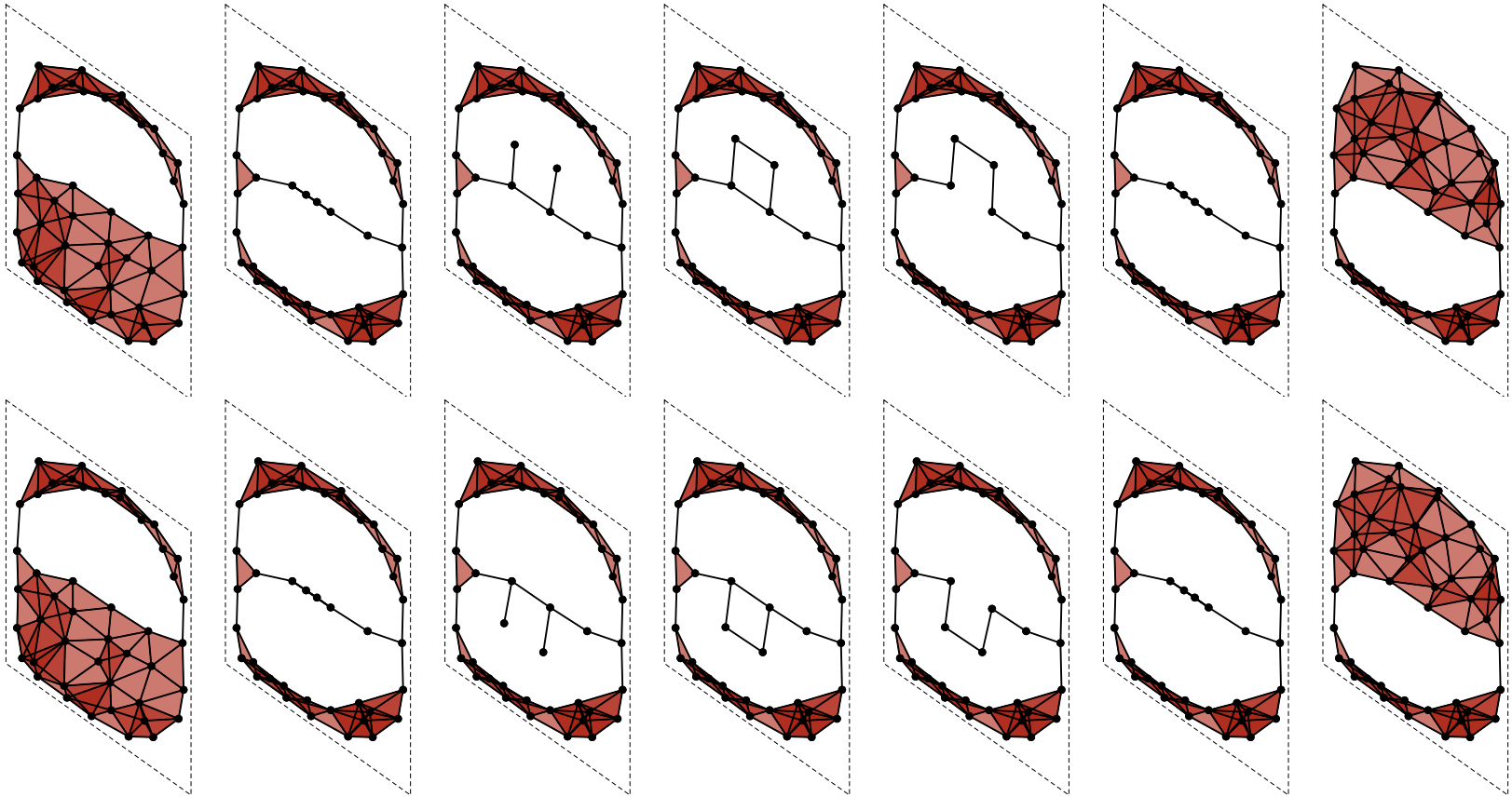

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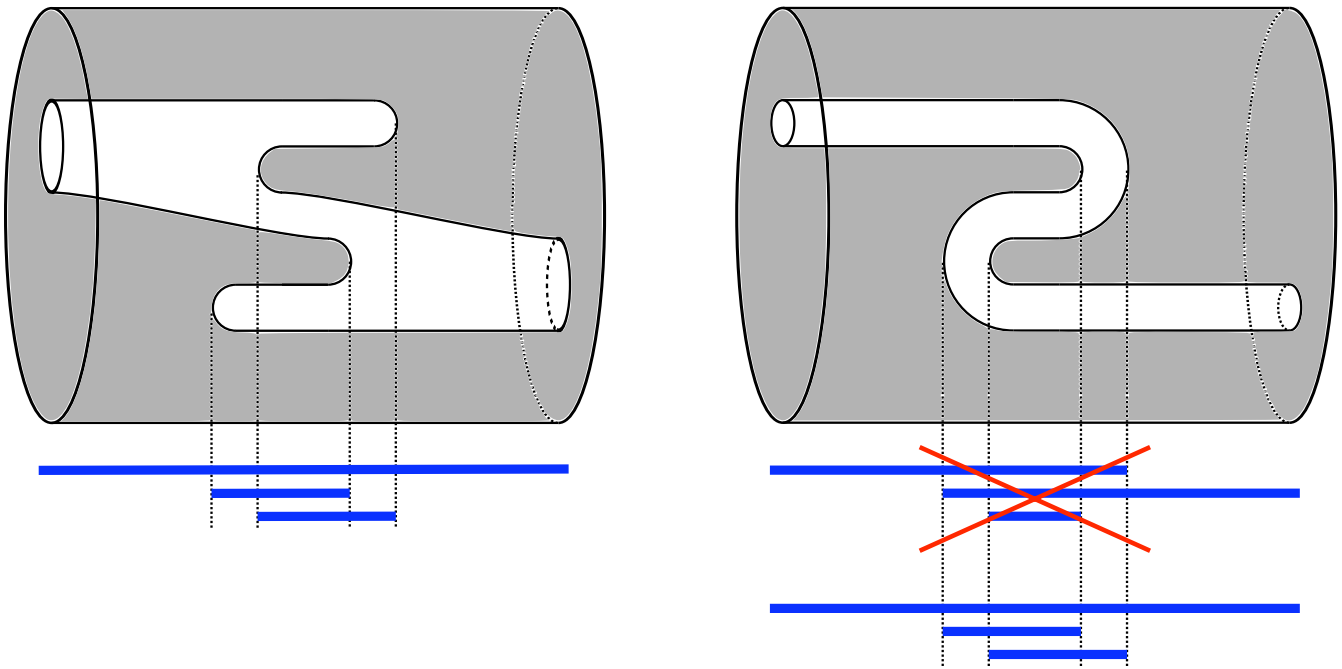
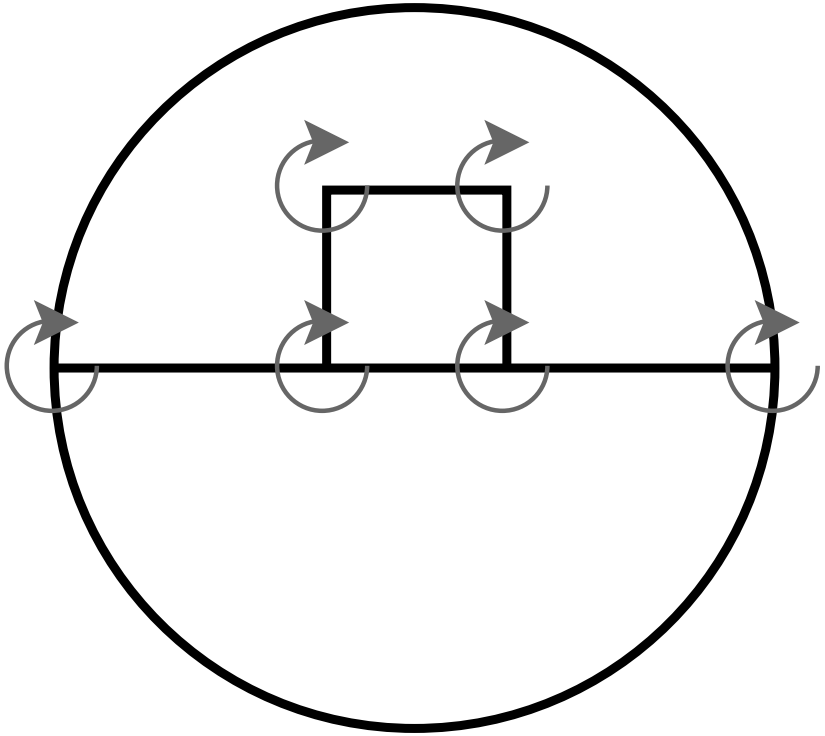


