An Introduction to Applied Topology

Henry Adams
University of Florida

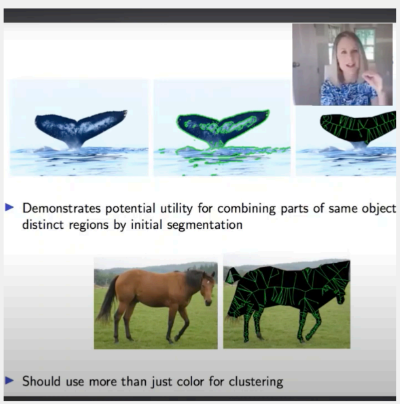
AATRN   Home   About   Seminar   Interviews   Interactive Events   Affiliated Activities   Participate   Ethics

AATRN

Applied Algebraic Topology Research Network

Bringing together researchers across the world to develop and use applied and computational topology.

Find out how you can participate and join AATRN (membership is free and so are all events), or check out all our content on our YouTube channel. We invite you to join the following activities, which we are currently organizing:



Seminar Series

We have a weekly Seminar Series (or sometimes a tea time) on Wednesdays.

Interview Series

See our upcoming interviews.

Tutorial-a-thon

Our next tutorial-a-thon is in 2023. Stay tuned!

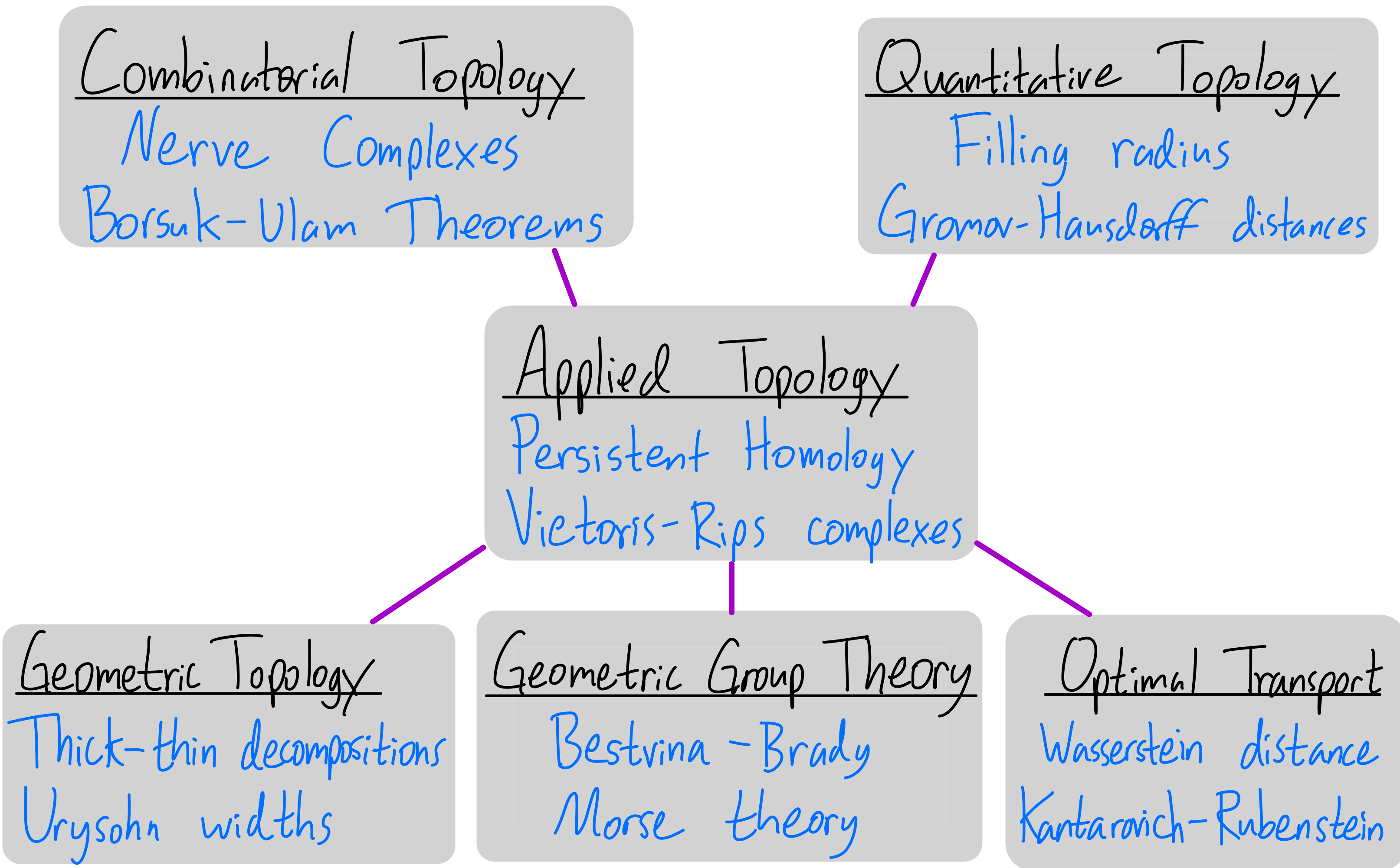
Service

www.aatrn.net

INTERVIEW
SERIES
2022 - 2023
Frédéric Chazal
interviewed by
Steve Oudot
SEP 14TH
Konstantin Mischaikow
interviewed by
Tomas Gedeon
OCT 26TH
Yusu Wang
interviewed by
Tamal Dey
DEC 7TH
Claudia Landi
interviewed by
Barbara Giunti
FEB 1ST
For Zoom coordinates, become a member at AATRN.NET
Leonidas Guibas
interviewed by
Primoz Skraba
JUN 21ST
AATRN
Applied Algebraic Topology Research Network

Meet Adetayo (Tayo for short), born January 20!

Research Themes



Bridging Applied and Quantitative Topology

An Introduction to Applied Topology

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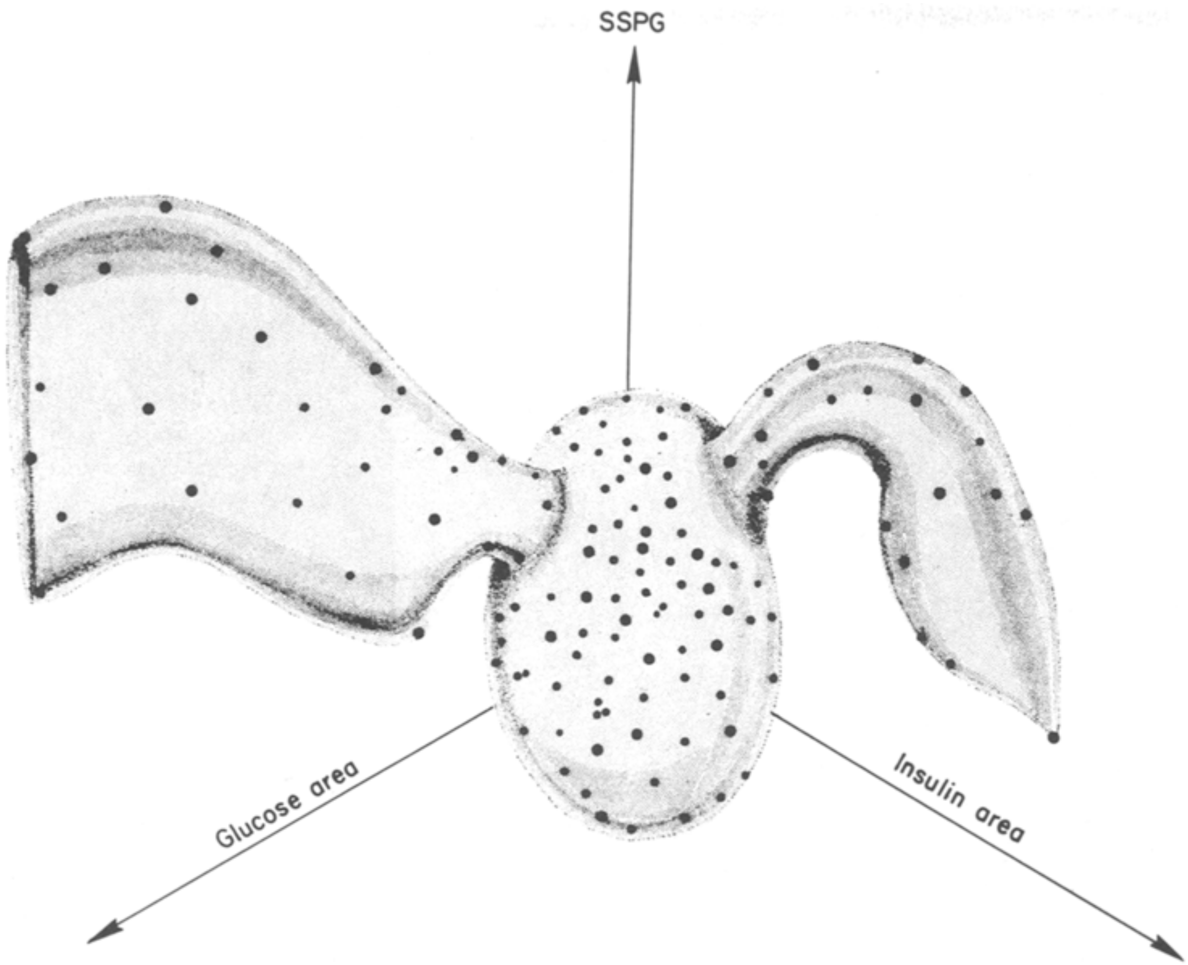
Datasets have shapes

Example: Diabetes study
145 points in 5-dimensional space

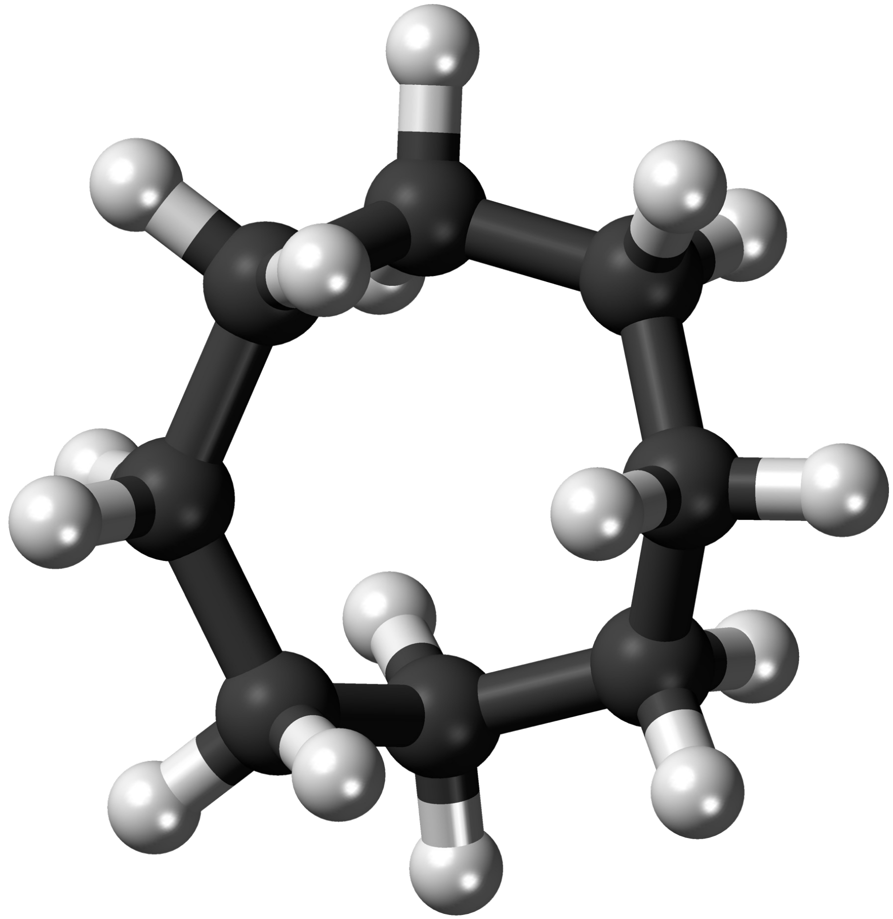


An attempt to define the nature of chemical diabetes using a multidimensional analysis by G. M. Reaven and R. G. Miller, 1979

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space



Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

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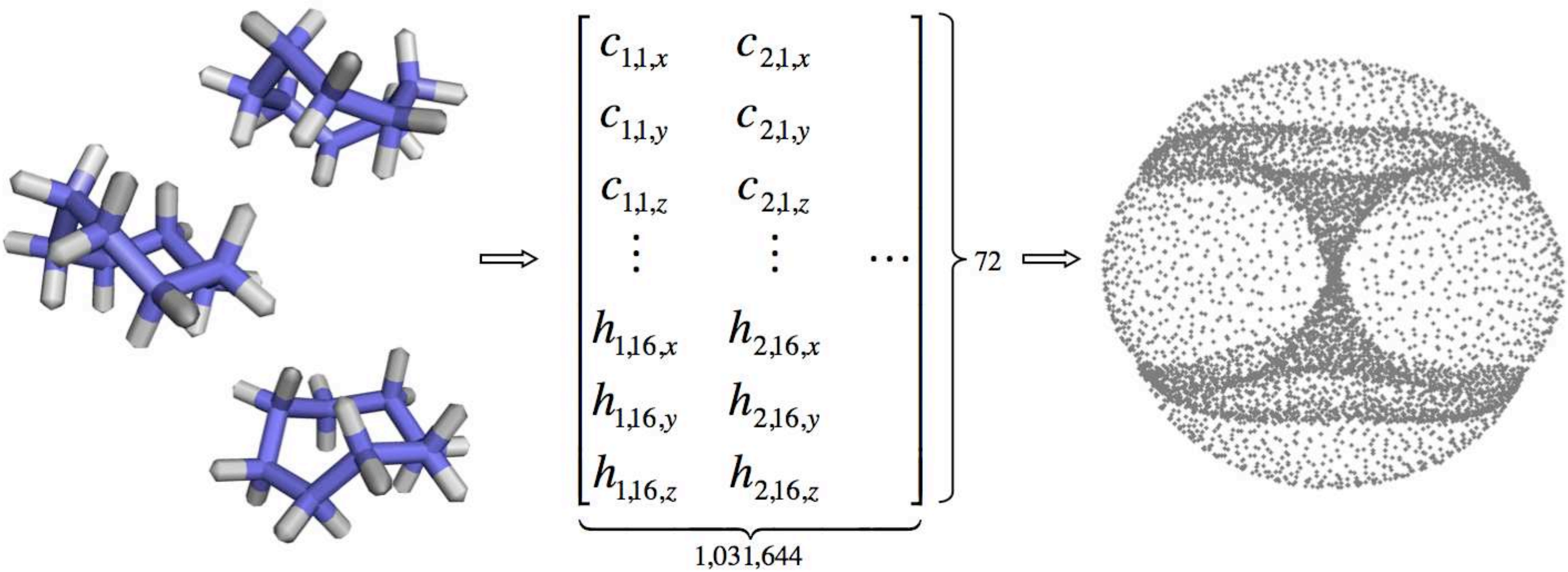
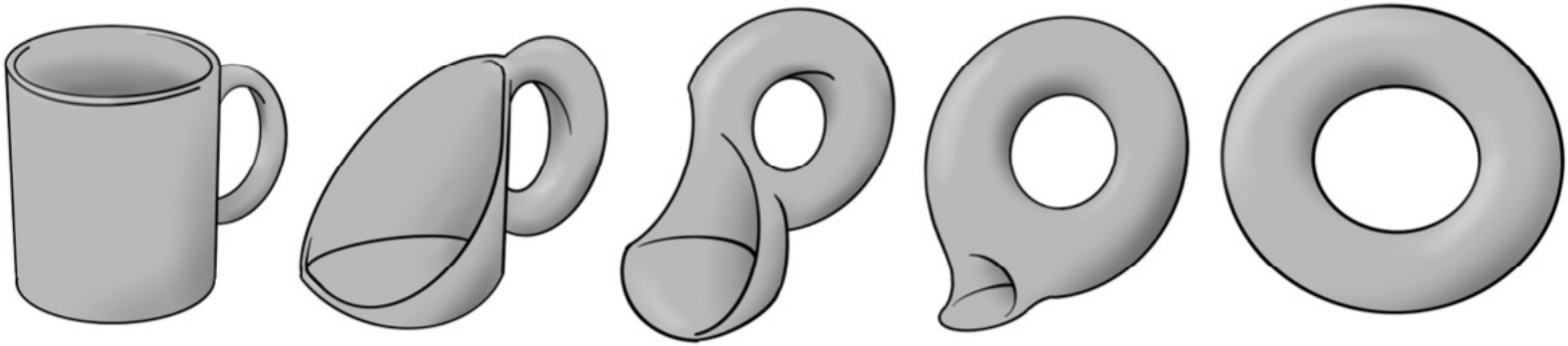


Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Datasets have shapes

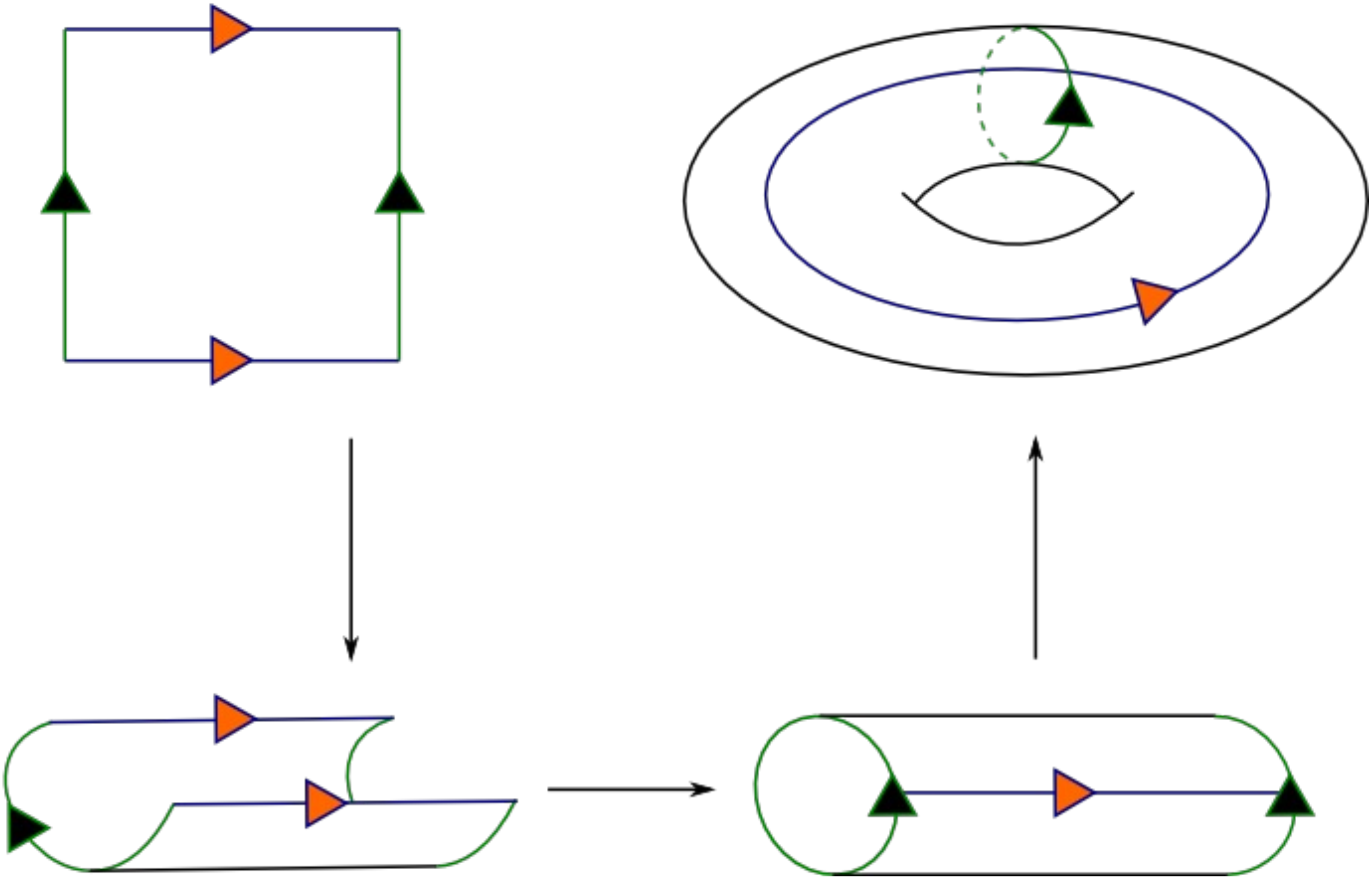
Topology studies shapes

A donut and coffee mug are “homotopy equivalent”, and considered to be the same shape. You can bend and stretch (but not tear) one to get the other.