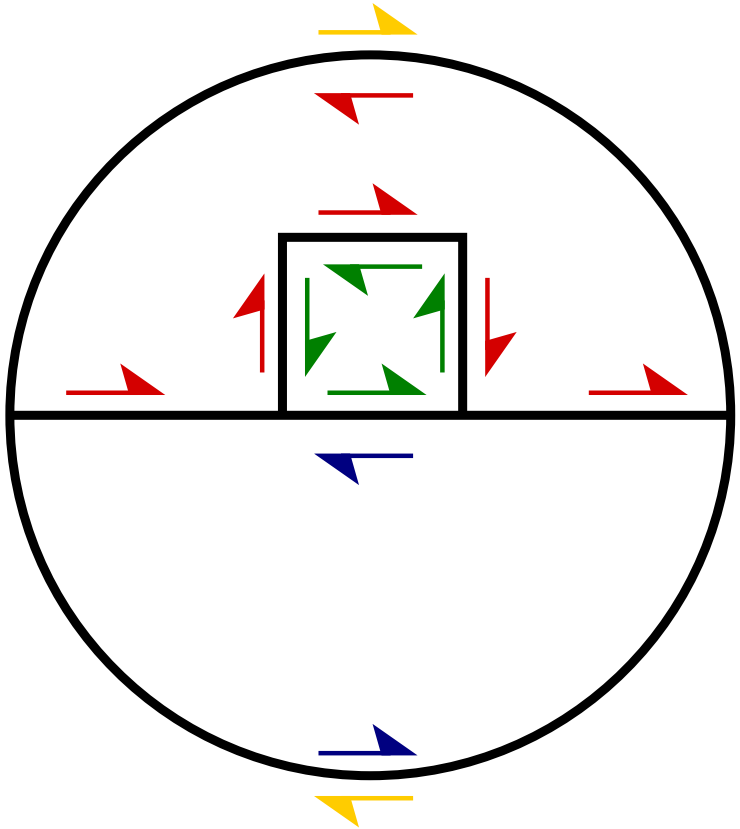
- Covered regions are fibrewise homotopy equivalent but their complements are not.

Planar sensors measuring cyclic orders

- Each vertex measures the cyclic ordering of adjacent edges.
- Equivalent to a set of boundary cycles.



Cyclic orderings



Boundary cycles

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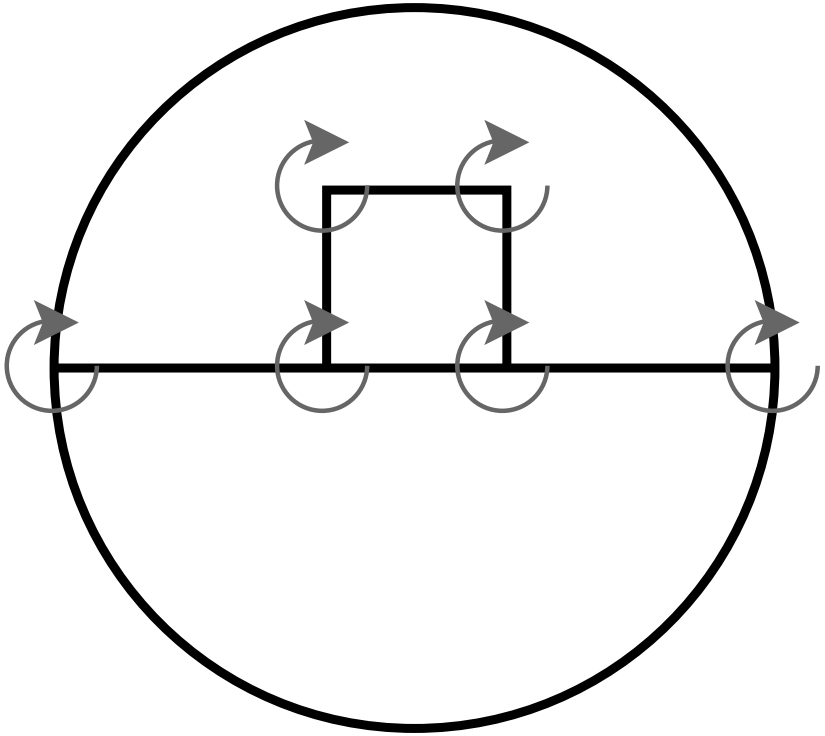
- <u>Theorem.</u> In a planar sensor network that remains connected, the time-varying alpha complex with rotation information determines if an evasion path exists.



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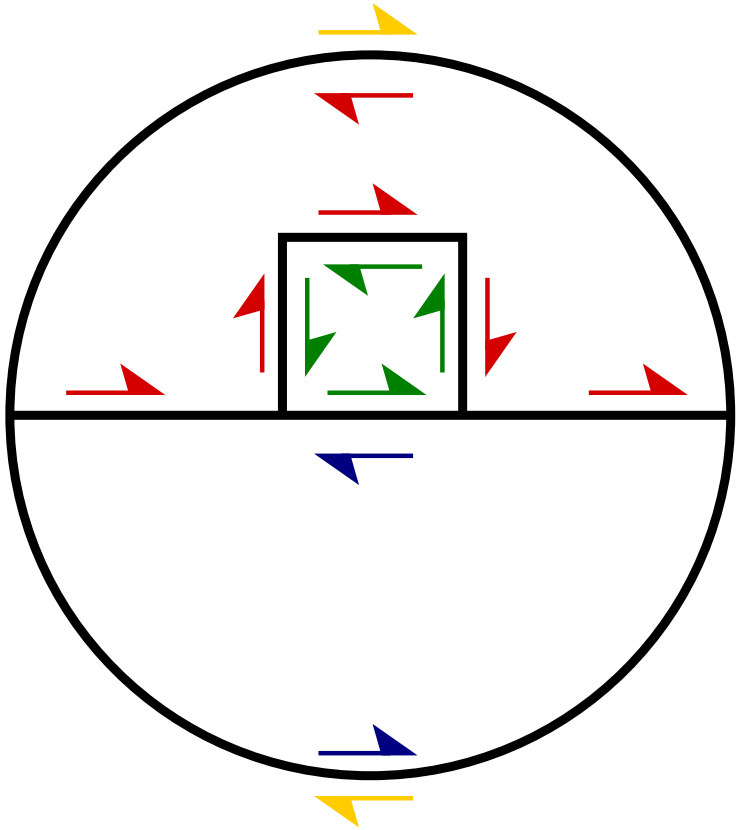