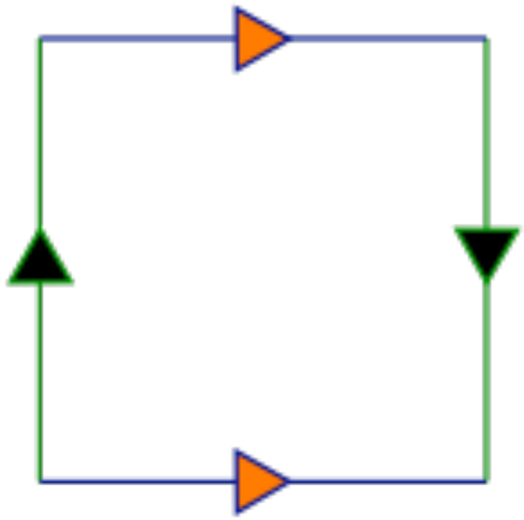
Topology studies shapes

Torus



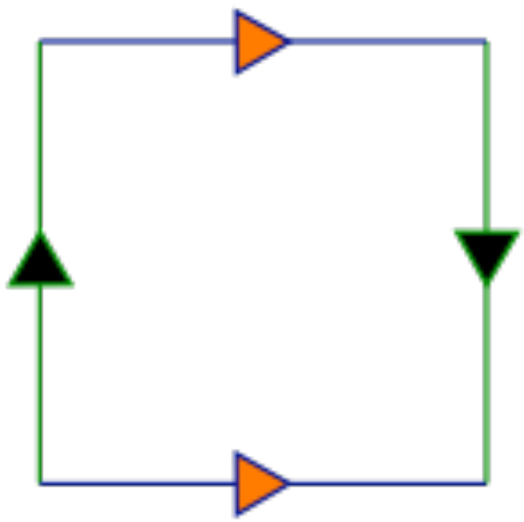
Topology studies shapes

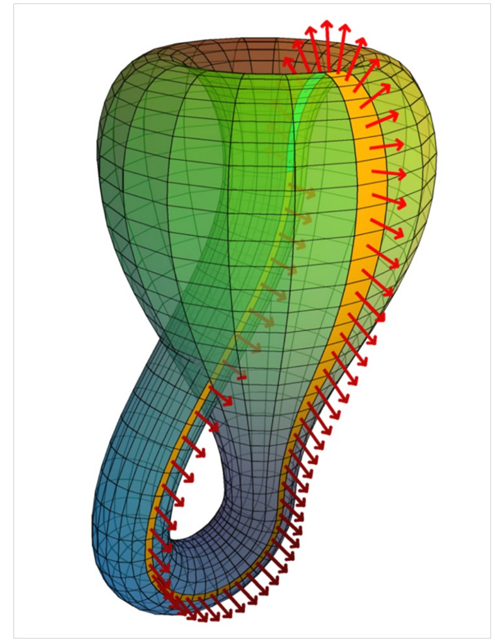
Klein bottle



Topology studies shapes

Klein bottle





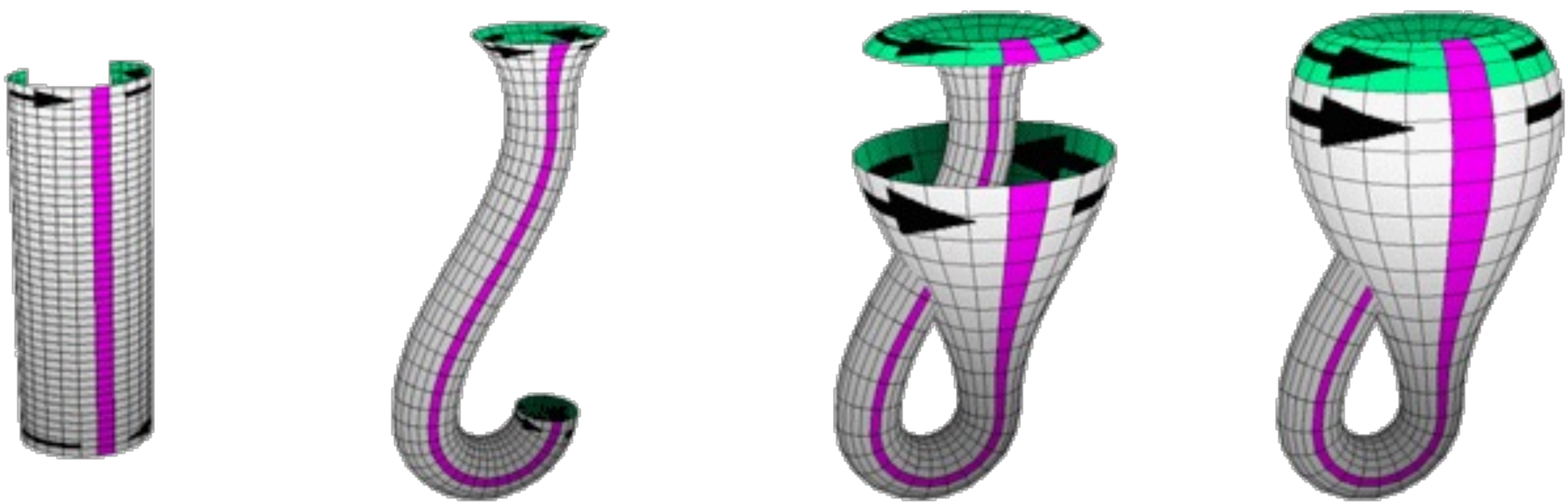
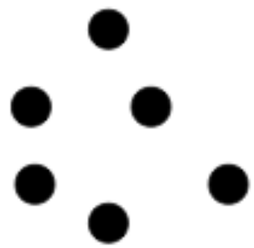


Image credit: https://plus.maths.org/content/imaging-maths-inside-klein-bottle

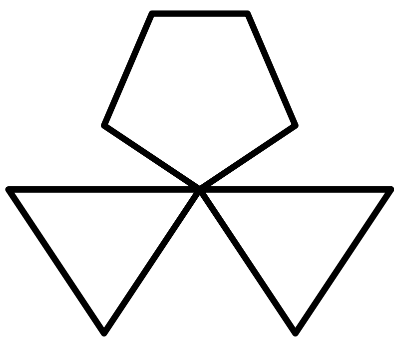
Homology

- i-dimensional homology H_i "counts the number of i-dimensional holes"
- i-dimensional homology H_i actually has the structure of a vector space!



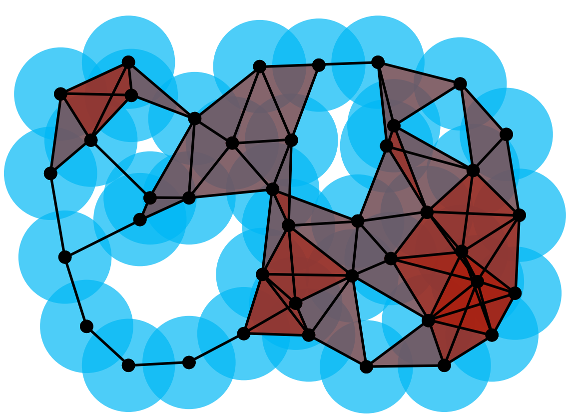
0-dimensional homology H_0: rank 6

1-dimensional homology H_1: rank 0



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 3

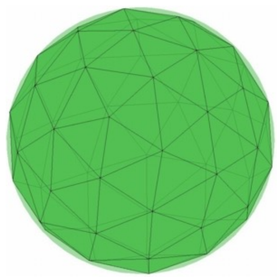


0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 6

Homology

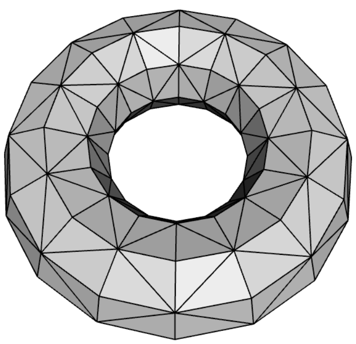
- *i*-dimensional homology "counts the number of *i*-dimensional holes"
- *i*-dimensional homology actually has the structure of a vector space!



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 0

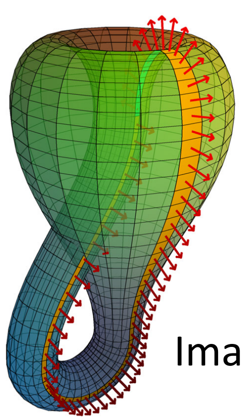
2-dimensional homology H_2: rank 1



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 2

2-dimensional homology H_2: rank 1

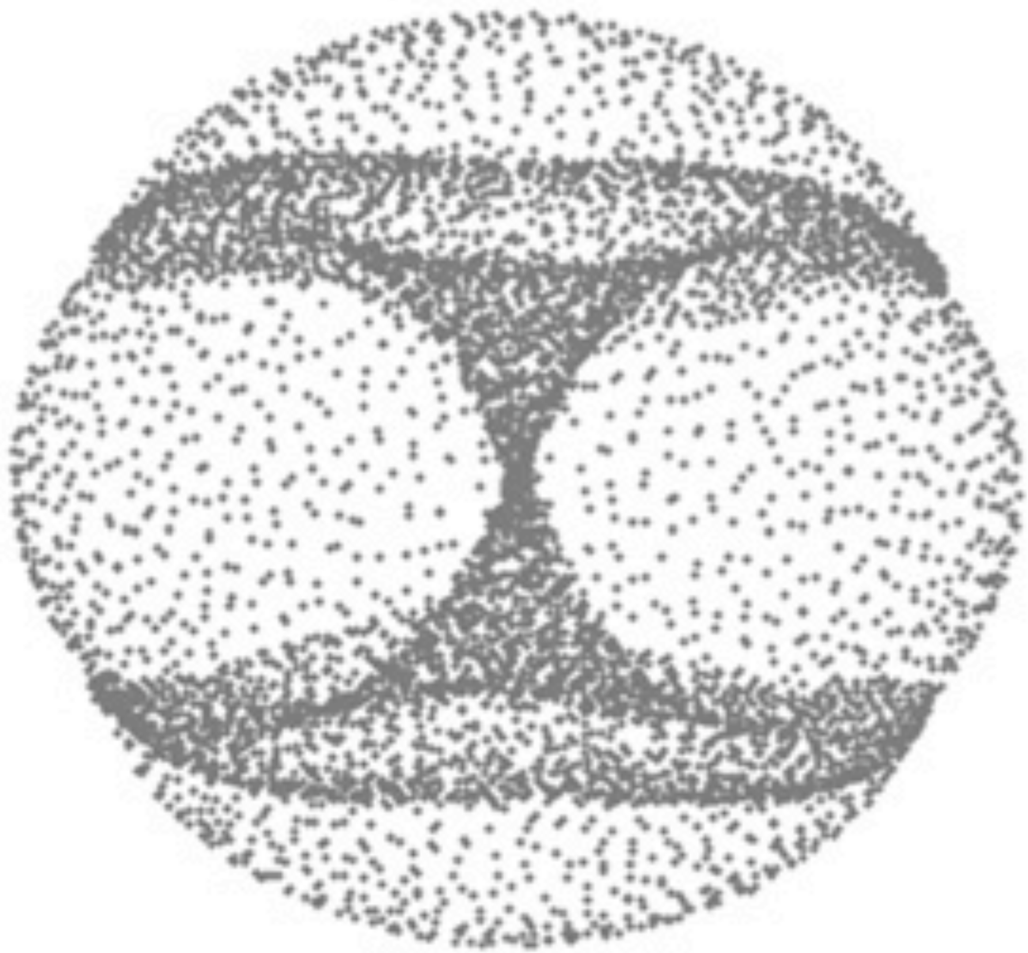


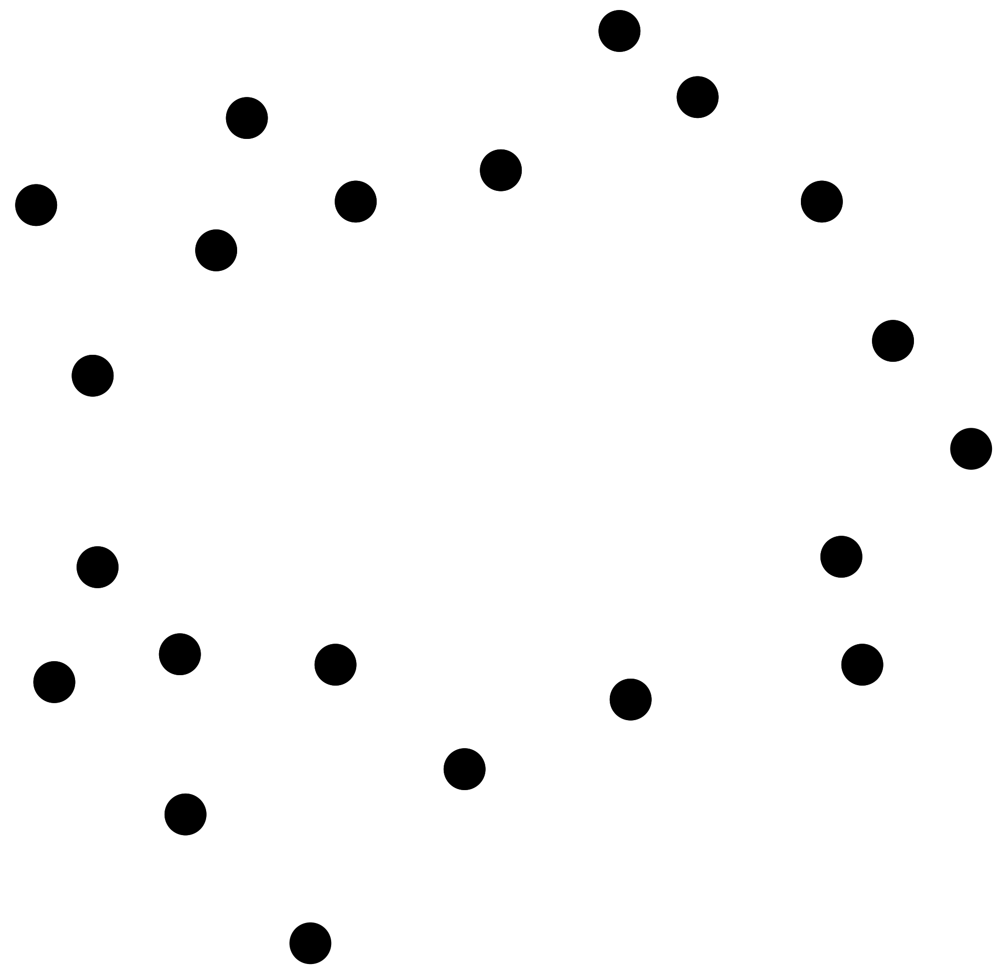
Be careful! (Same as torus over $\mathbb{Z}/2\mathbb{Z}$)

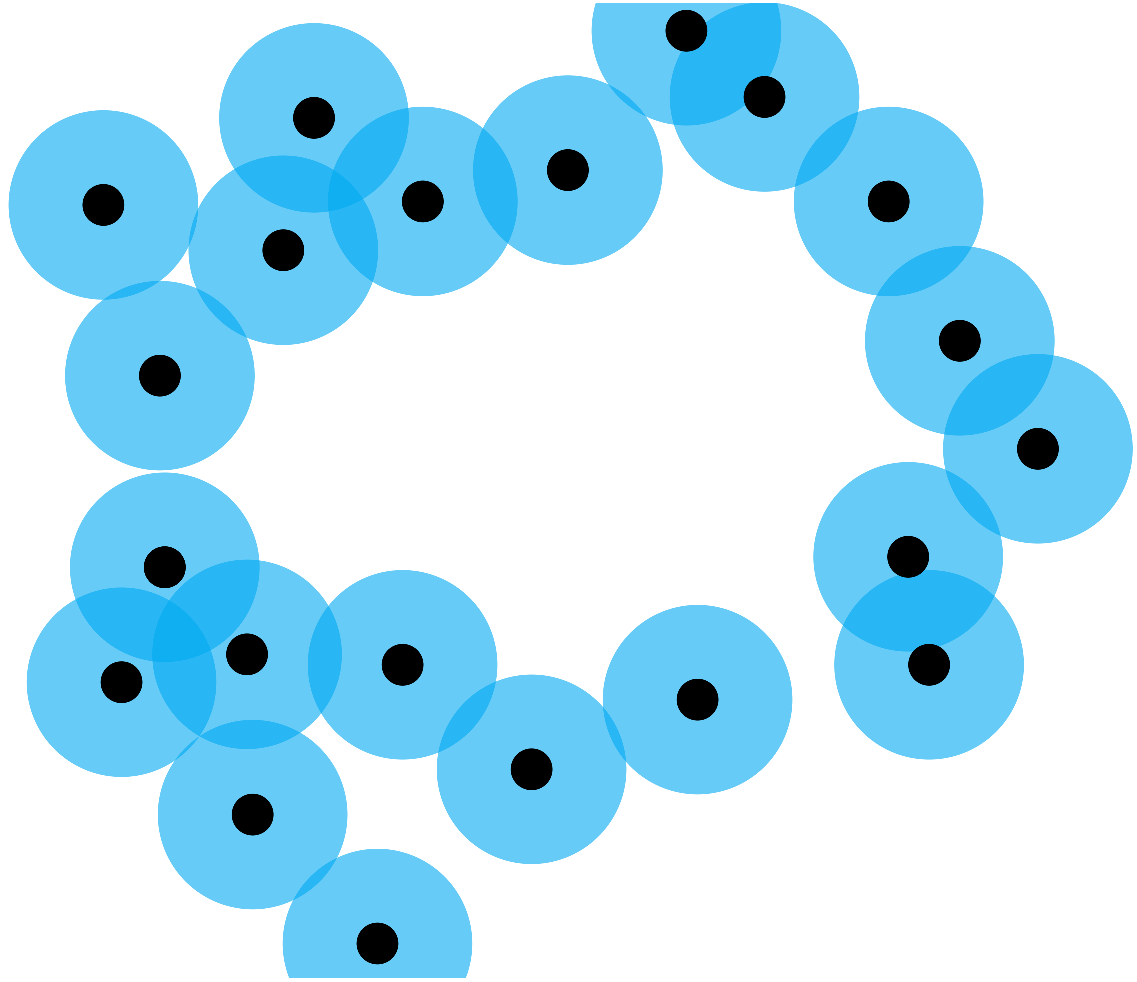
Image credit: https://plus.maths.org/content/imaging-maths-inside-klein-bottle

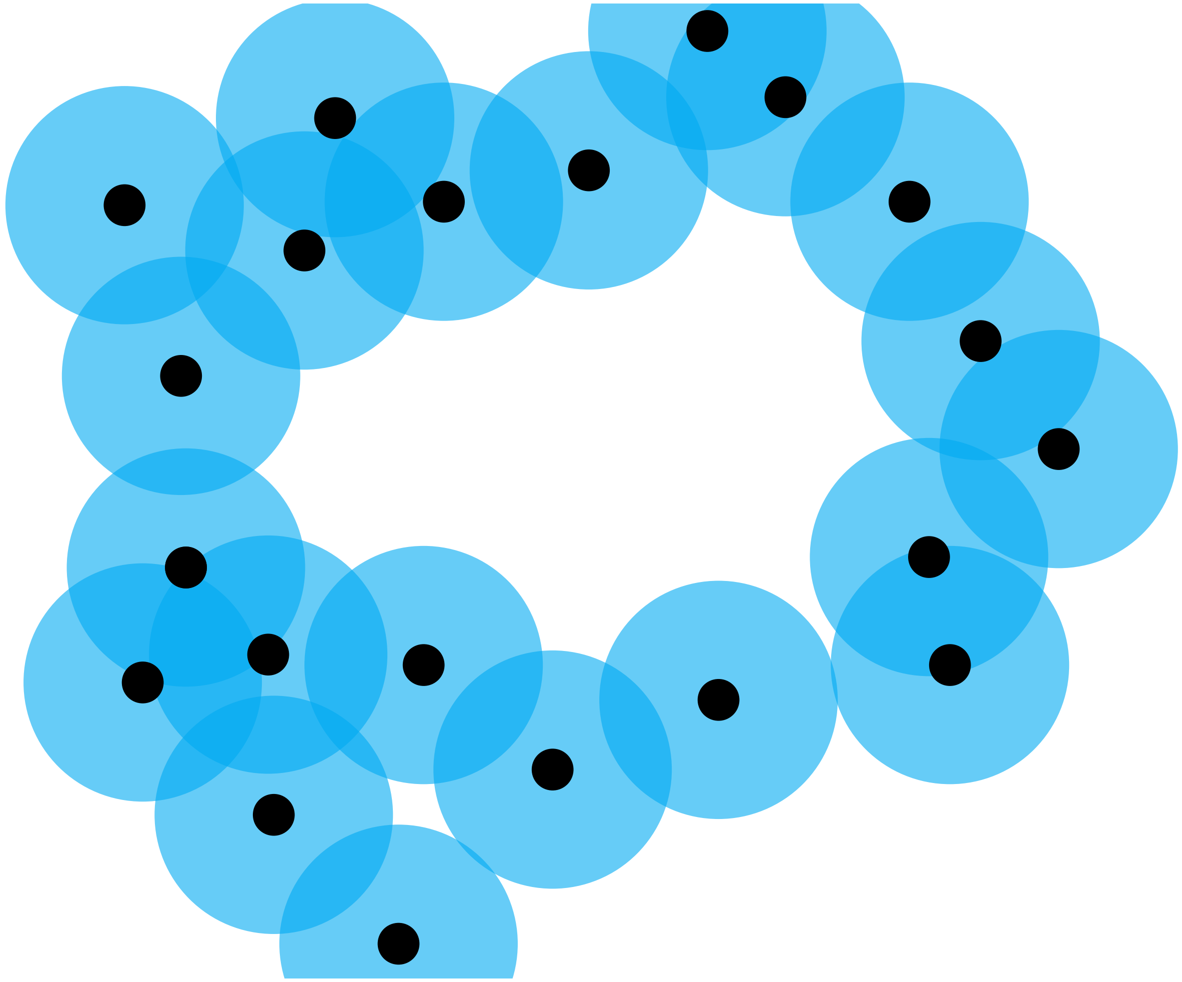
Topology studies shapes

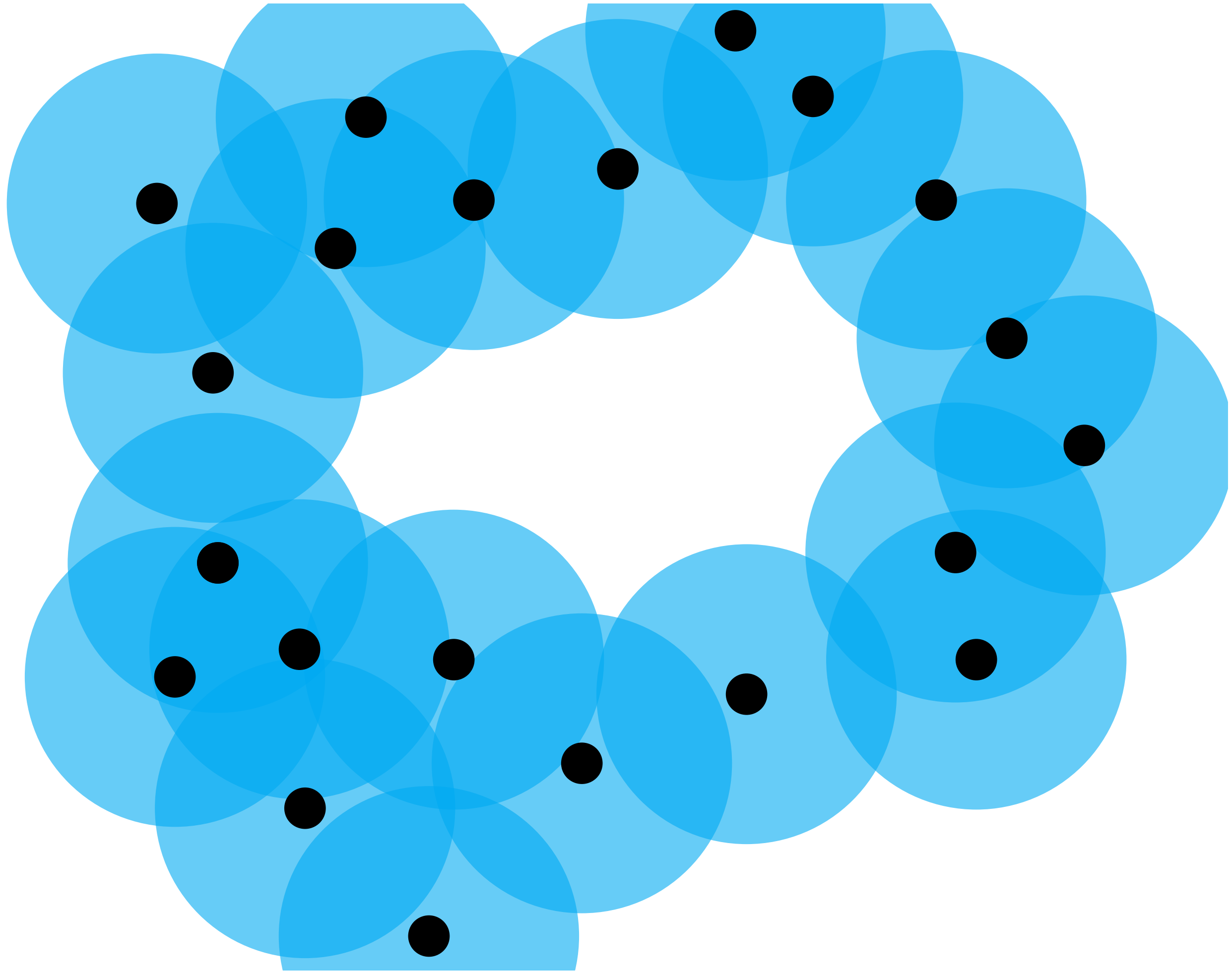
What shape is this?