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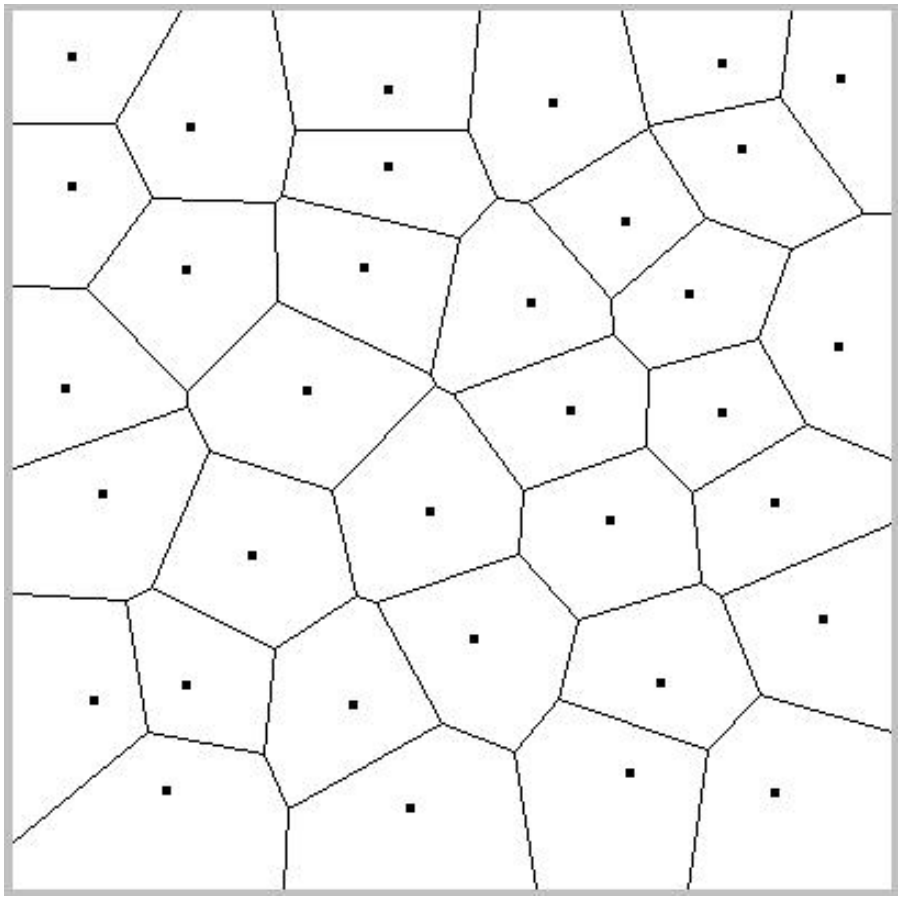
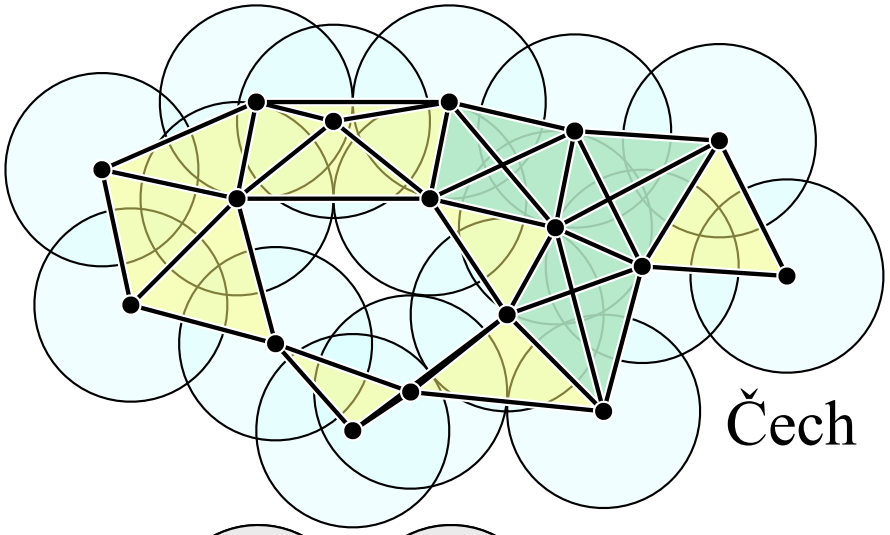
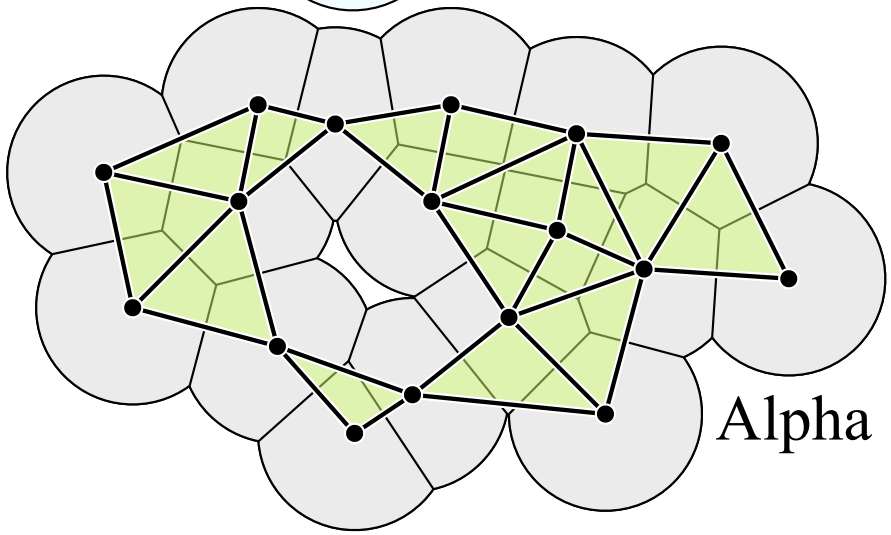


Voronoi regions





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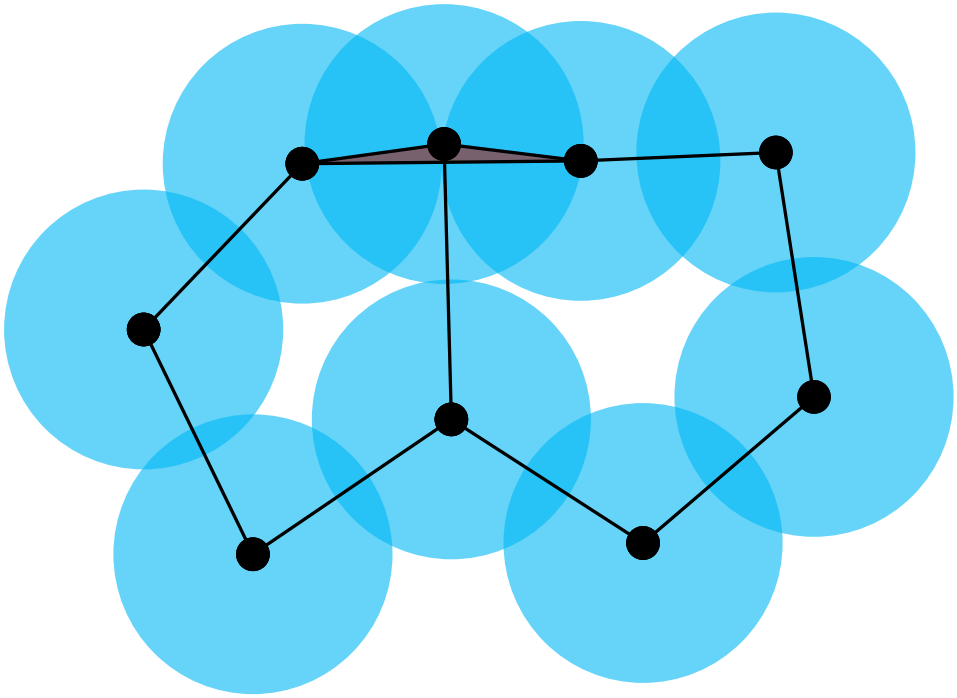
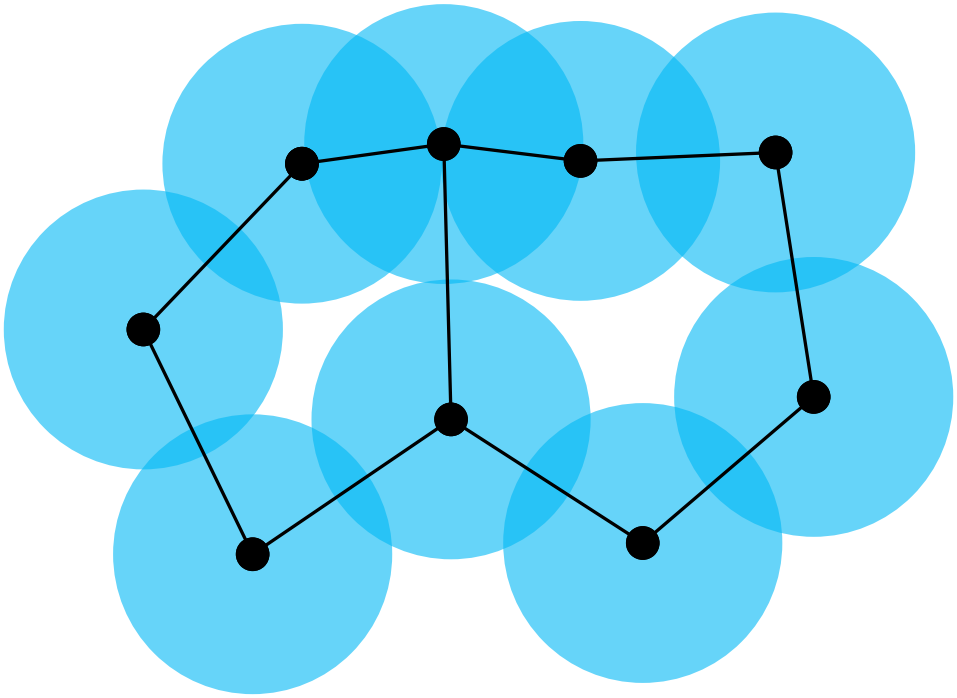


Čech



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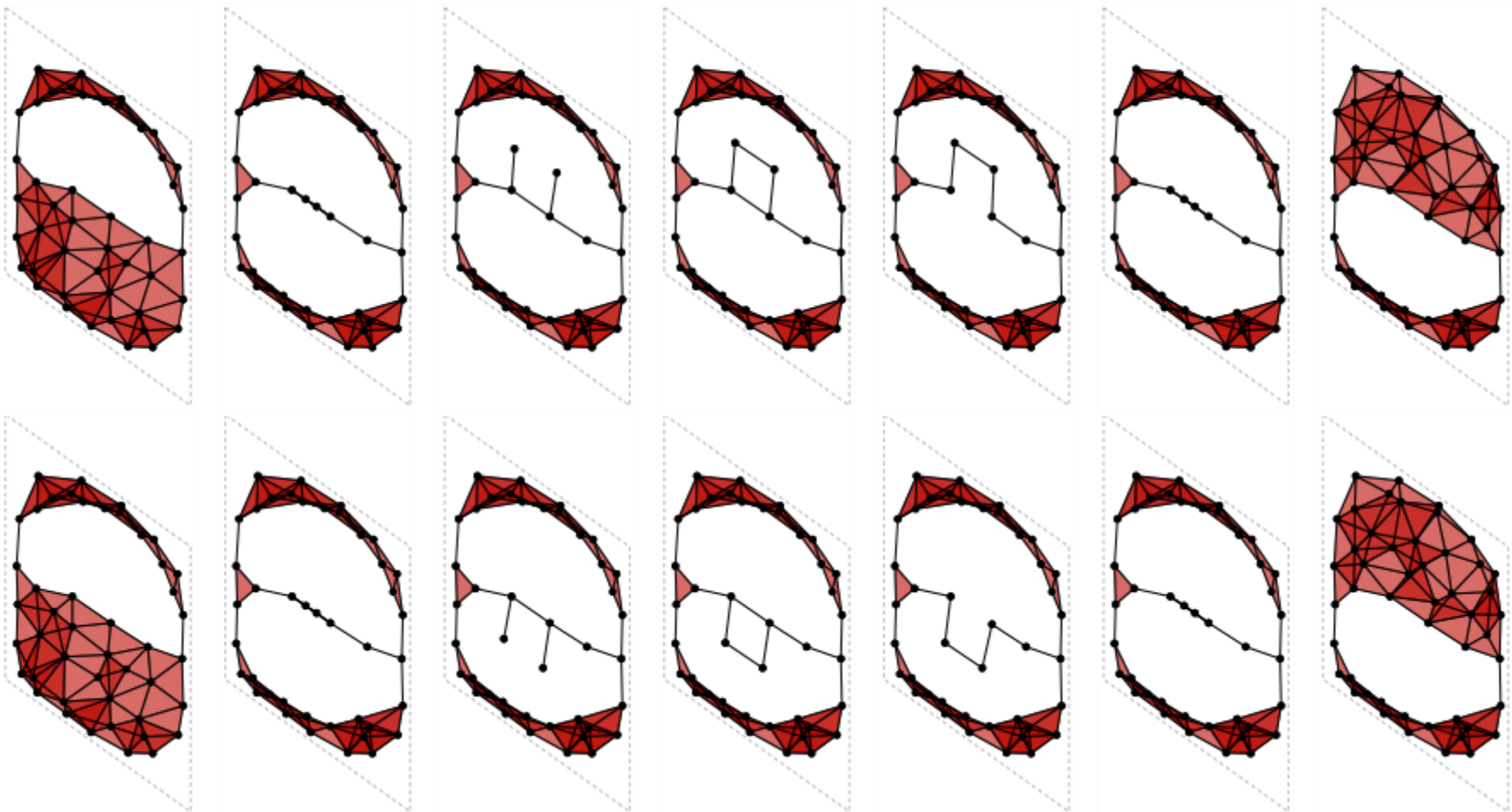

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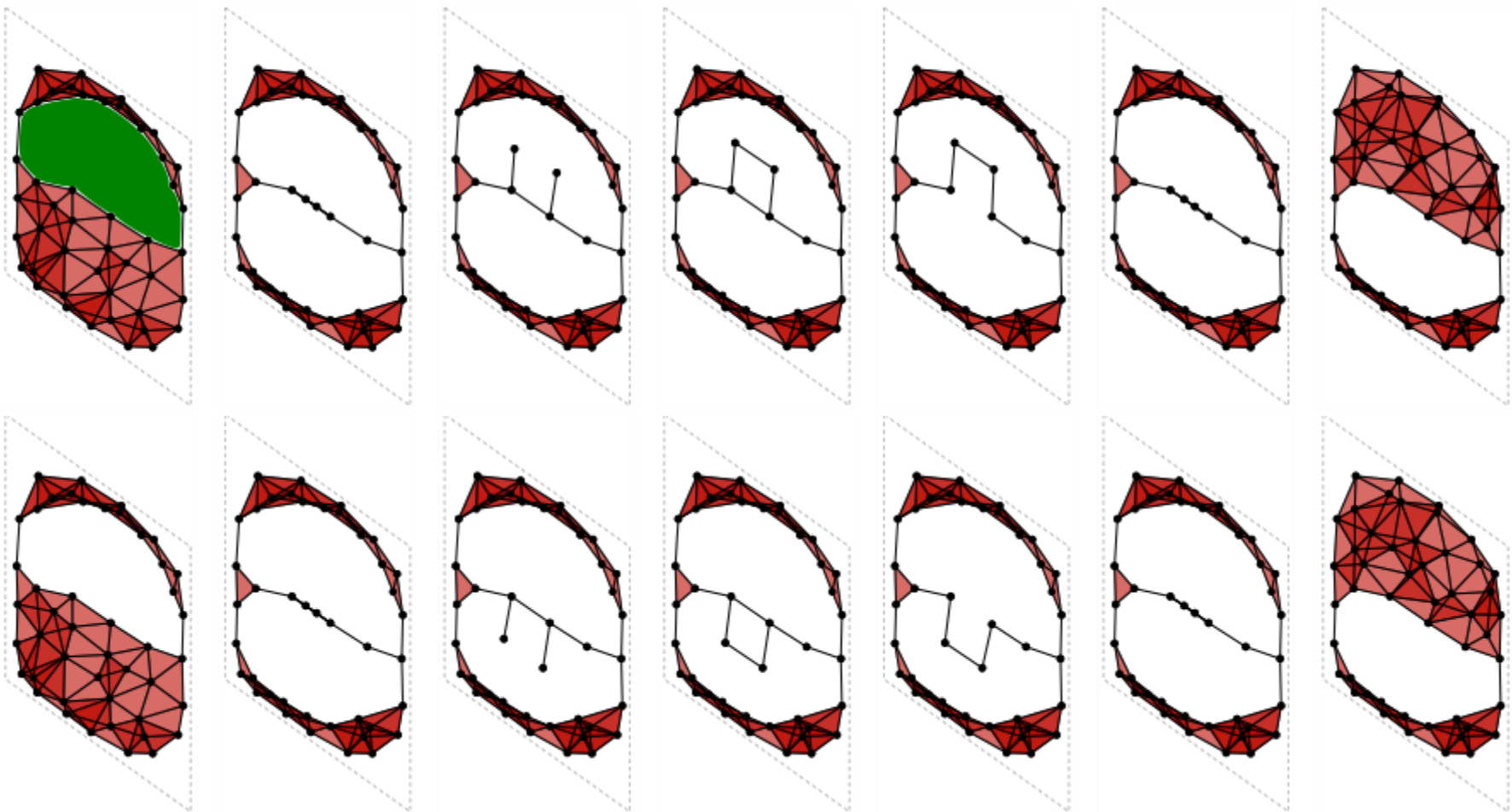

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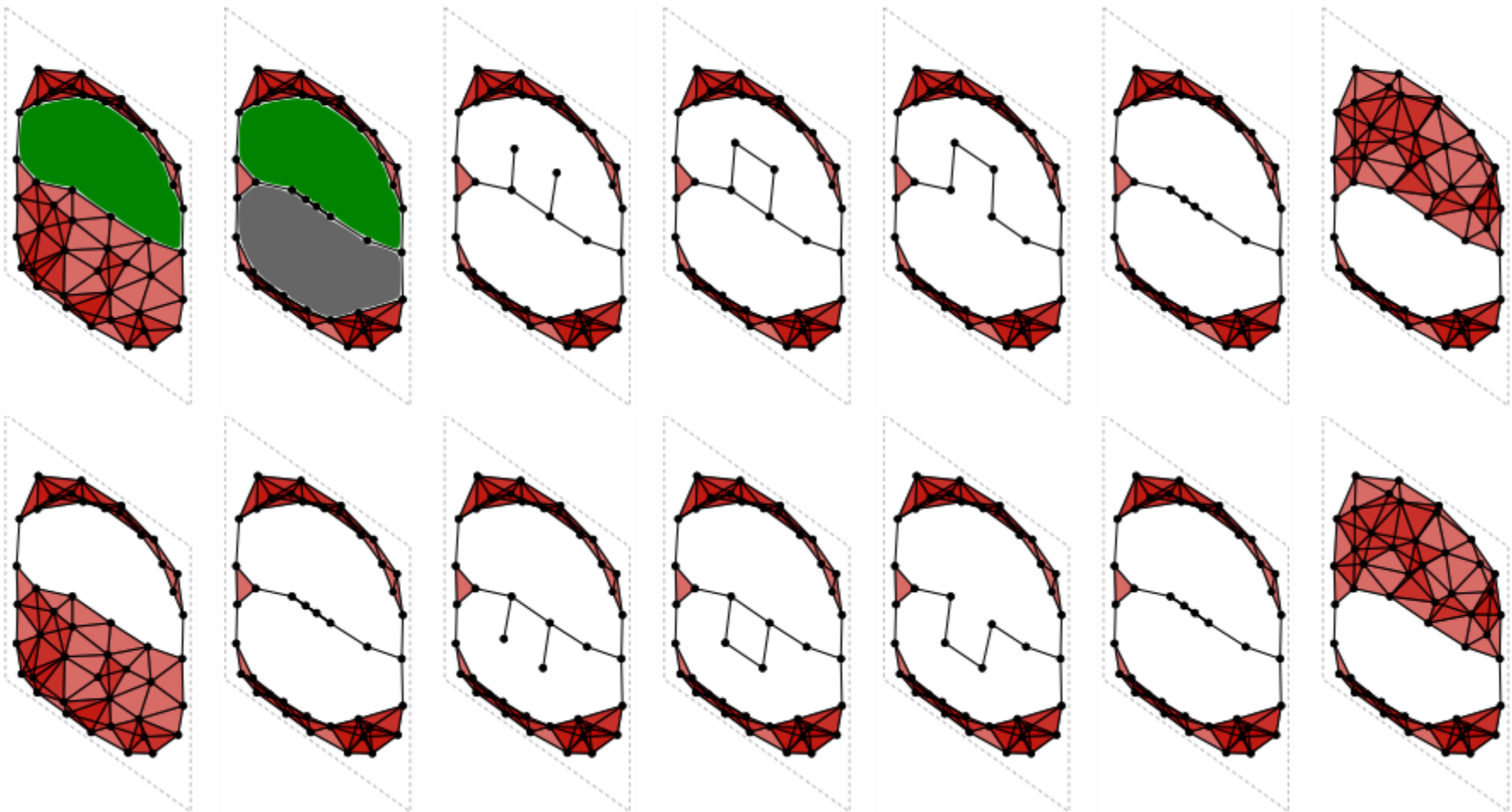