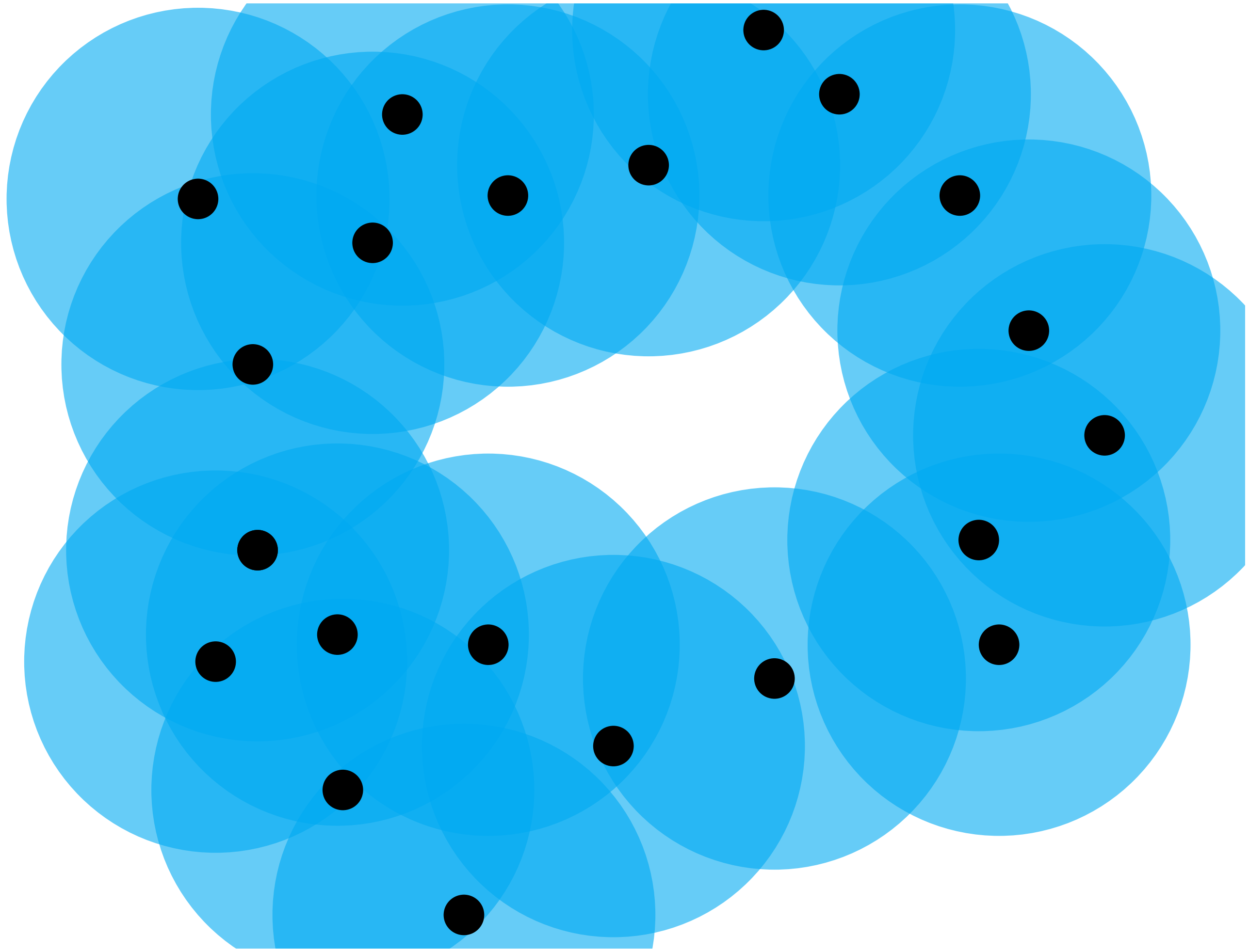


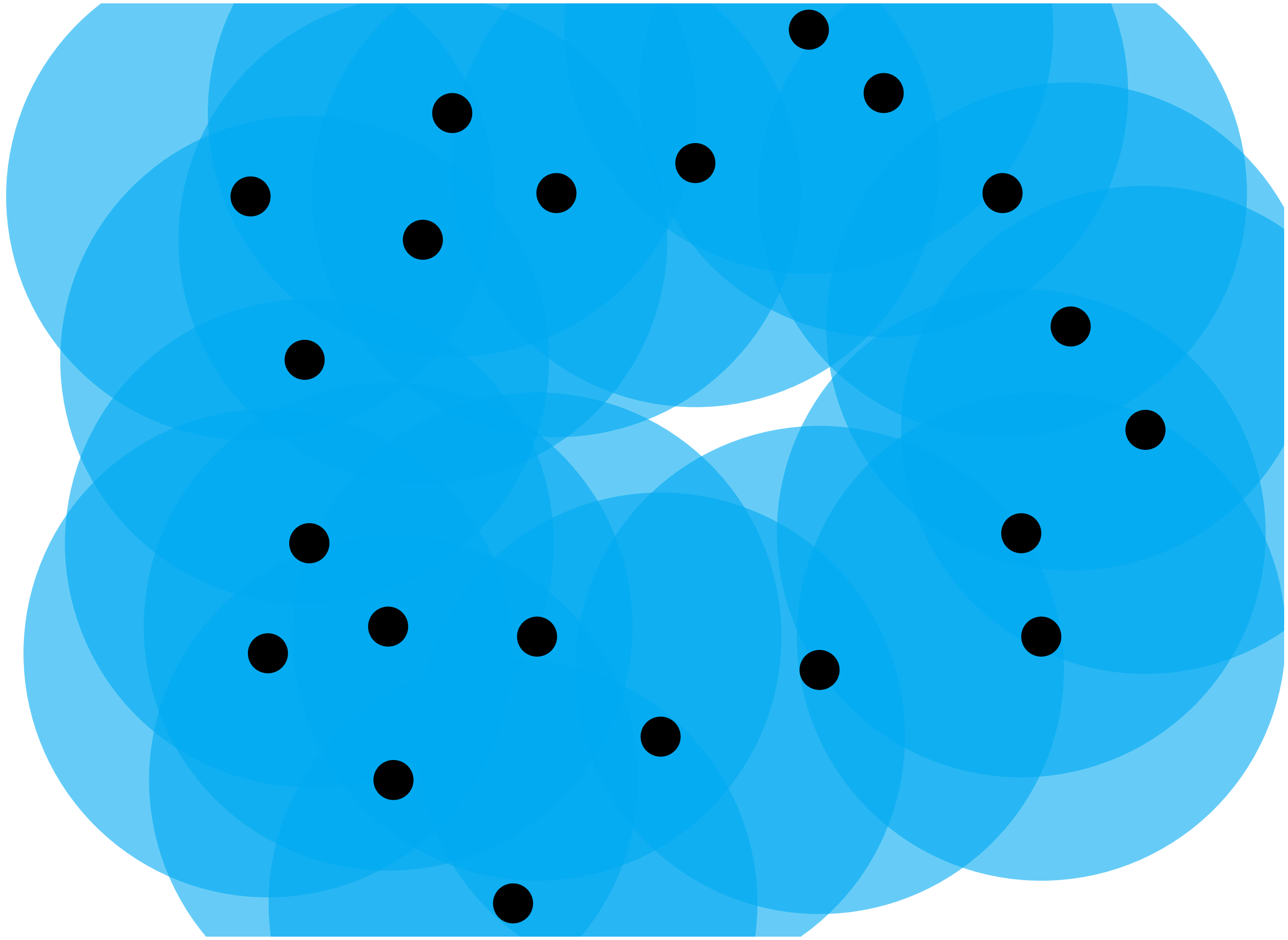


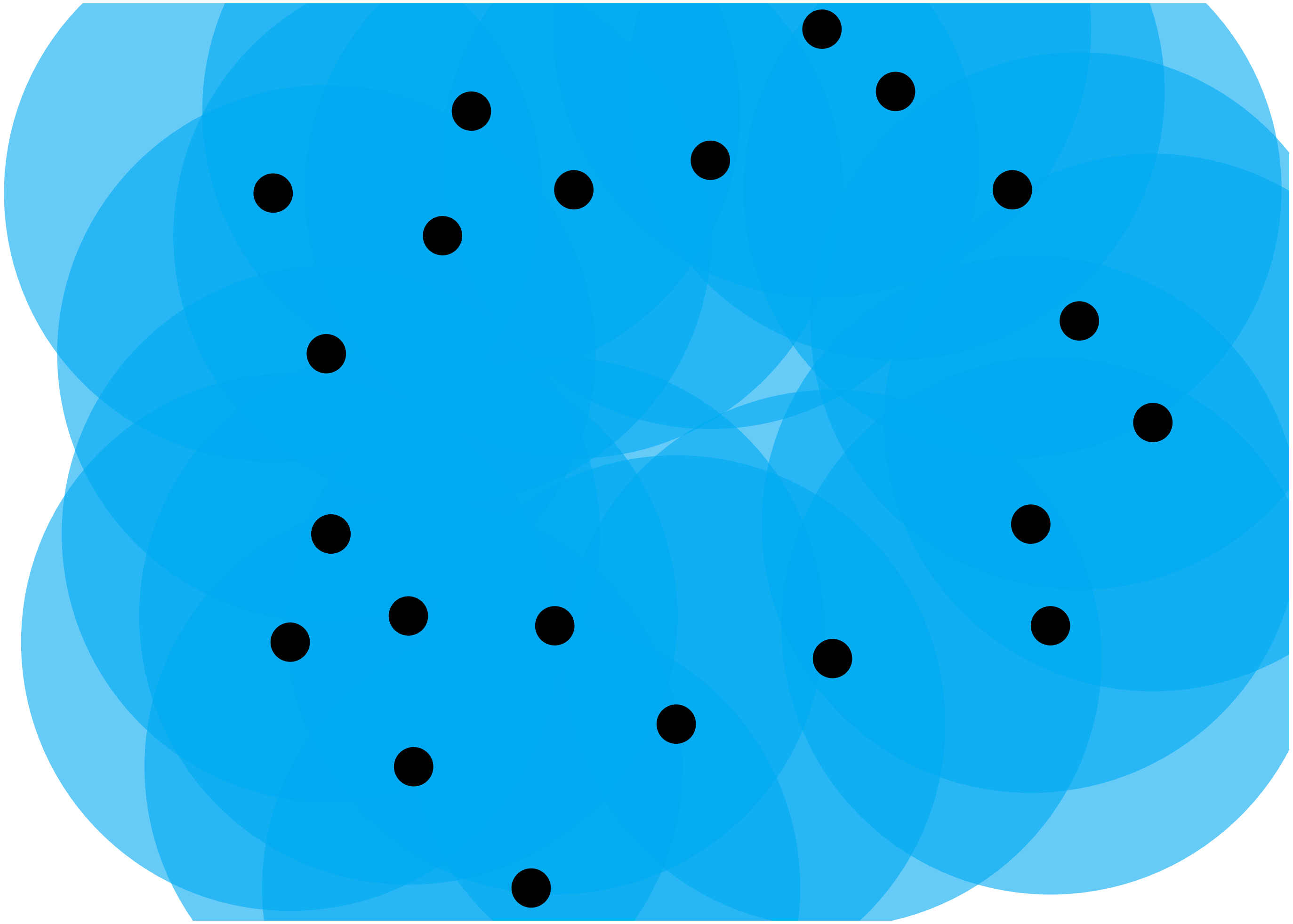


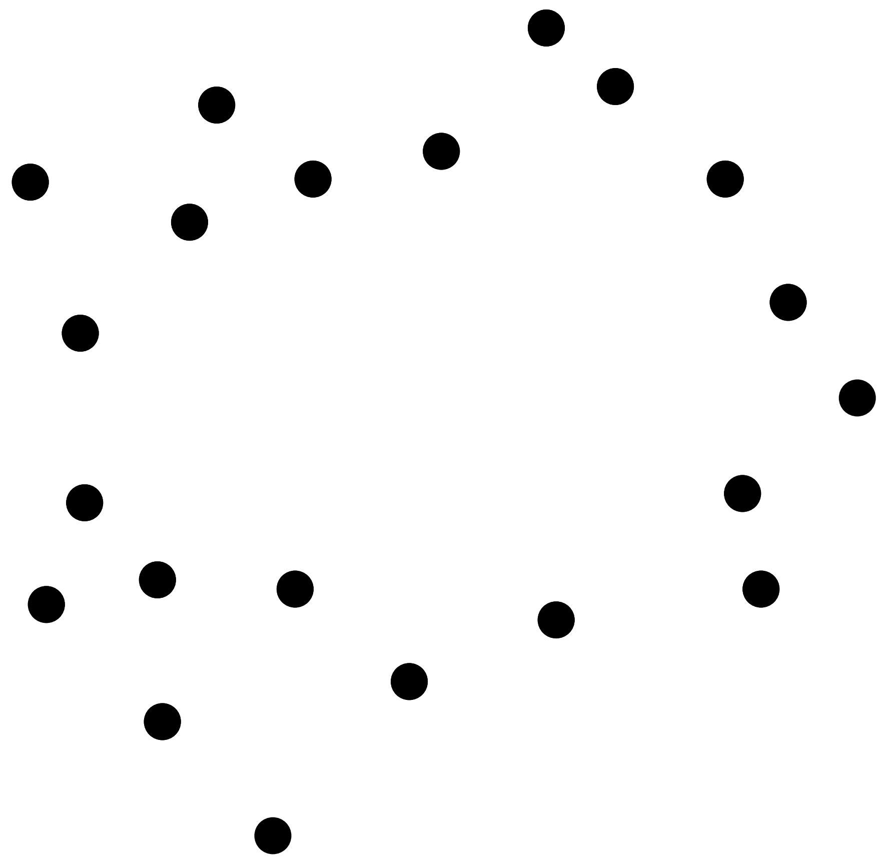








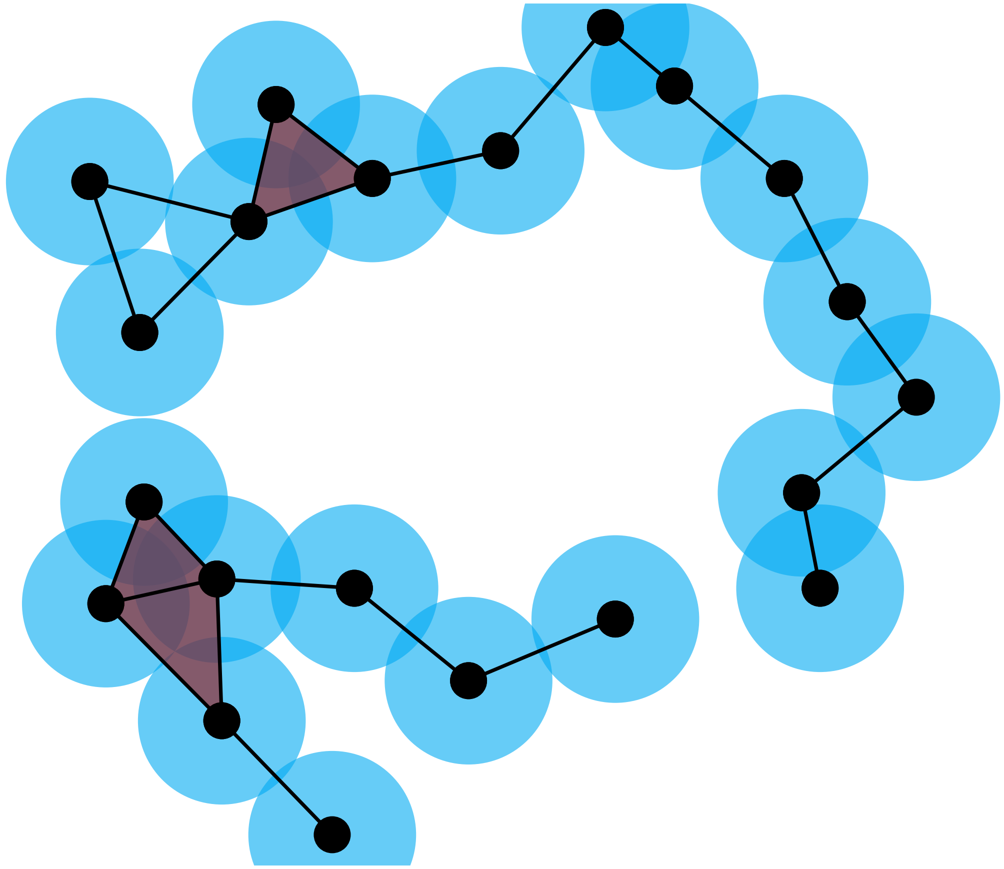




Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* Čech$(X; r)$ has

- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



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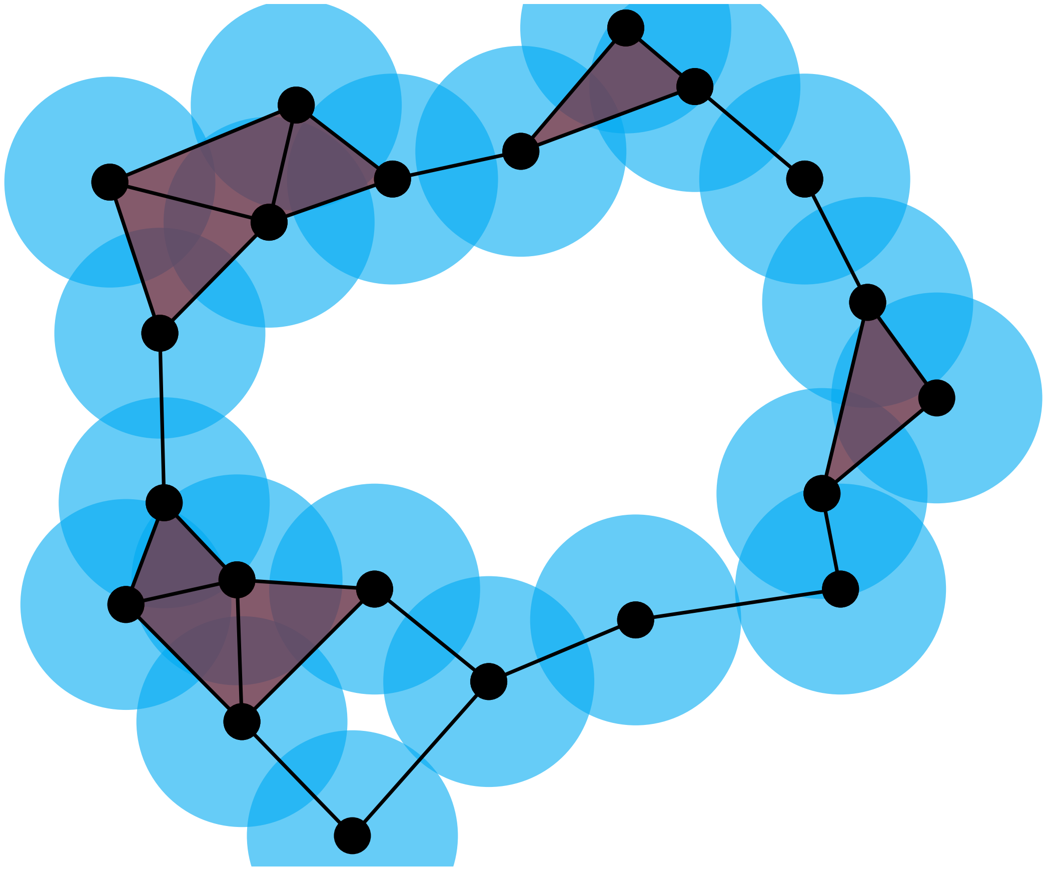

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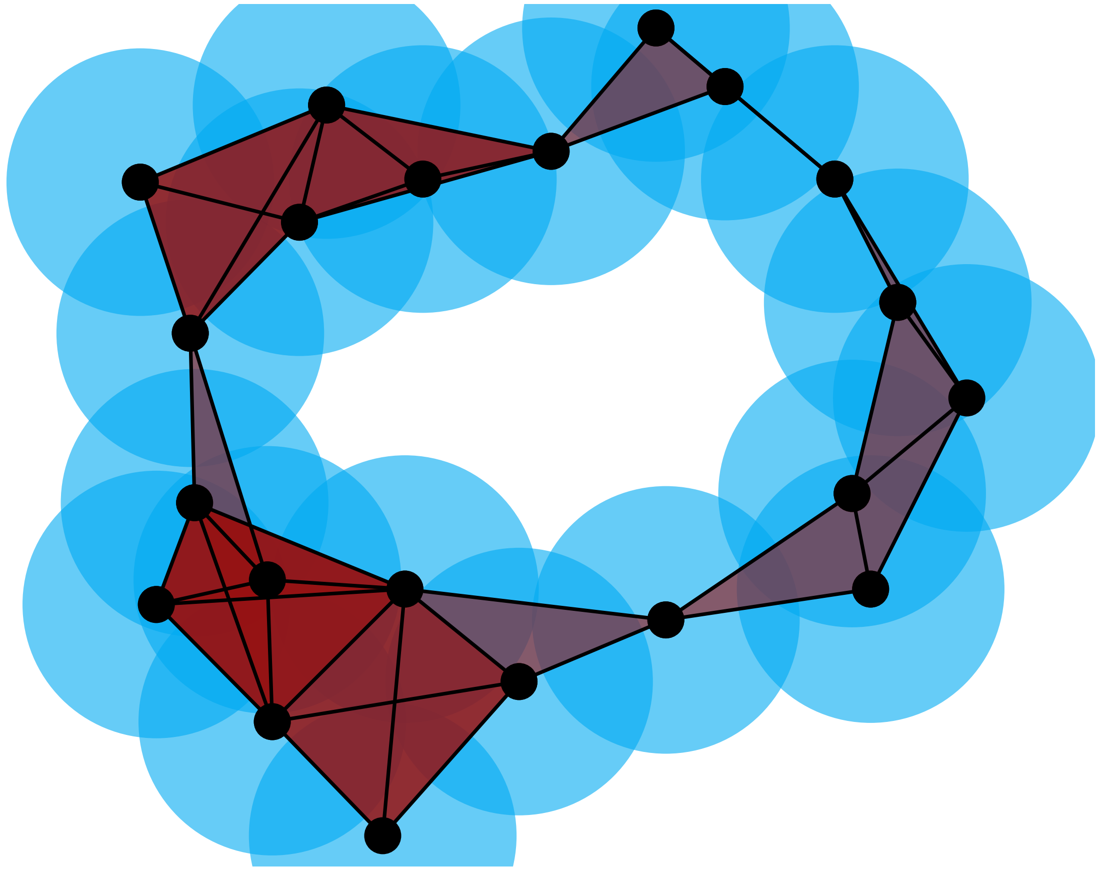

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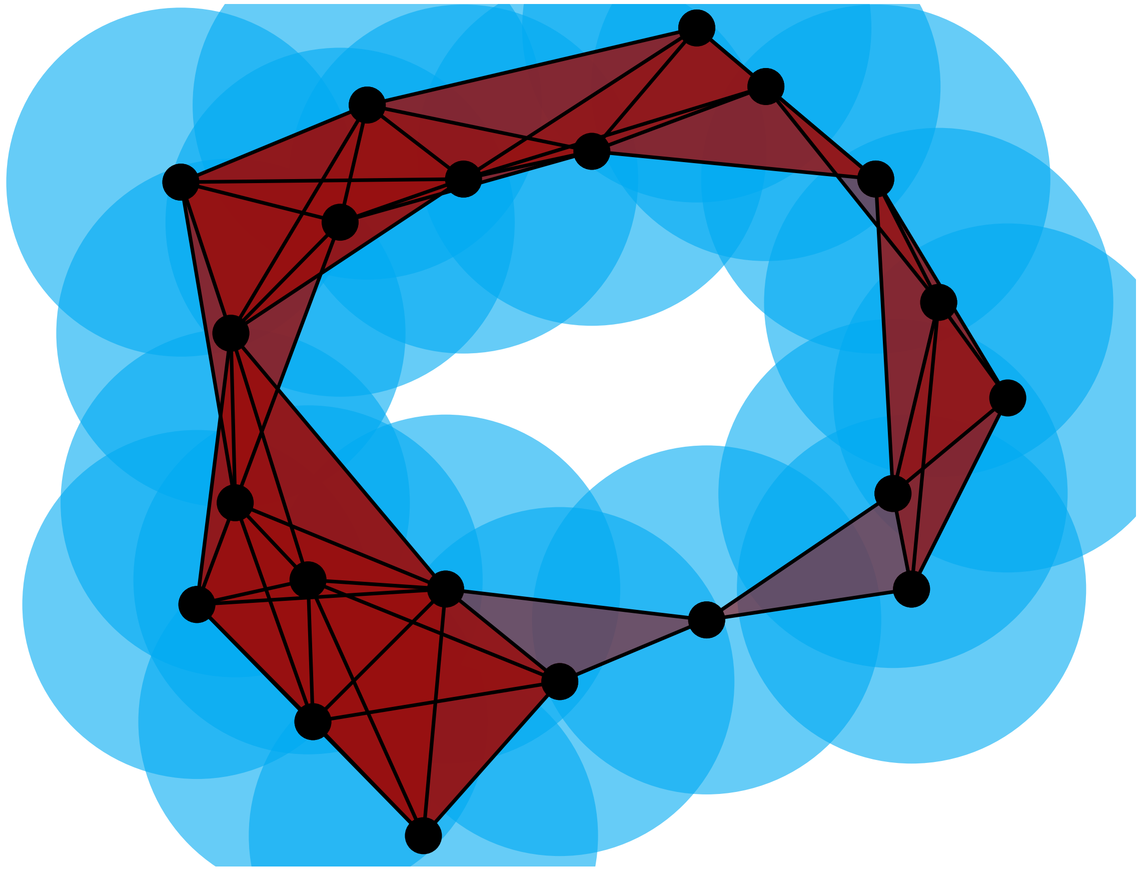