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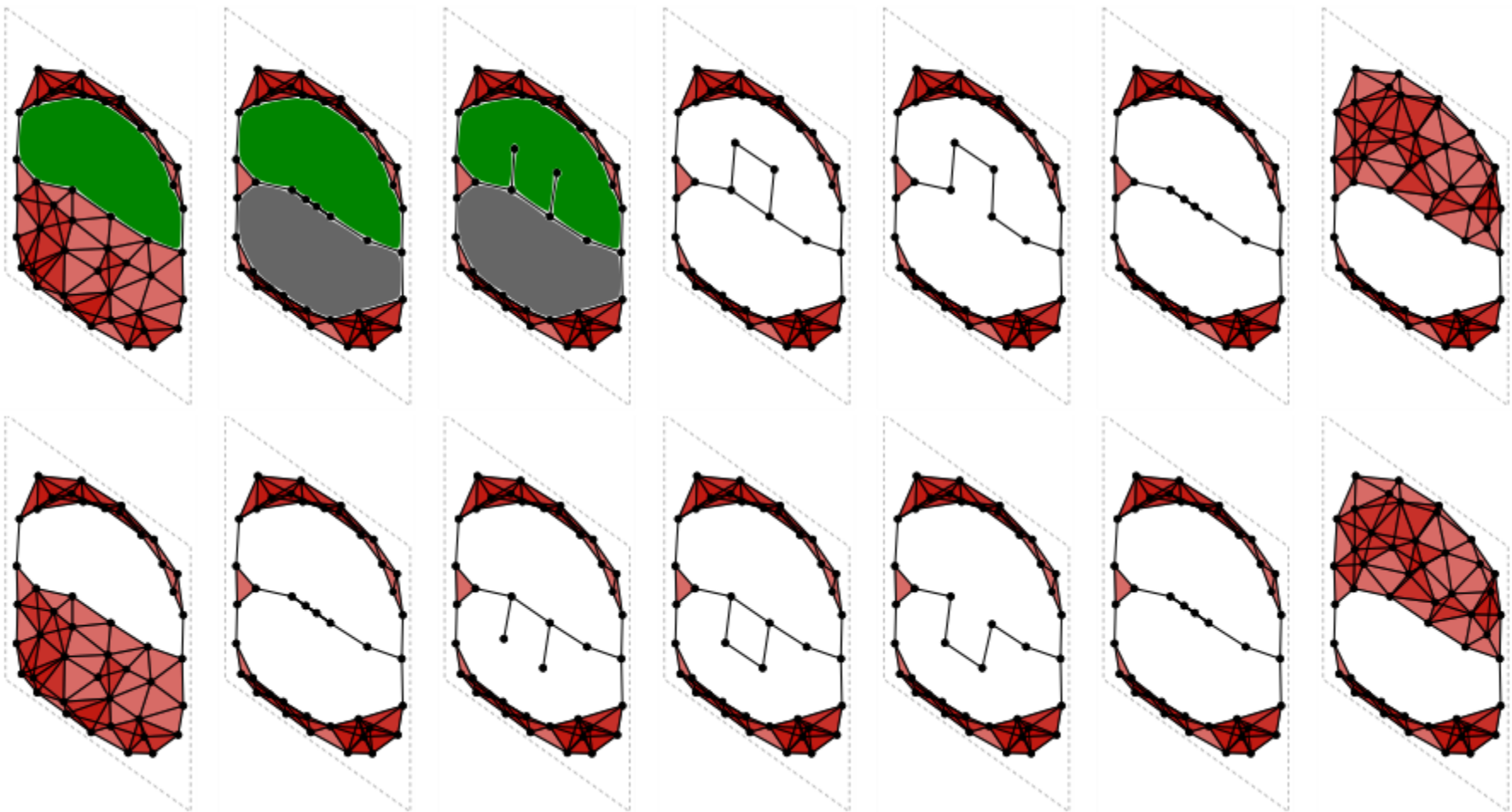

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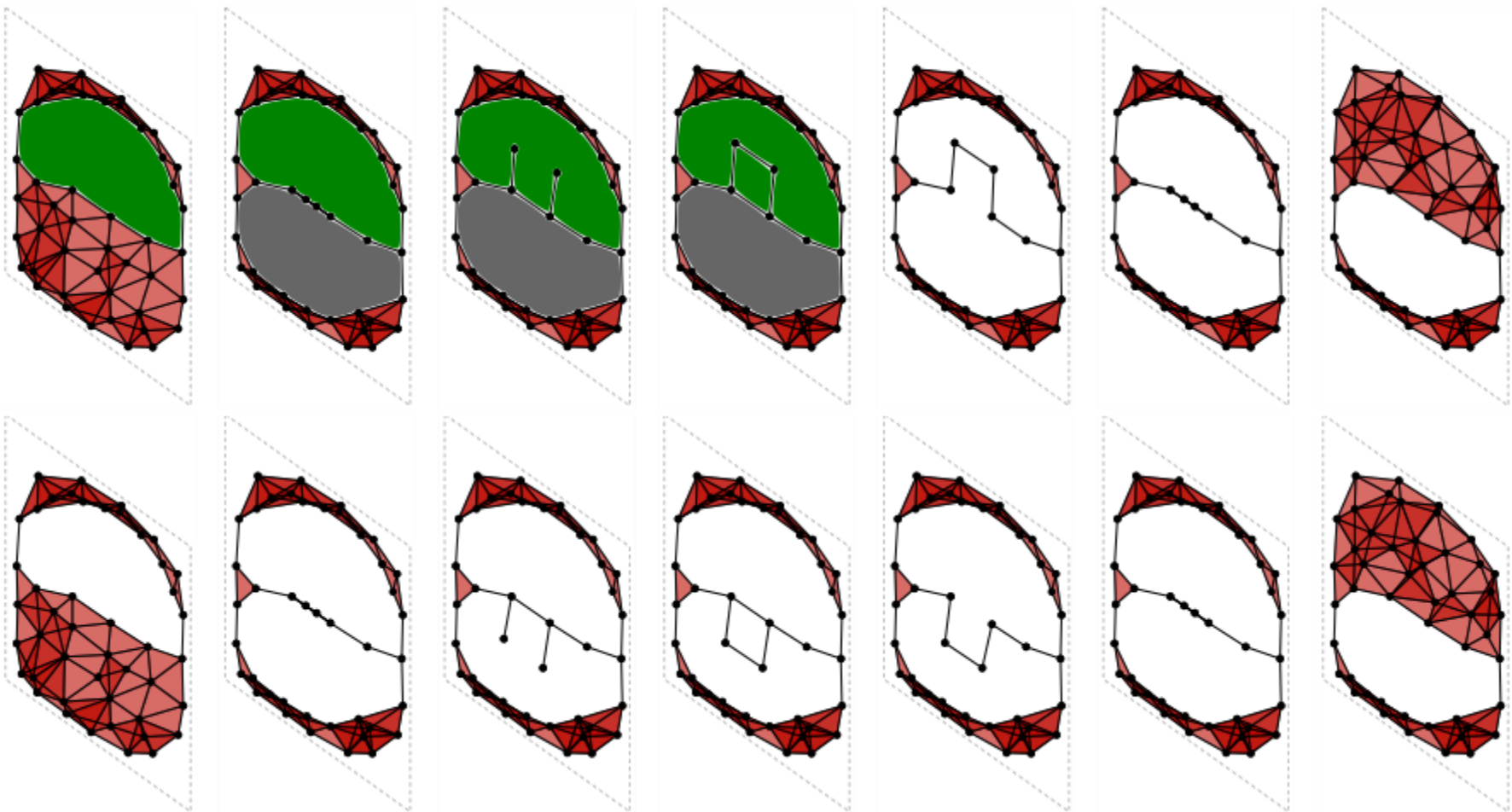

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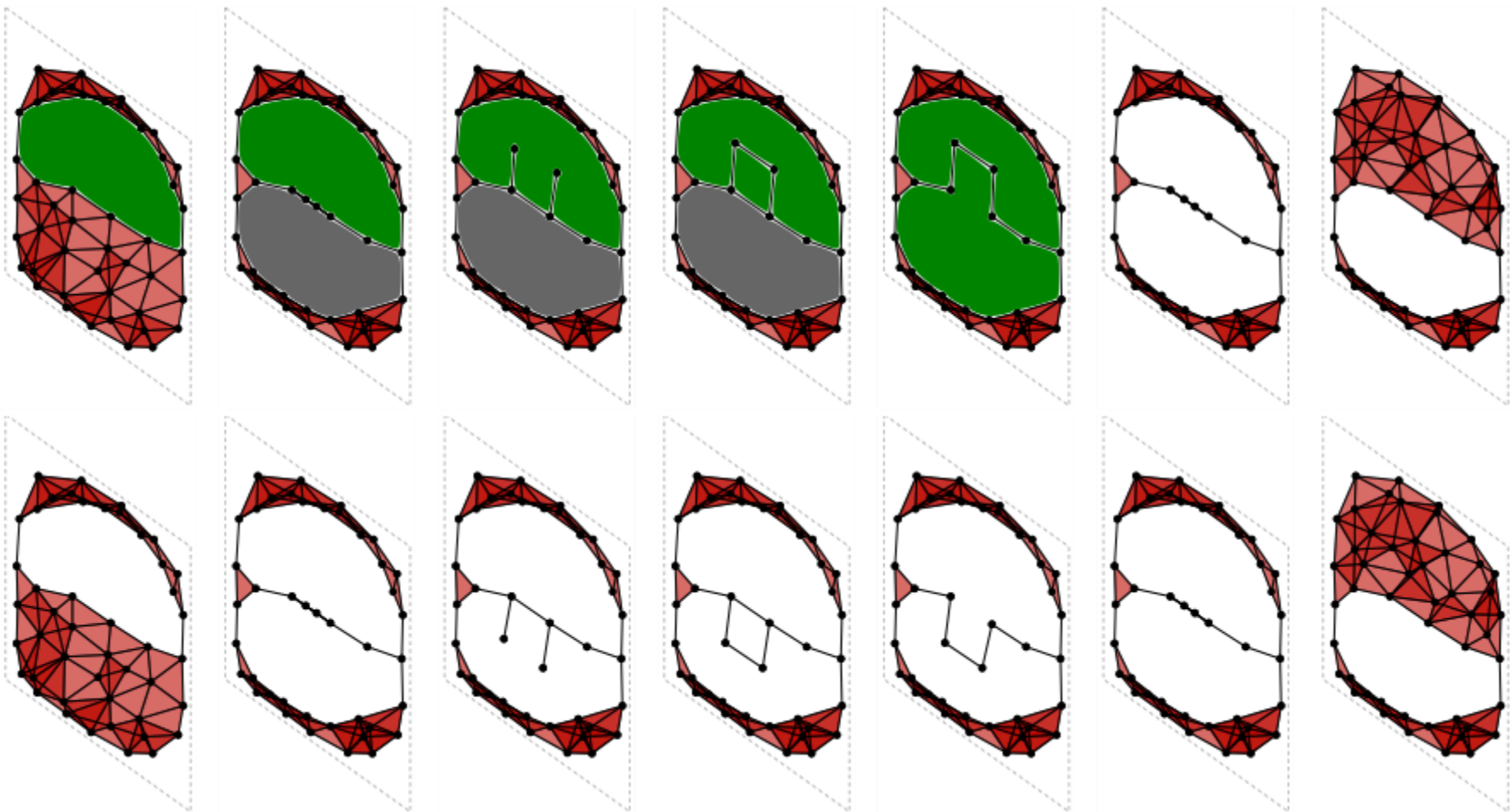

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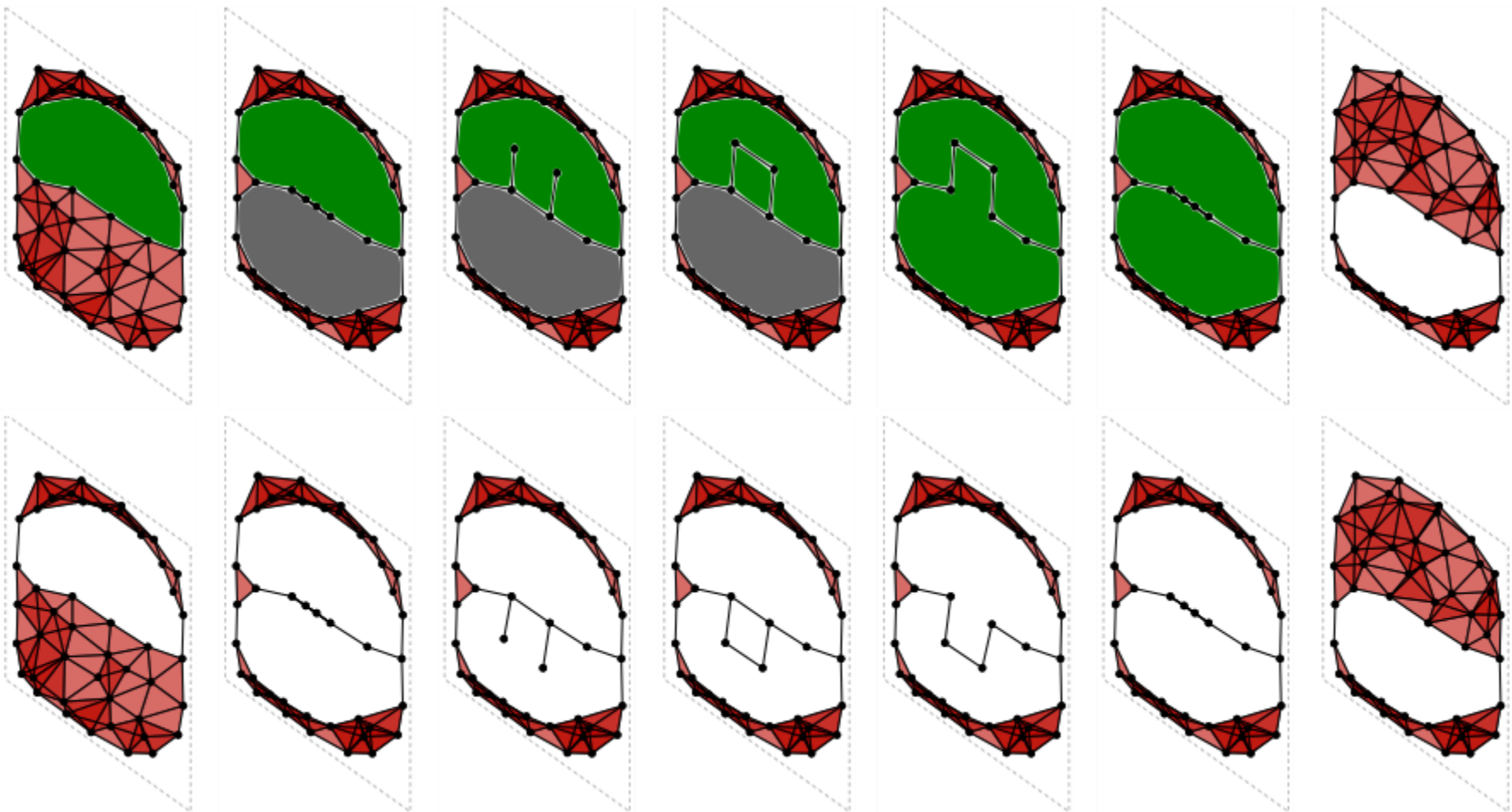