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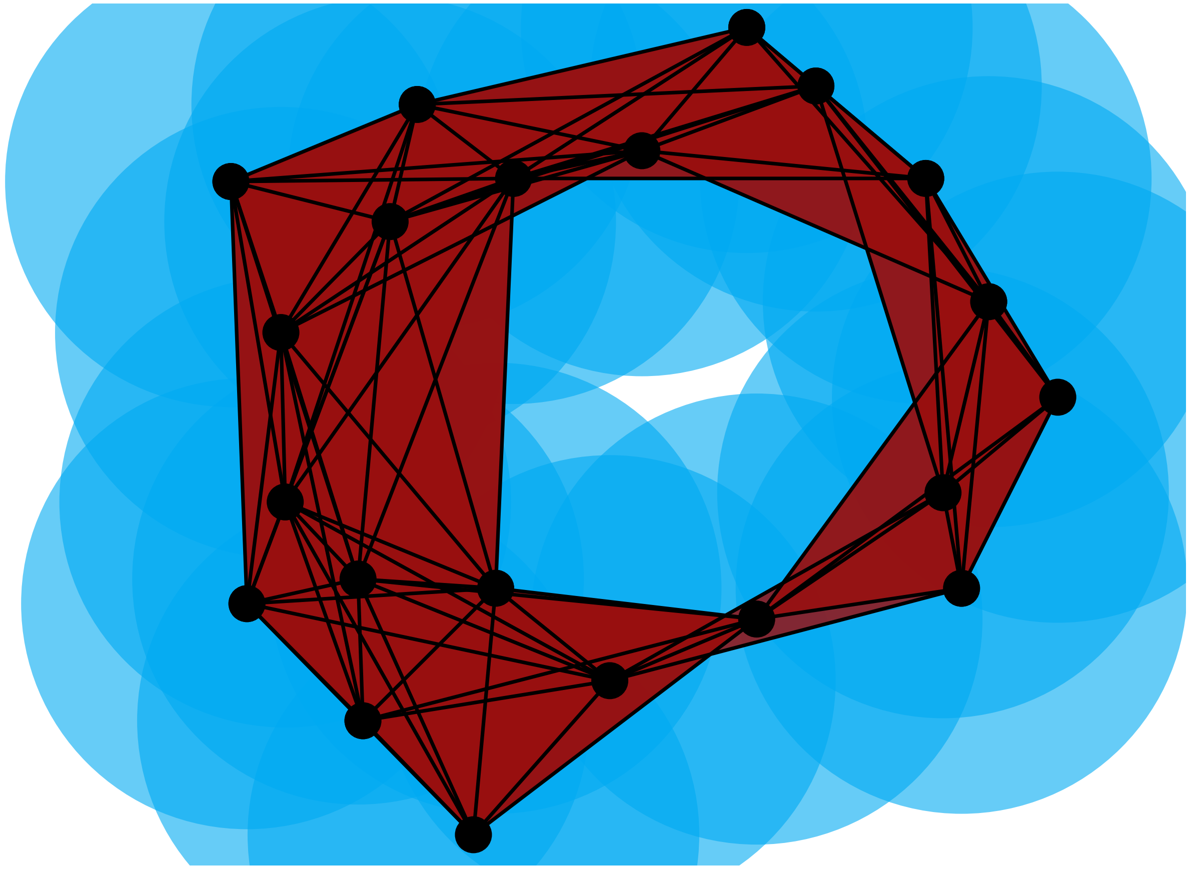

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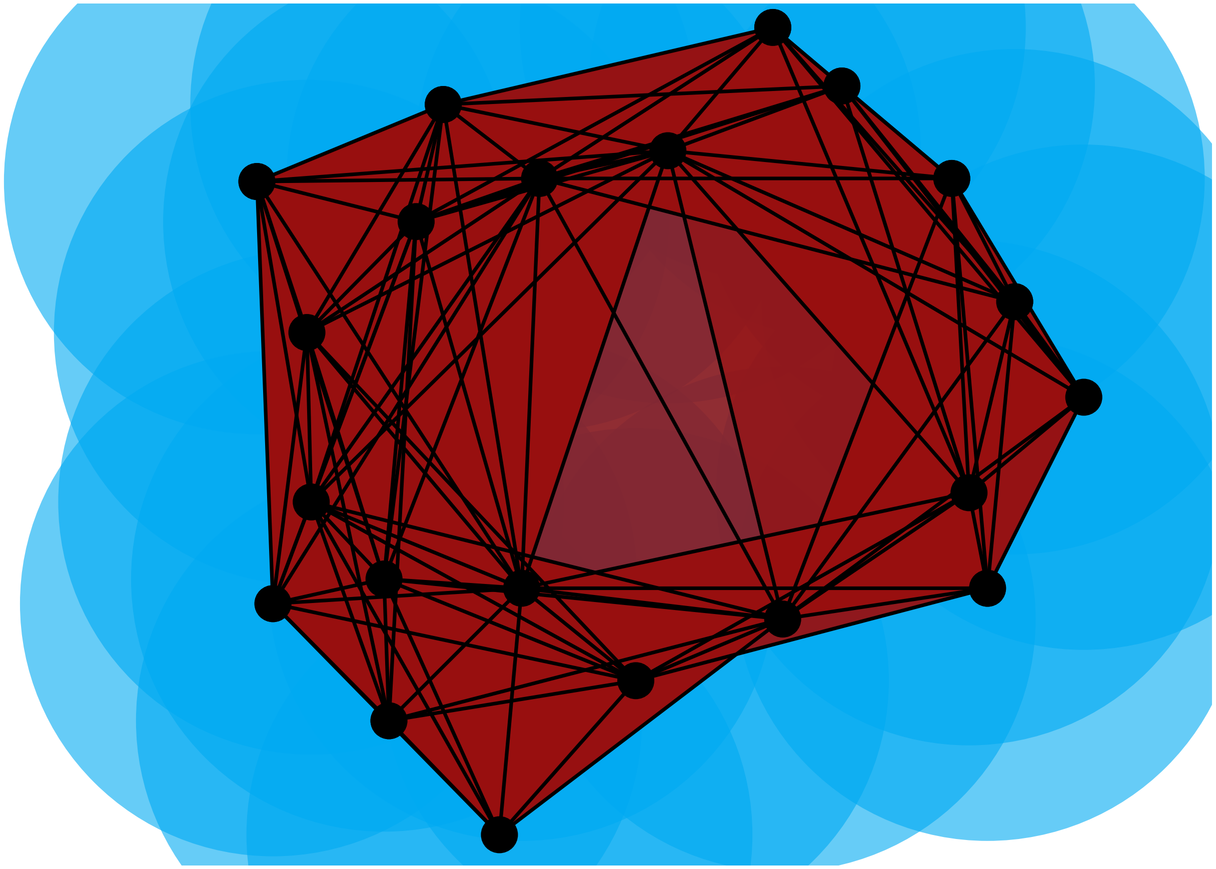

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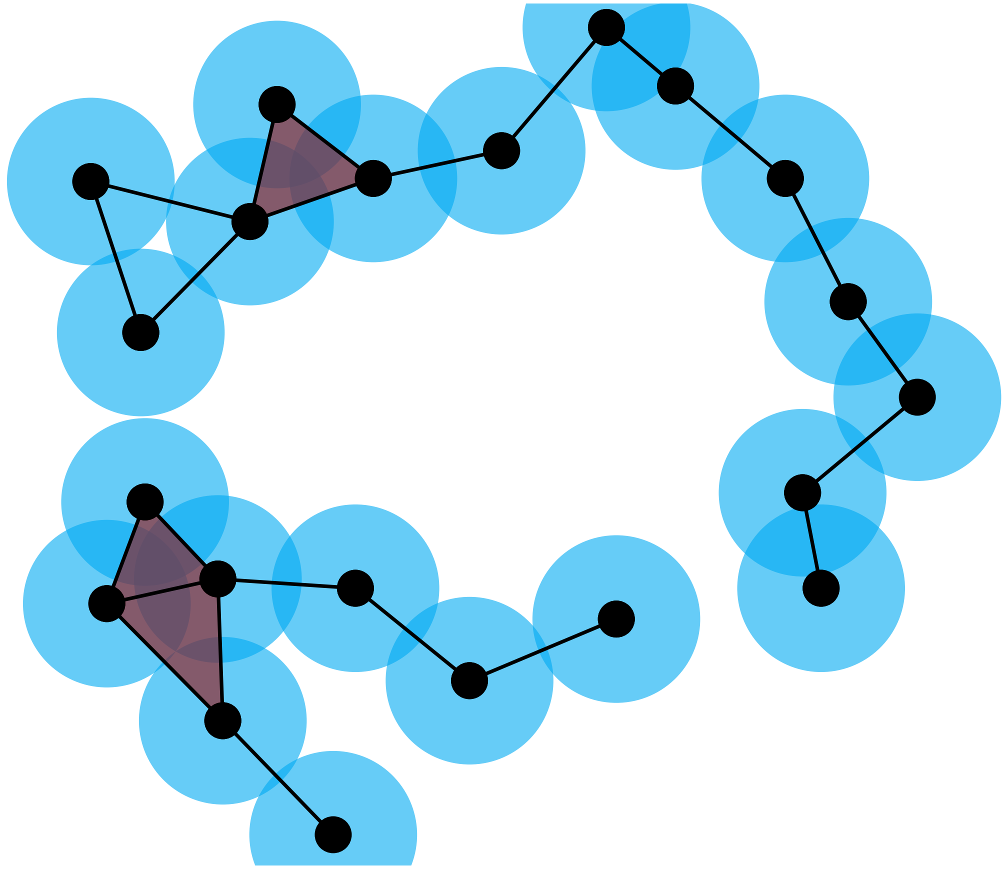

Nerve Lemma. $\check{\mathrm{C}}\mathrm{ech}(X; r) \simeq$ union of balls

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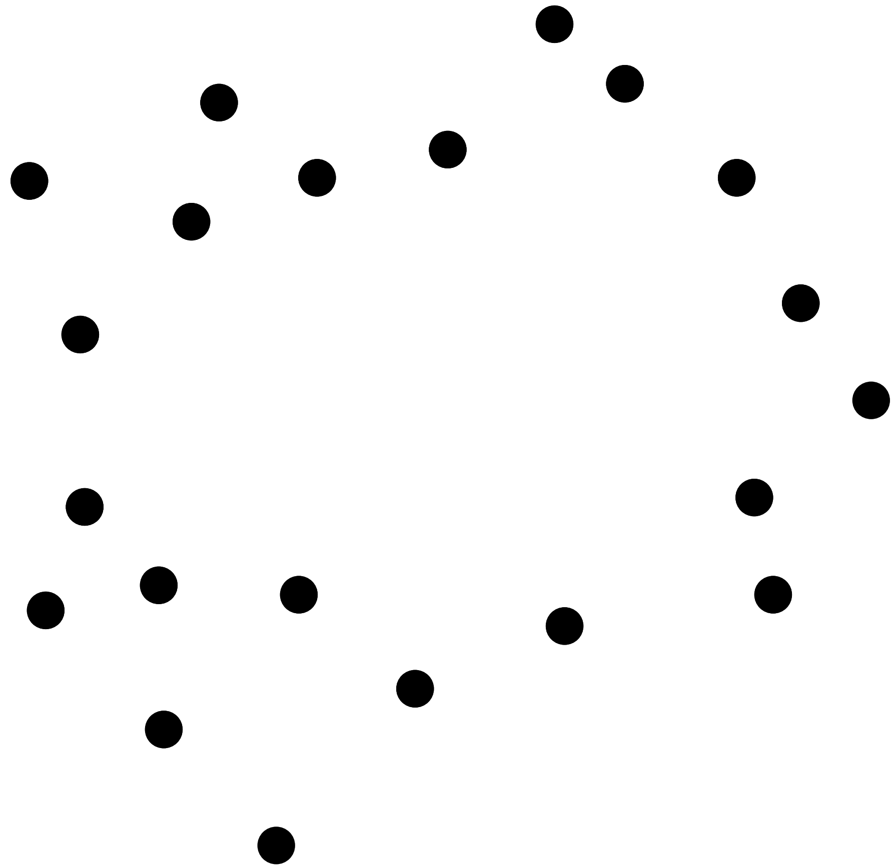

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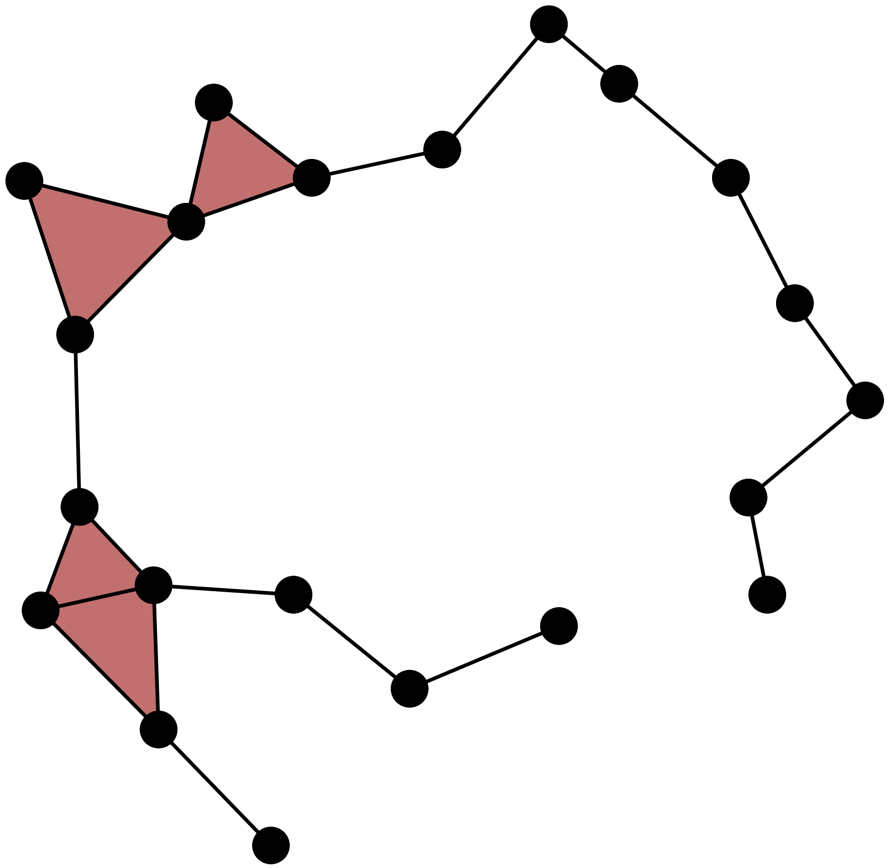

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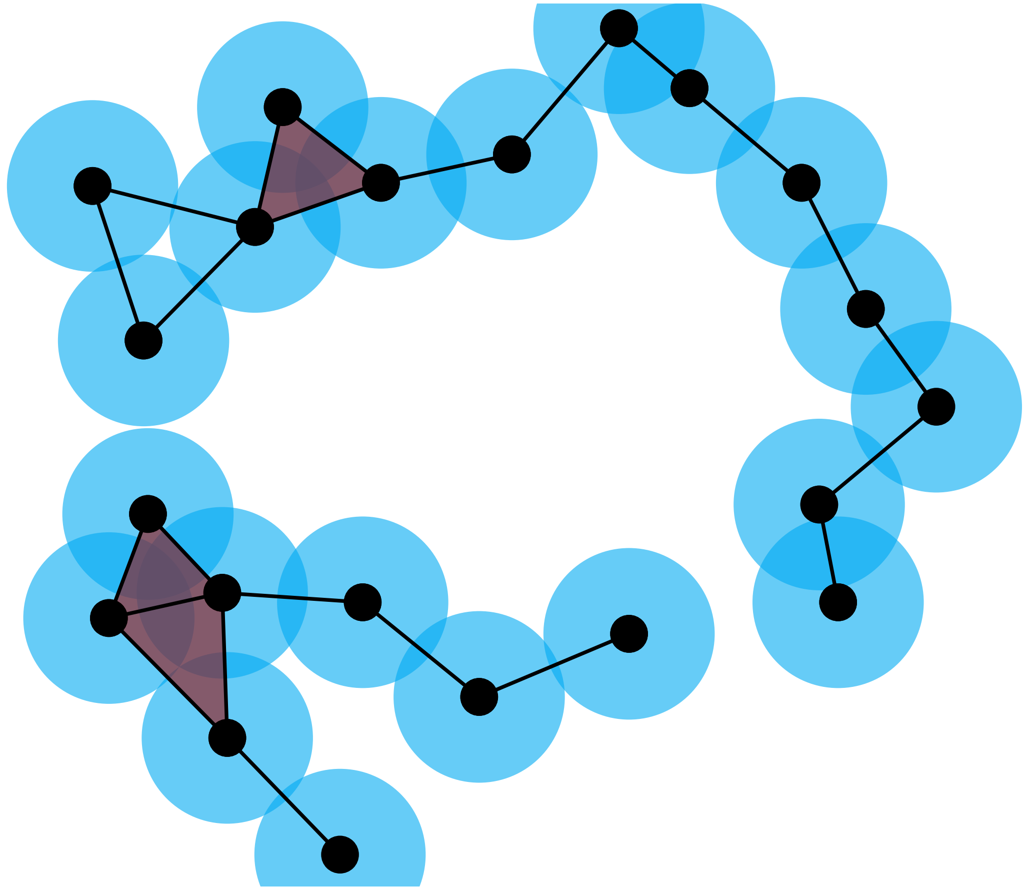

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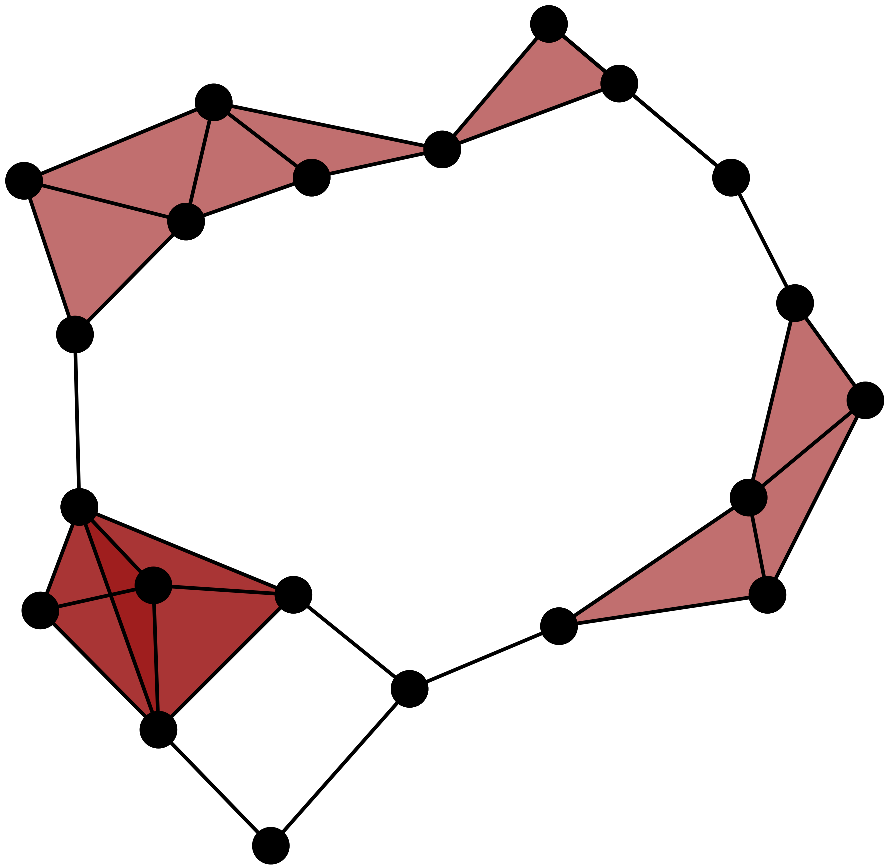

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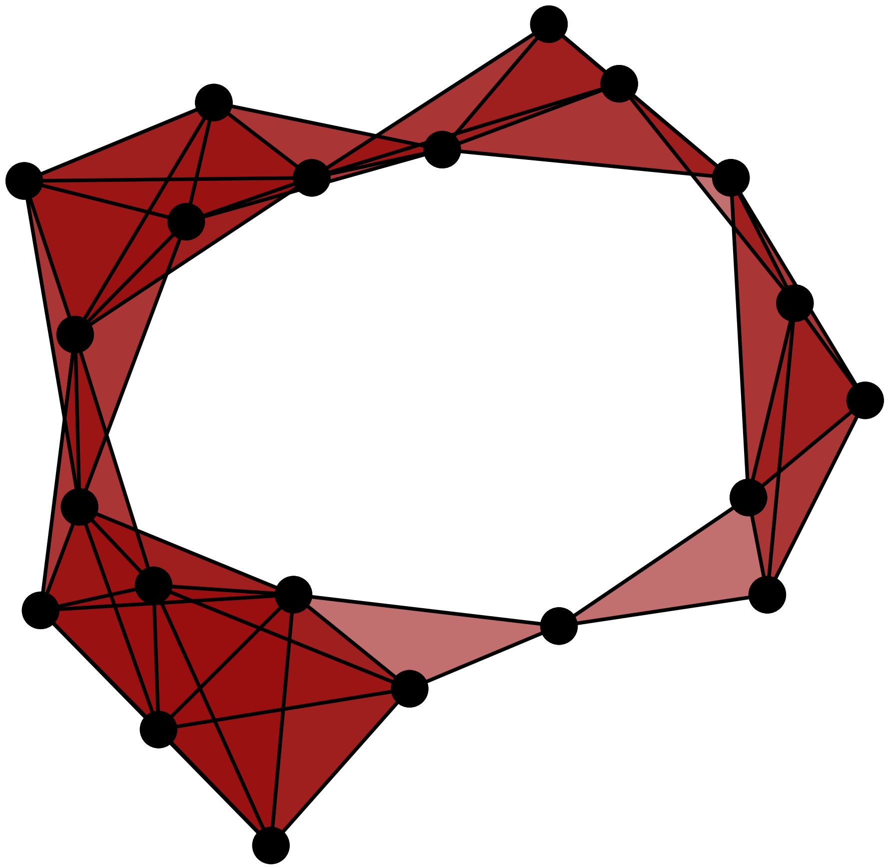

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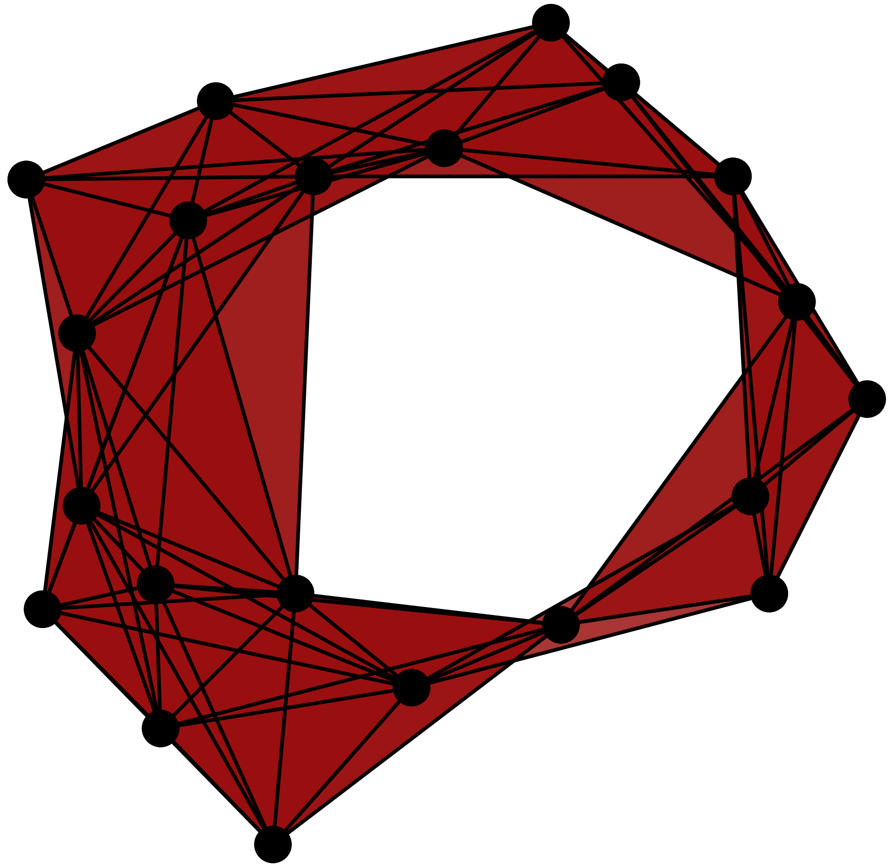

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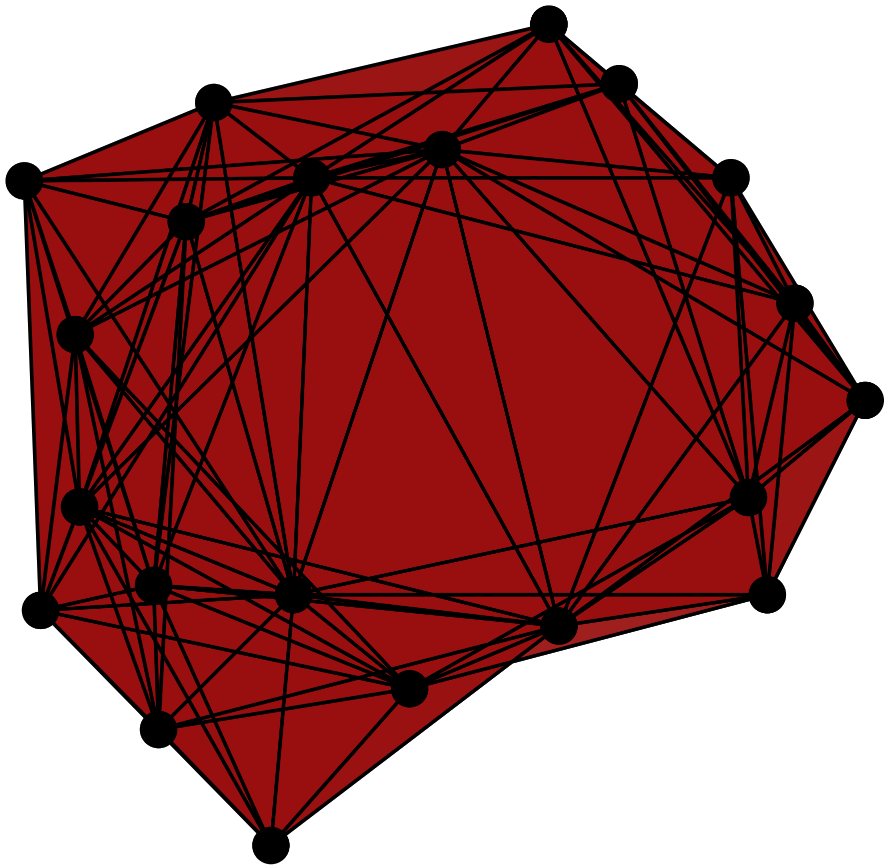

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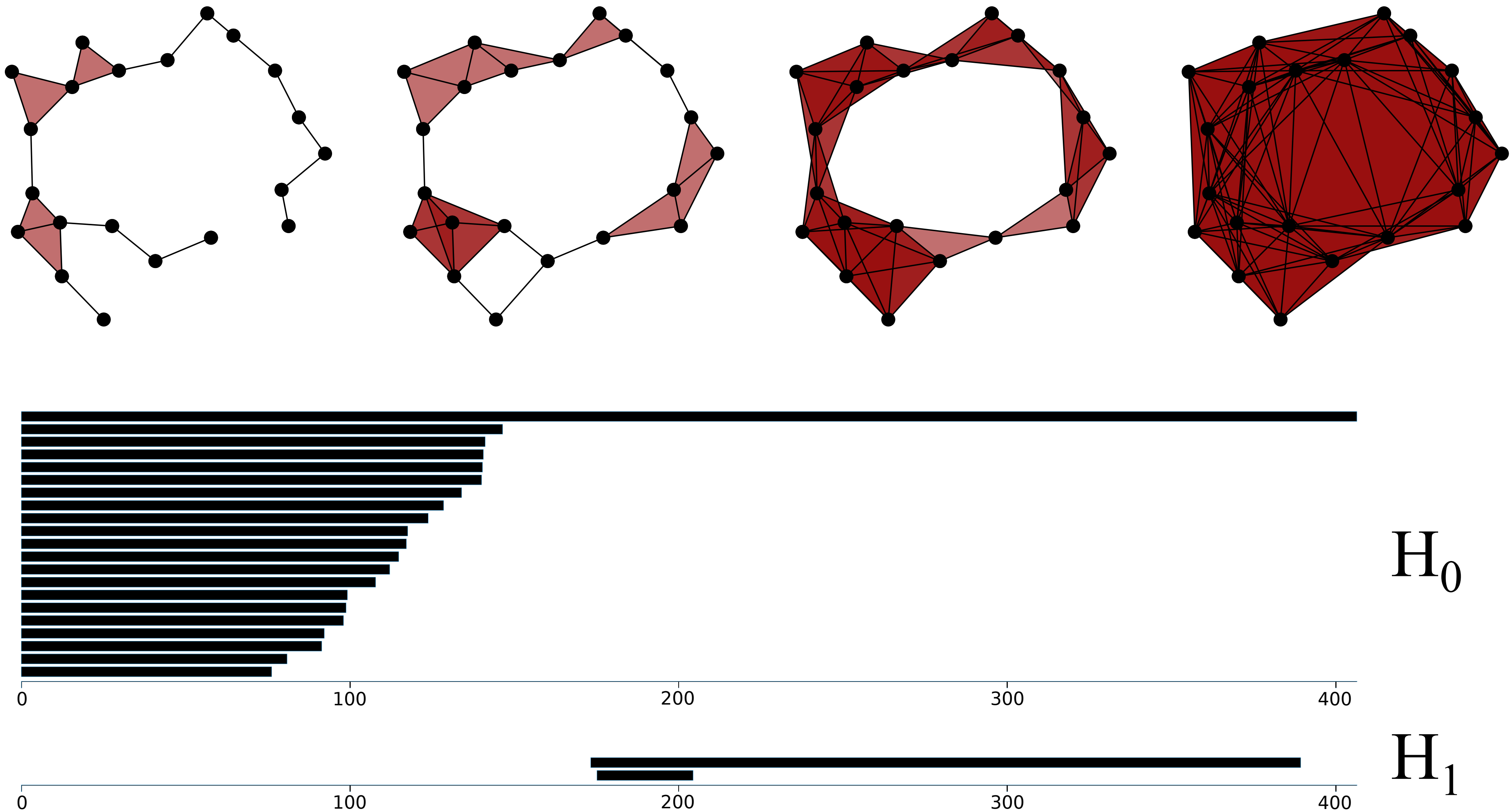
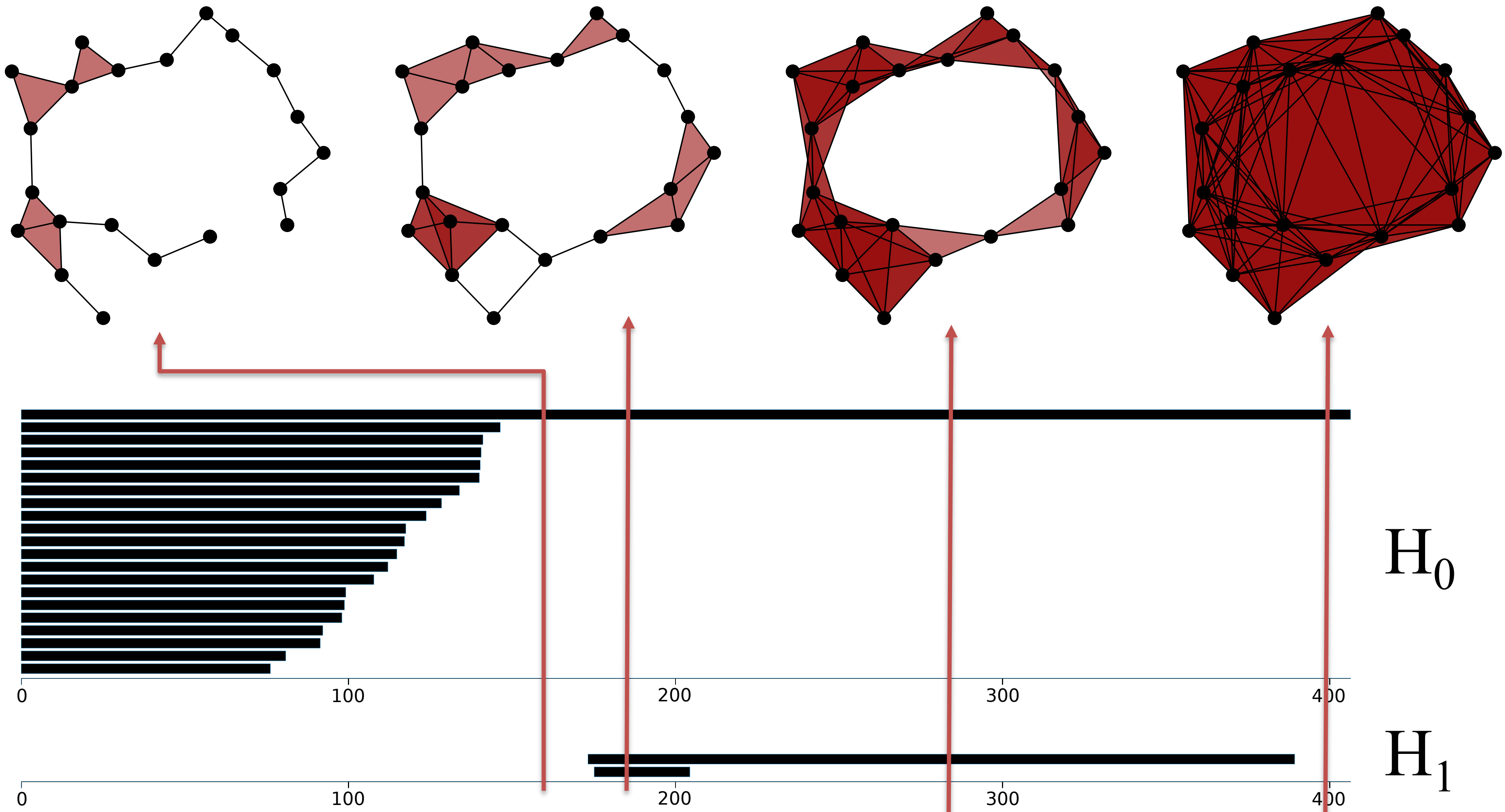
Persistent homology



- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

Persistent homology



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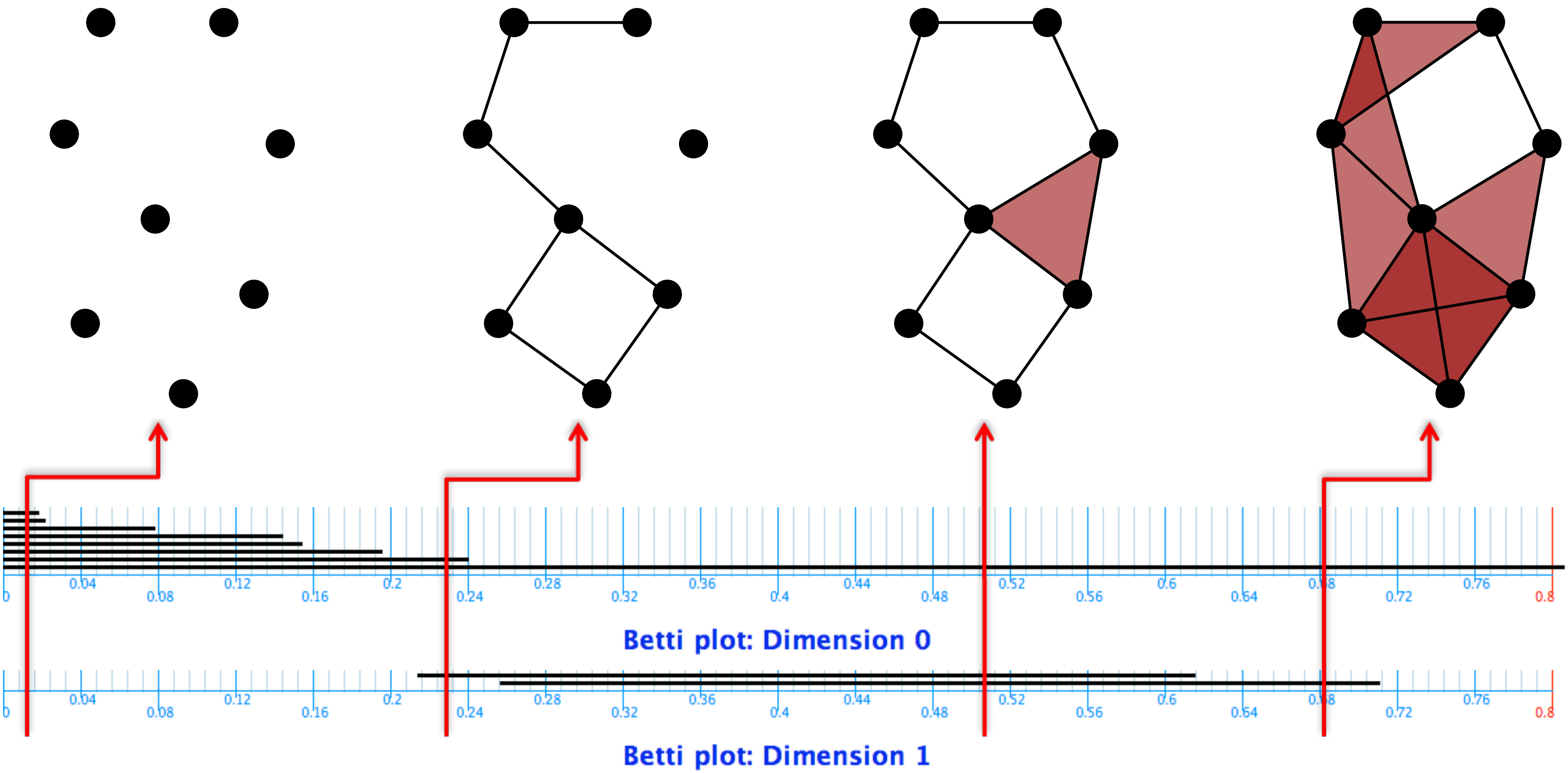