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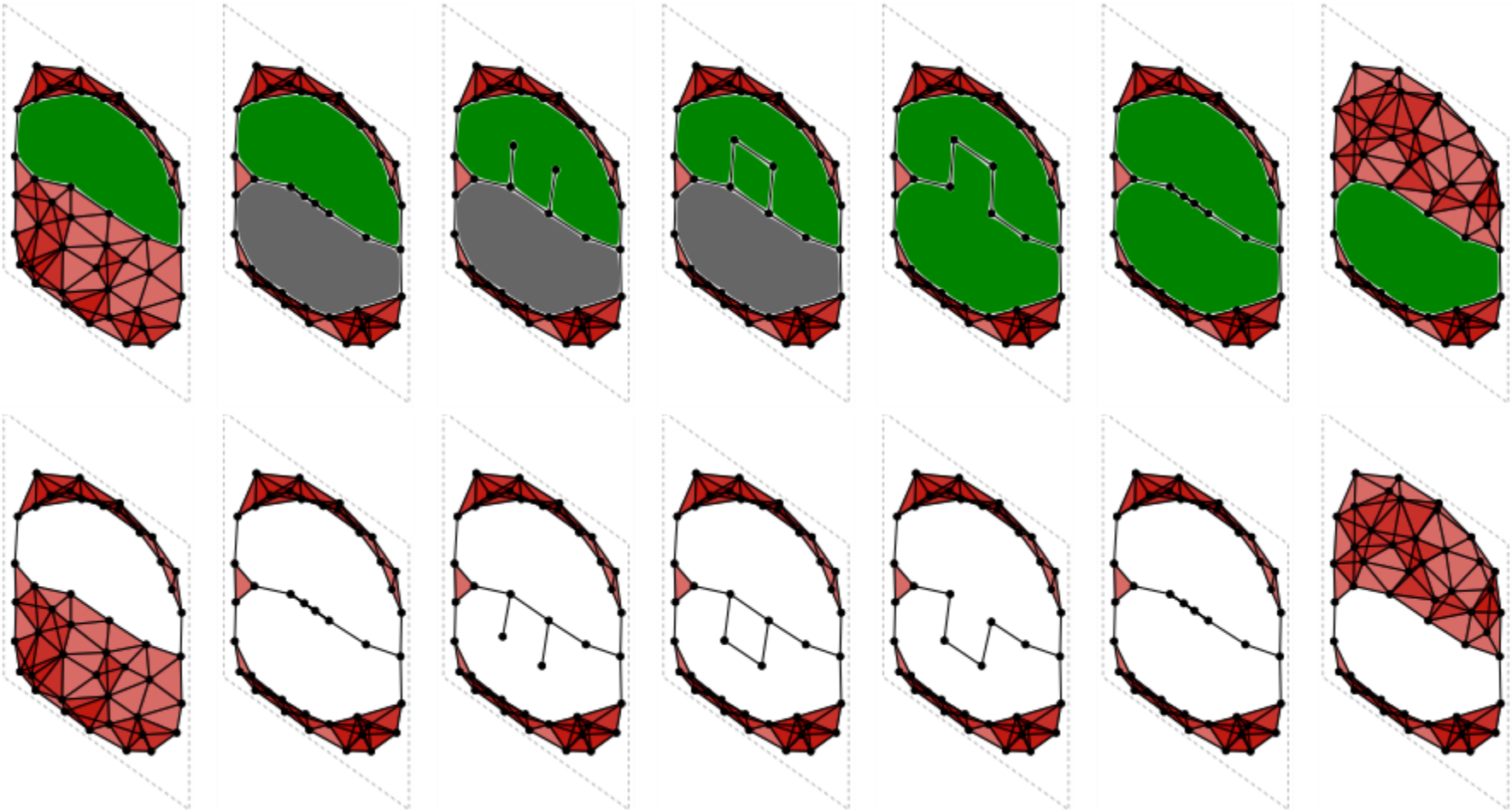

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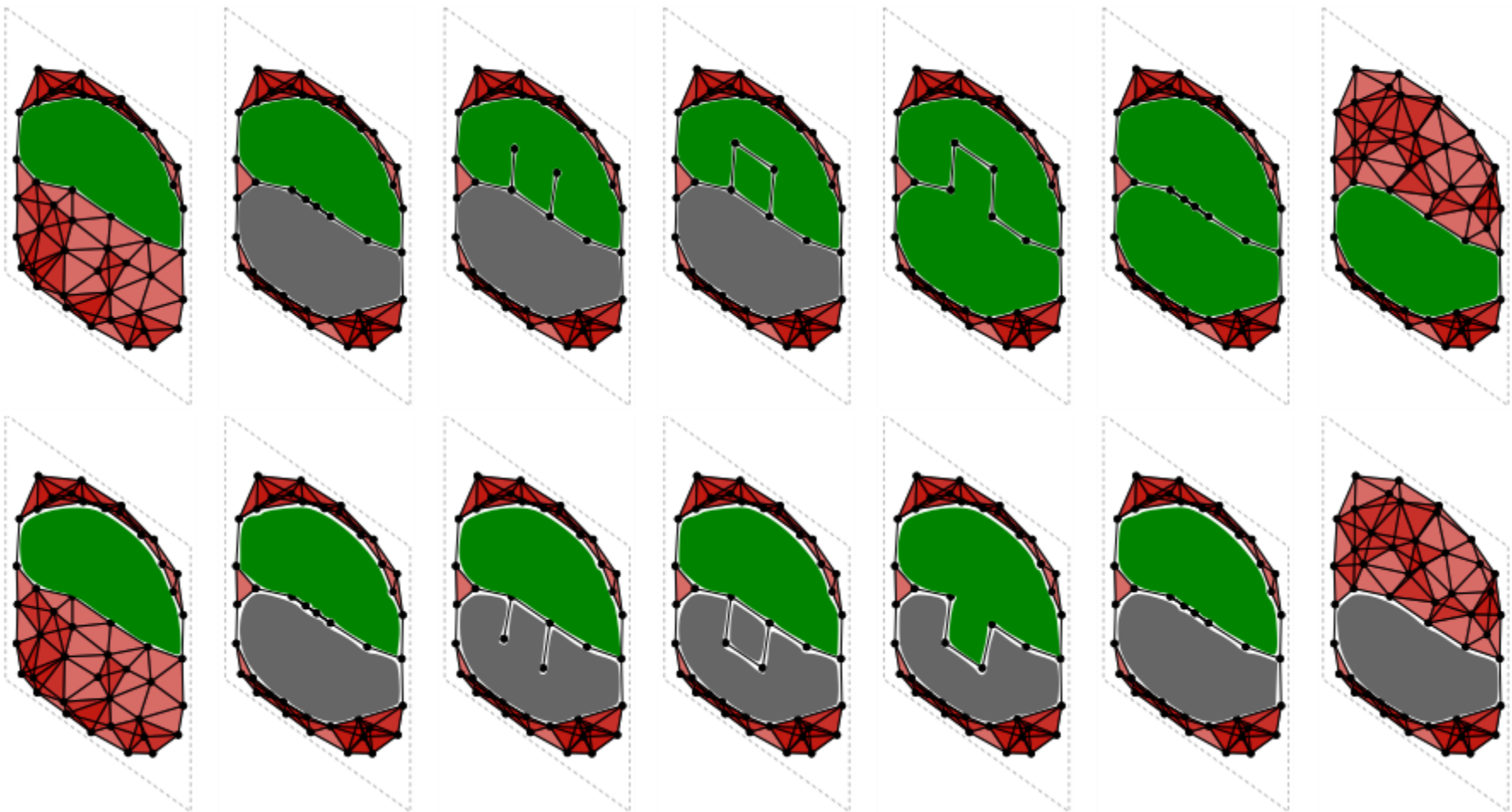

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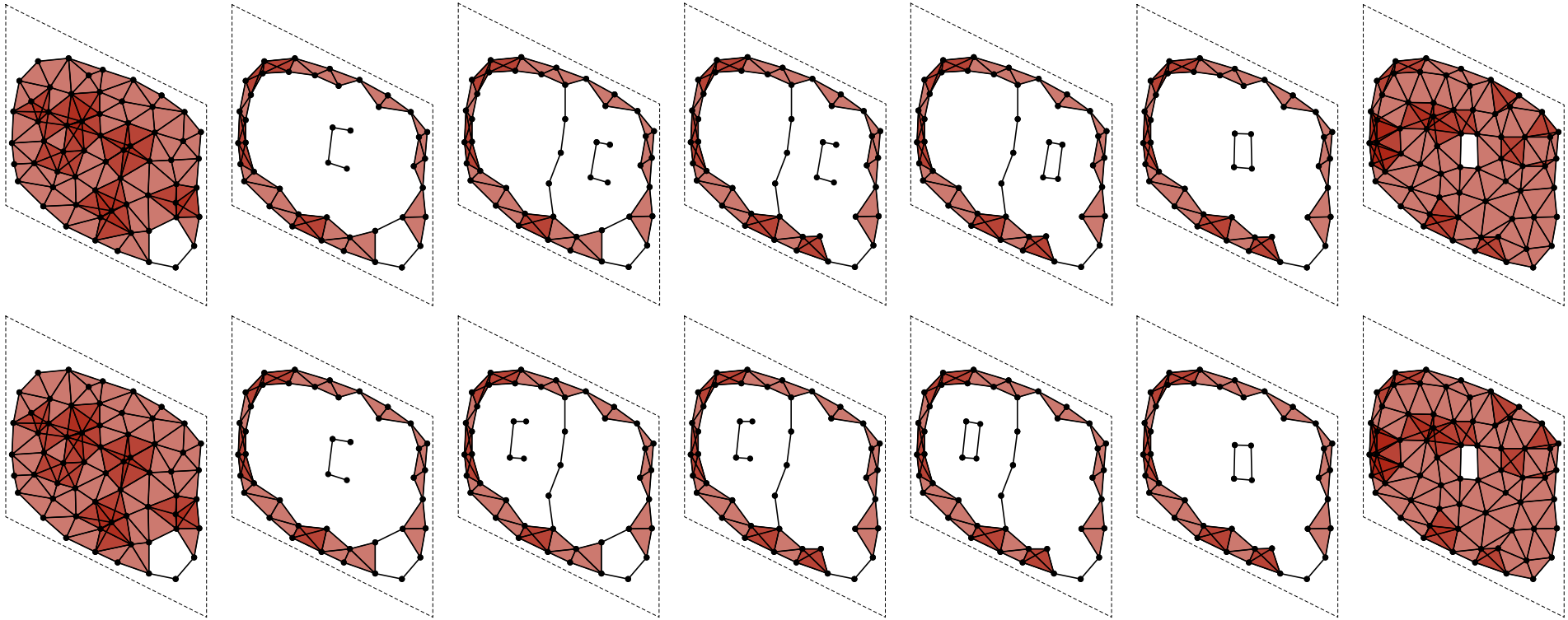
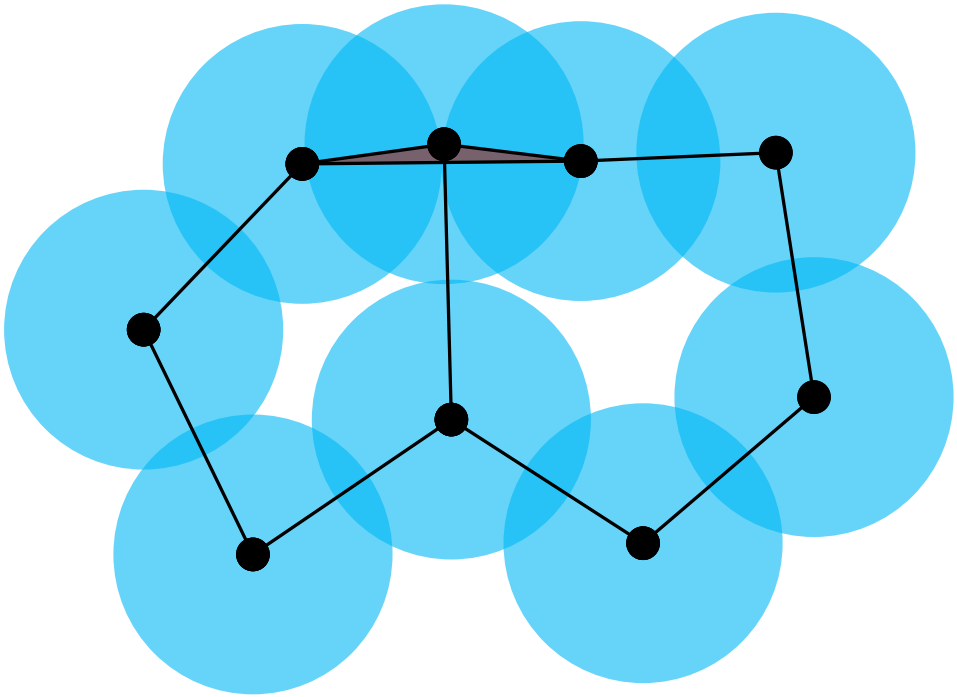


Planar sensors measuring cyclic orders

- <u>Theorem.</u> In a planar sensor network that remains connected, the time-varying alpha complex with rotation information determines if an evasion path exists.
- <u>Open question.</u> Is the Čech complex with rotation information sufficient?



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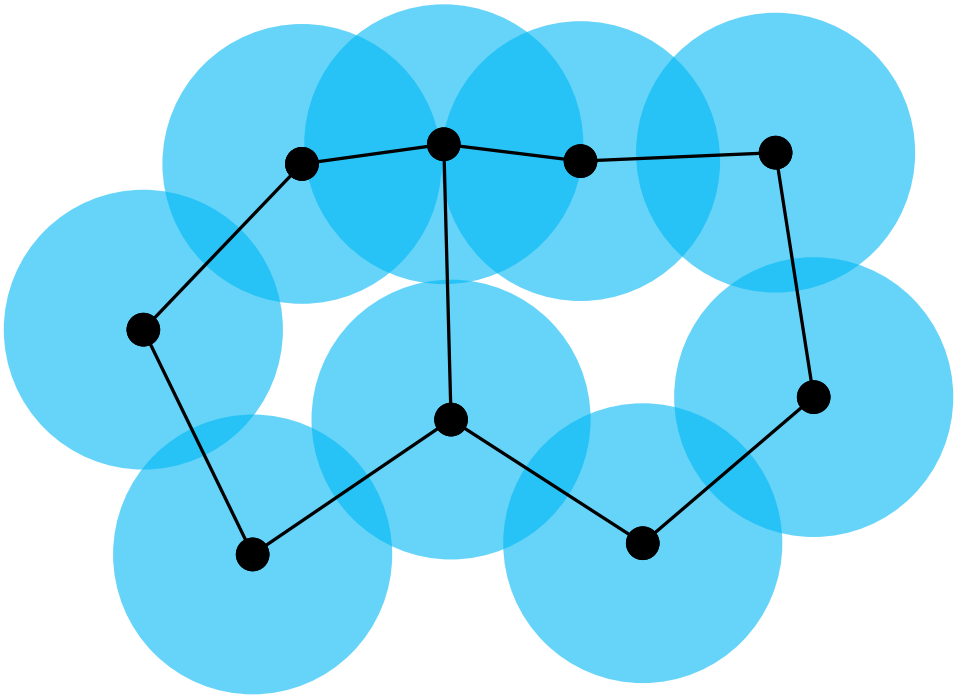

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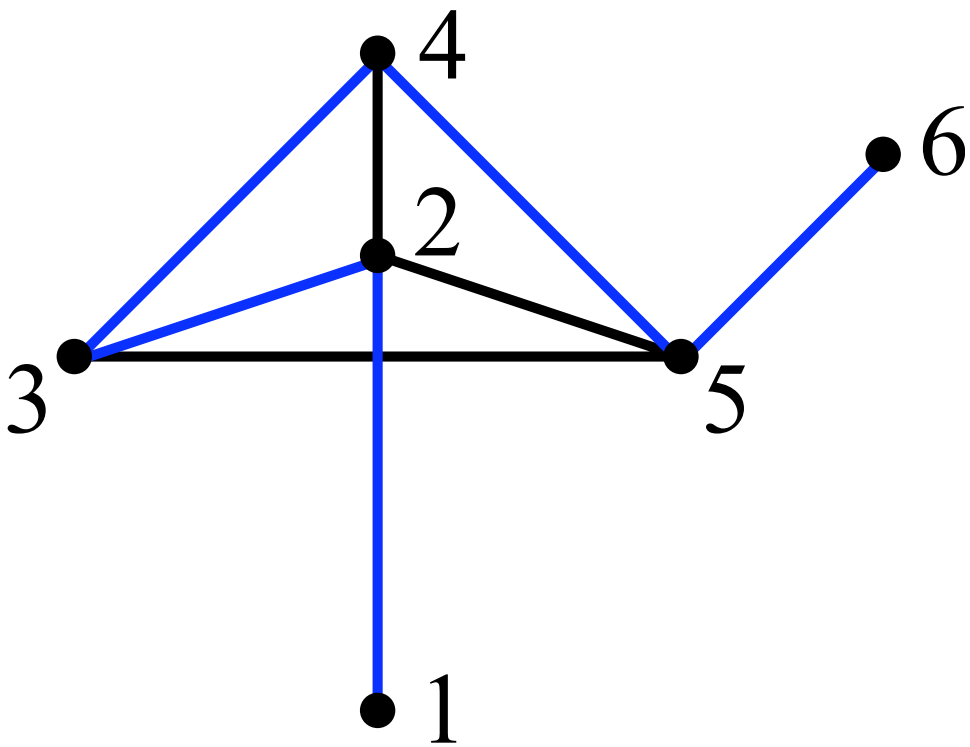

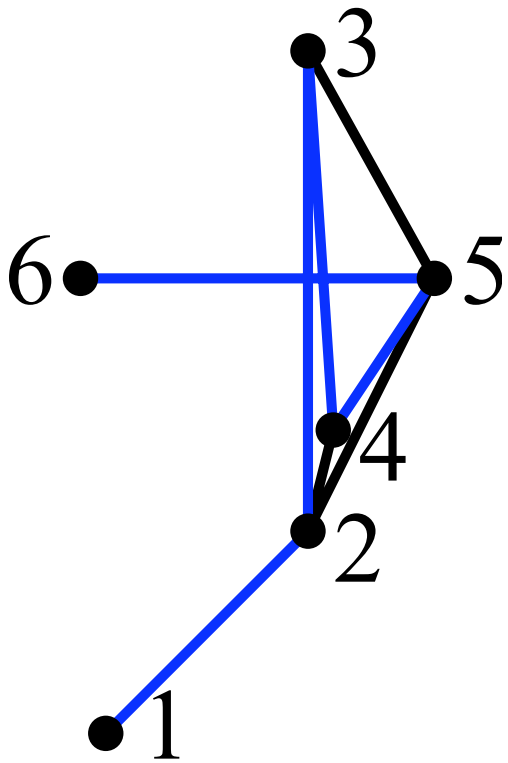

Conclusions

- Streaming zigzag persistence criterion.
- Cech complex insufficient.
  Alpha complex with rotation information suffices.
  What about the Čech complex with rotation information?

Vin de Silva and Robert Ghrist, *Coordinate-free coverage in sensor networks with controlled boundaries via homology*, International Journal of Robotics Research 25 (2006), 1205-1222.

Henry Adams and Gunnar Carlsson, *Evasion paths in mobile sensor networks*, International Journal of Robotics Research 34 (2015), 90-104.

Thank you!