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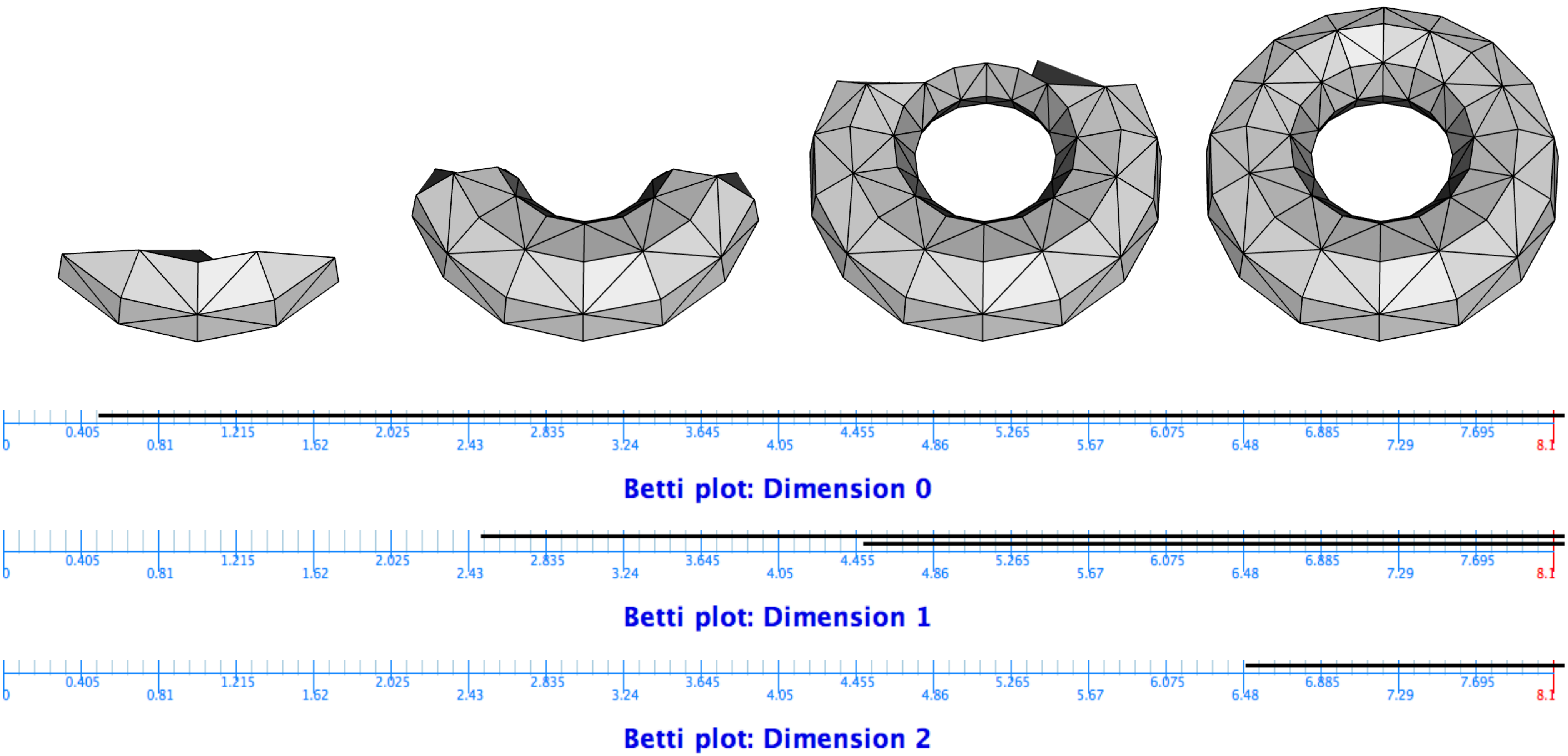

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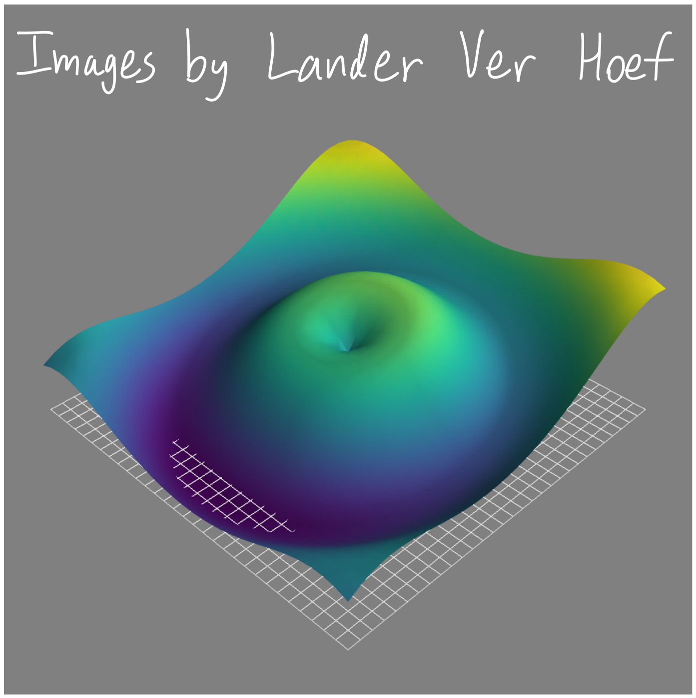
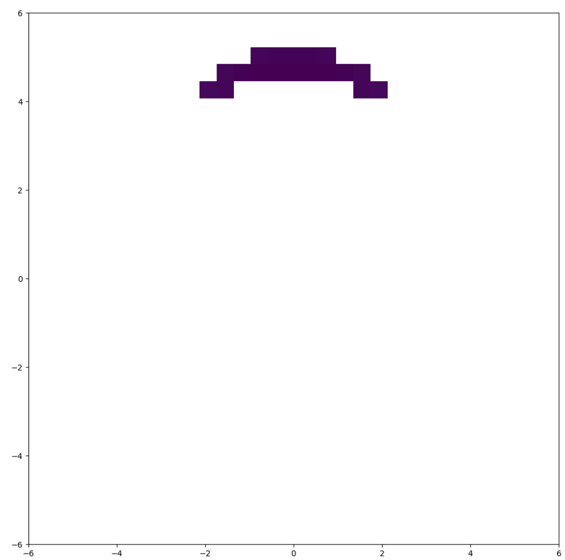
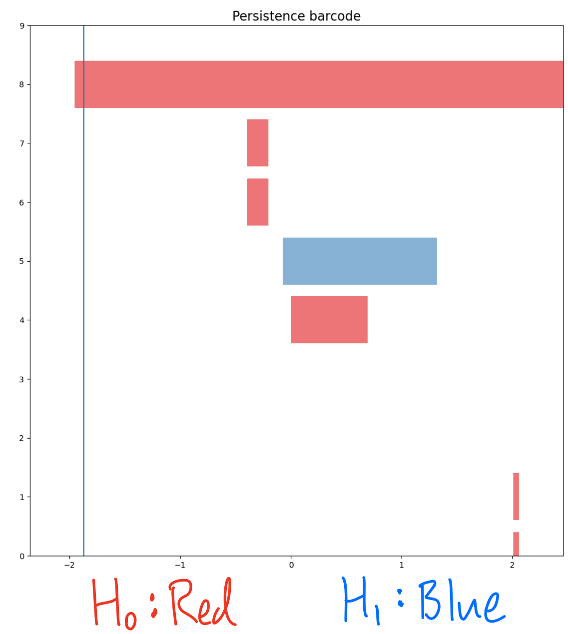
Persistent homology

Sublevelset persistence







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Persistent homology

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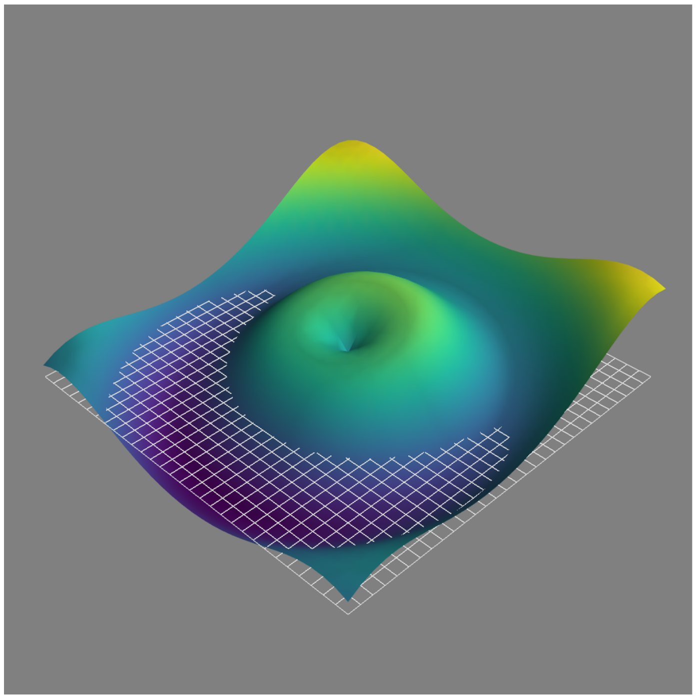

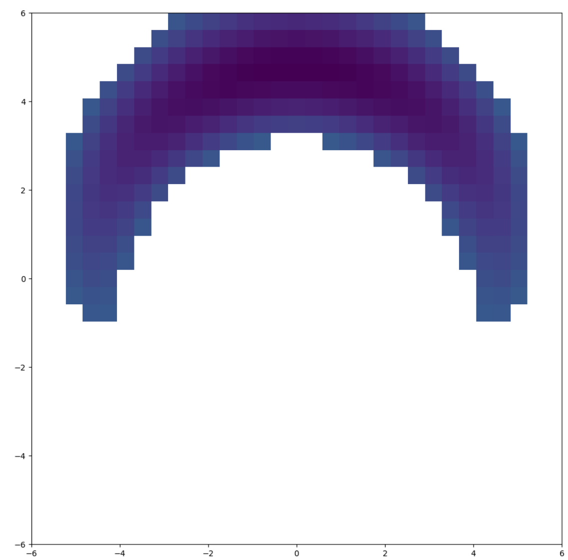

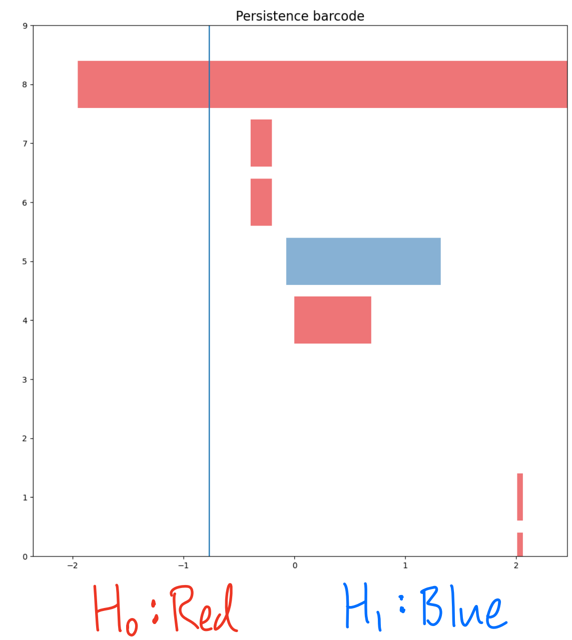

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Persistent homology

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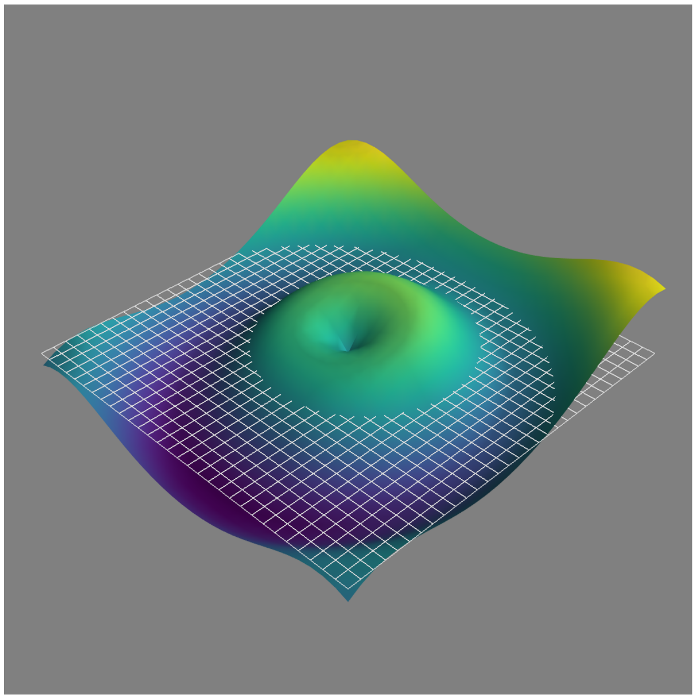

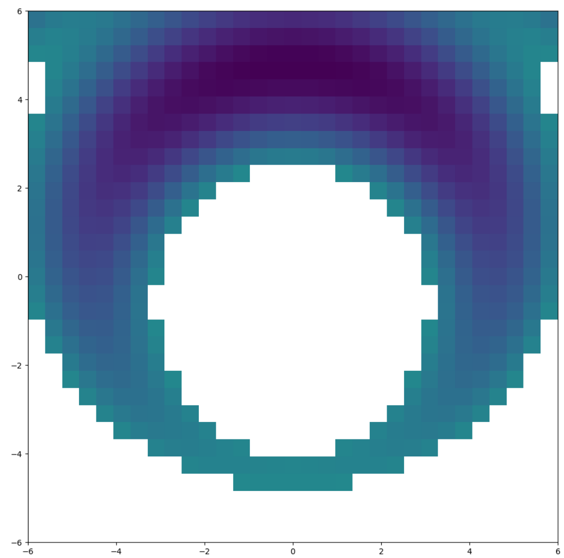

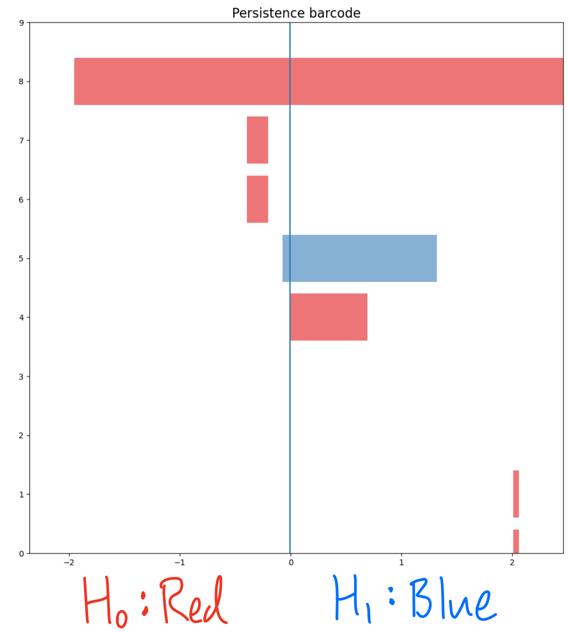

- Input: Increasing spaces. Output: barcode.
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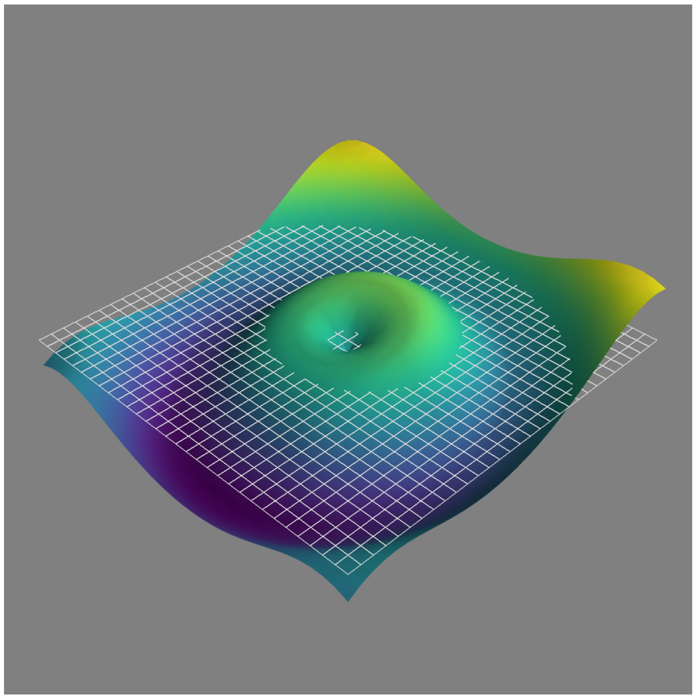
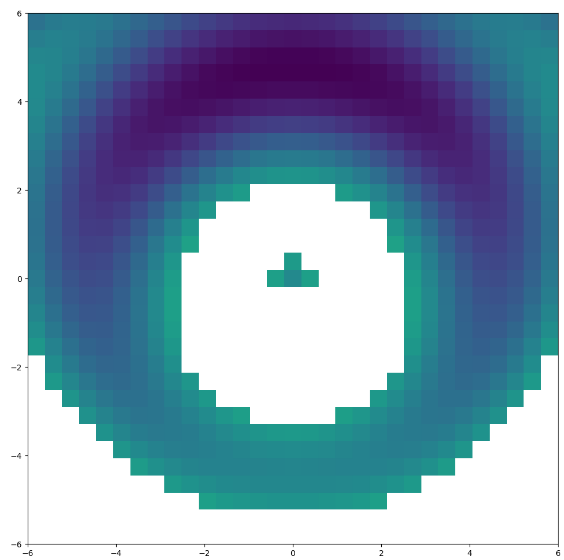
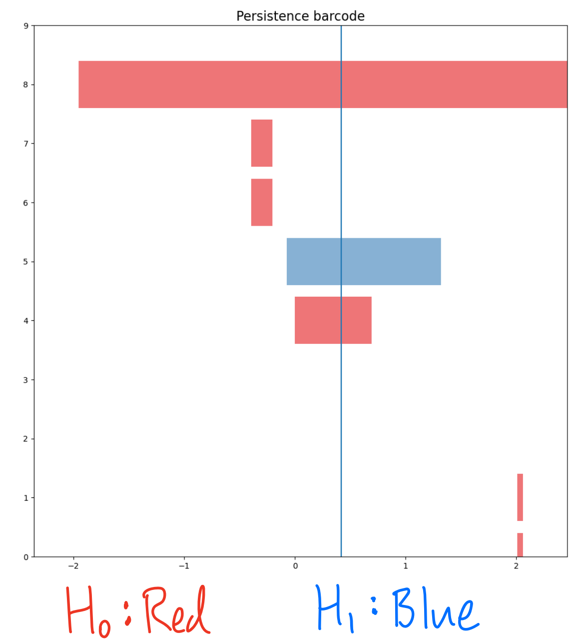
Persistent homology

Sublevelset persistence







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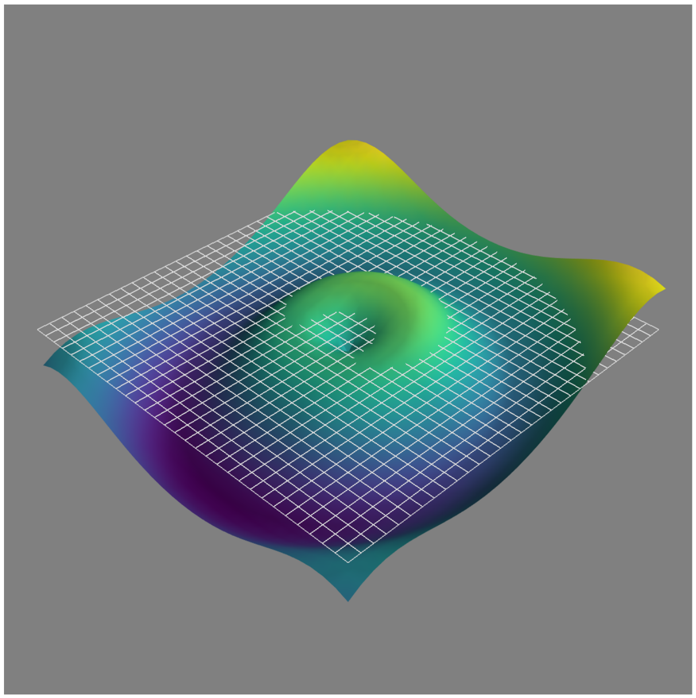

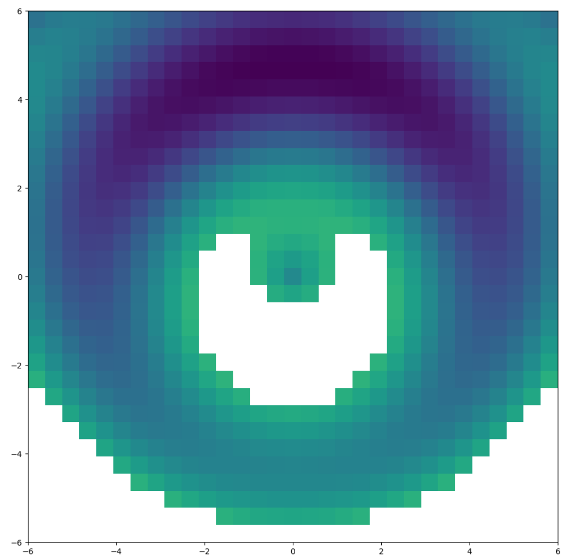

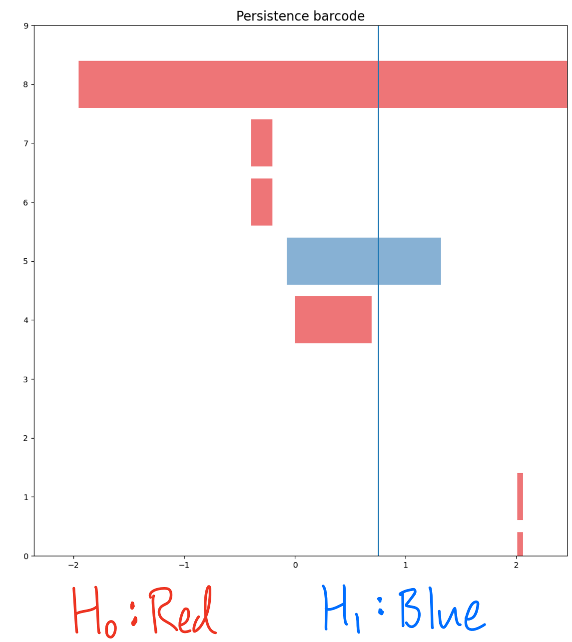

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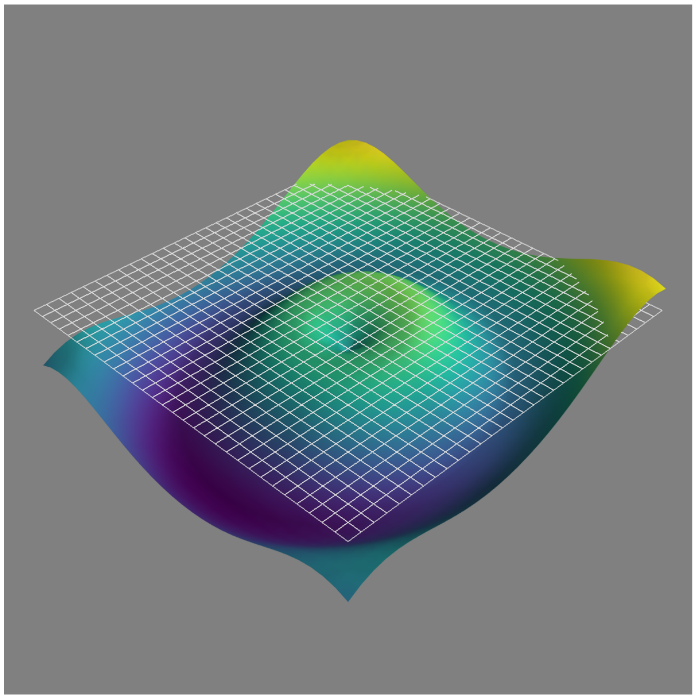

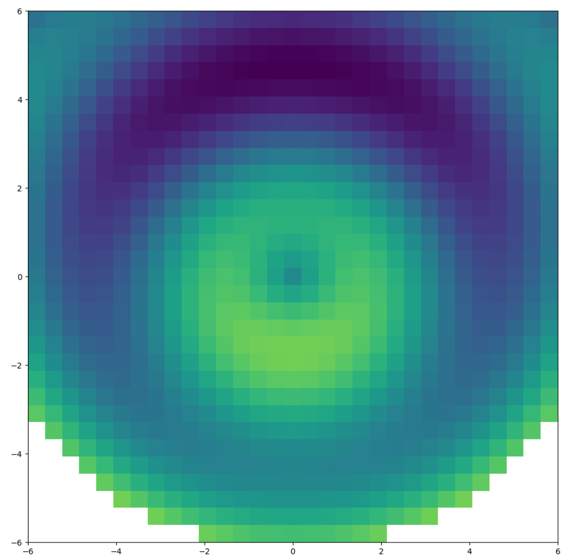

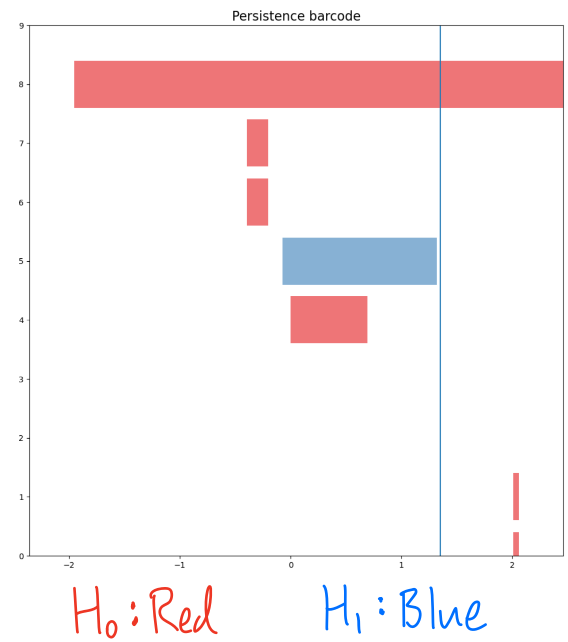

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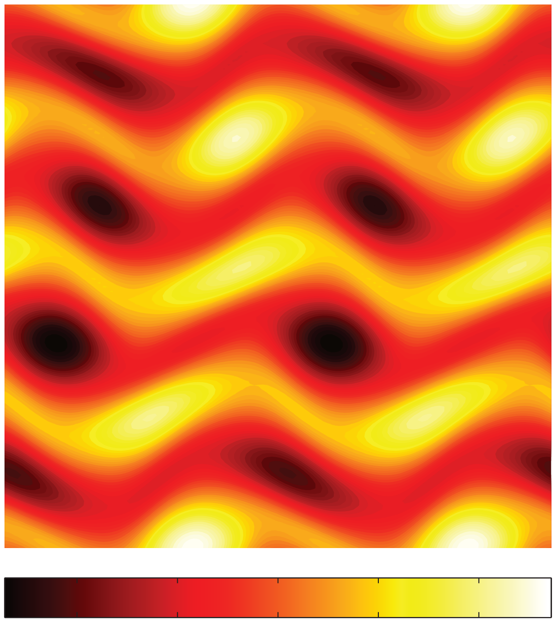
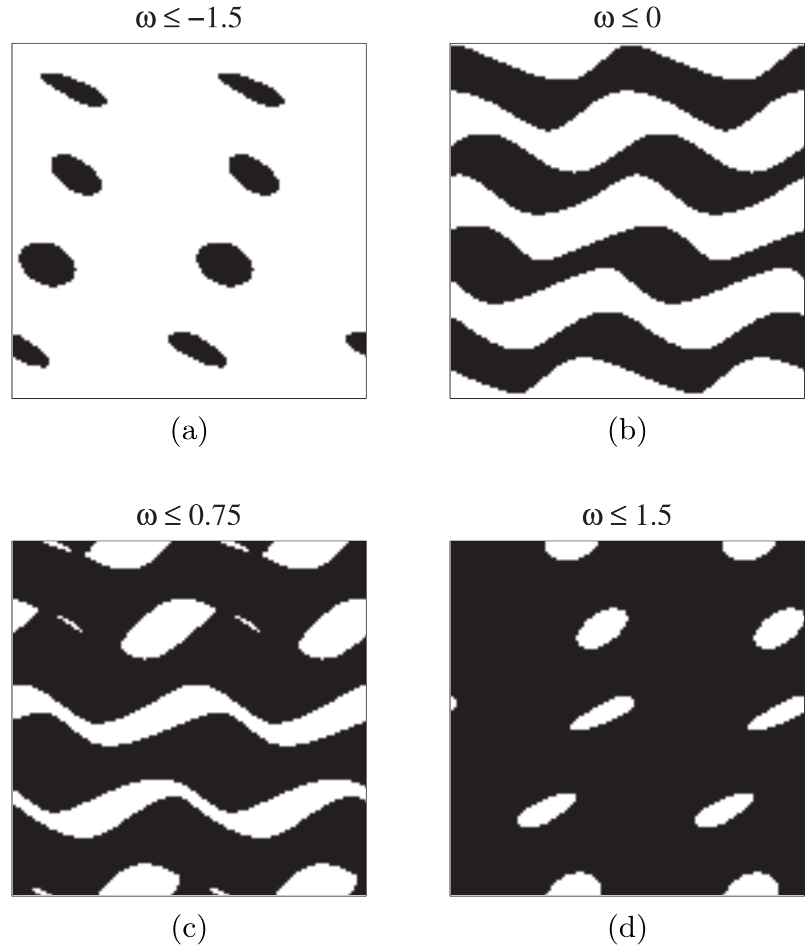


Analysis of Kolmogorov flow and Rayleigh–Bénard convection using persistent homology by Miroslav Kramár, Rachel Levanger, Jeffrey Tithof, Balachandra Suri, Mu Xu, Mark Paul, Michael F Schatz, Konstantin Mischaikow



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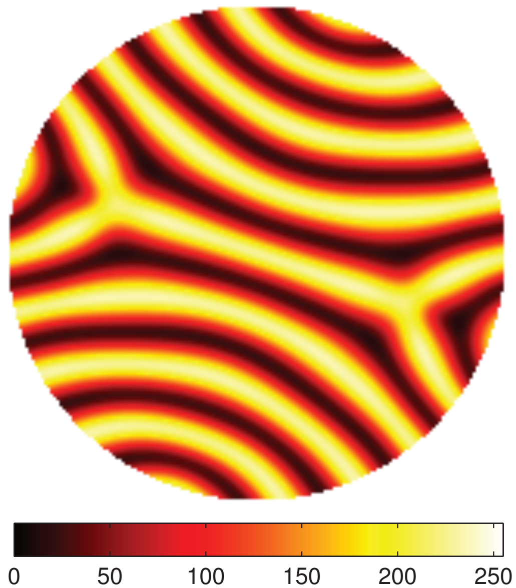

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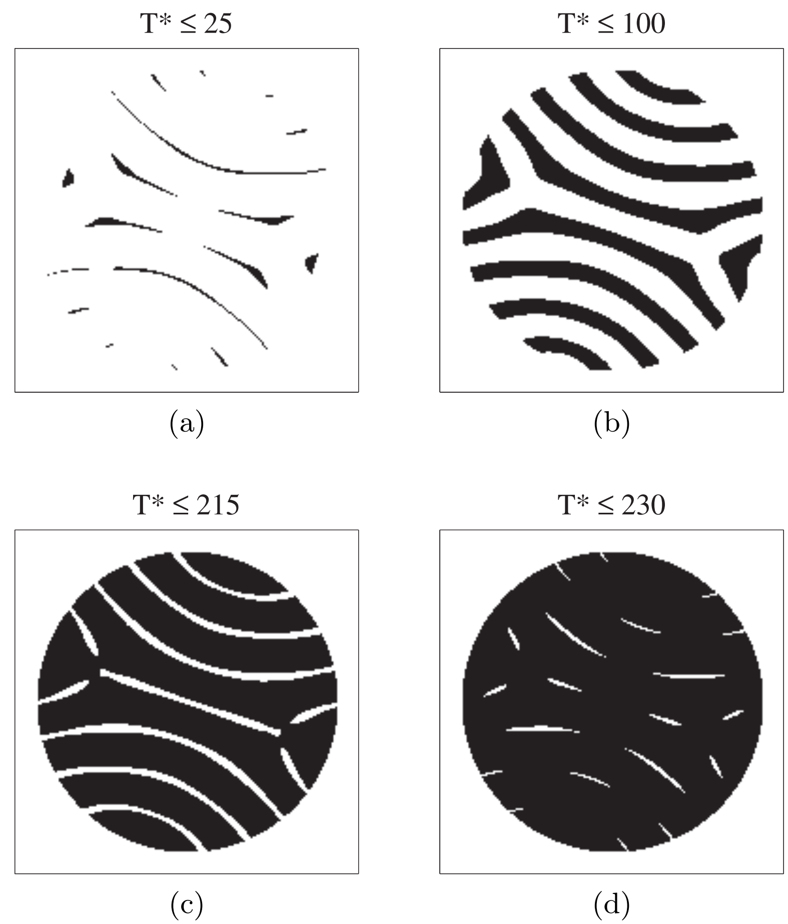

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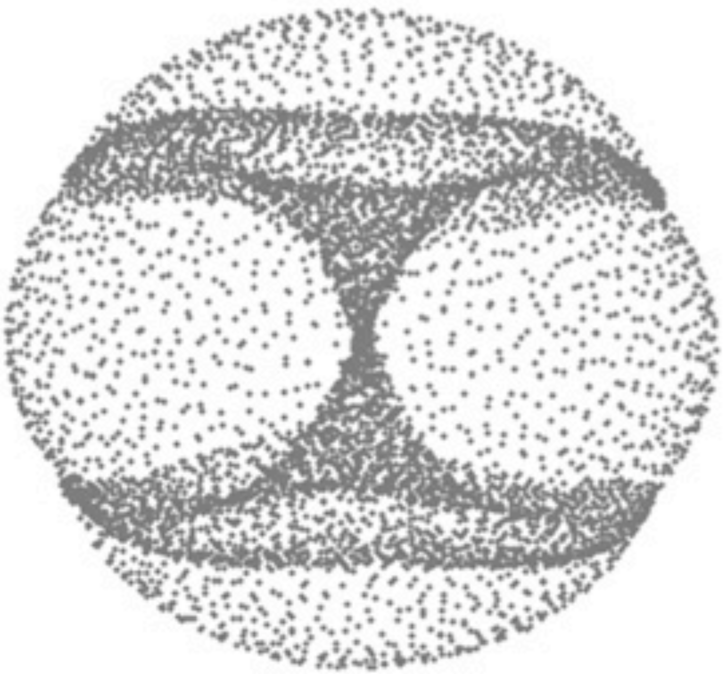
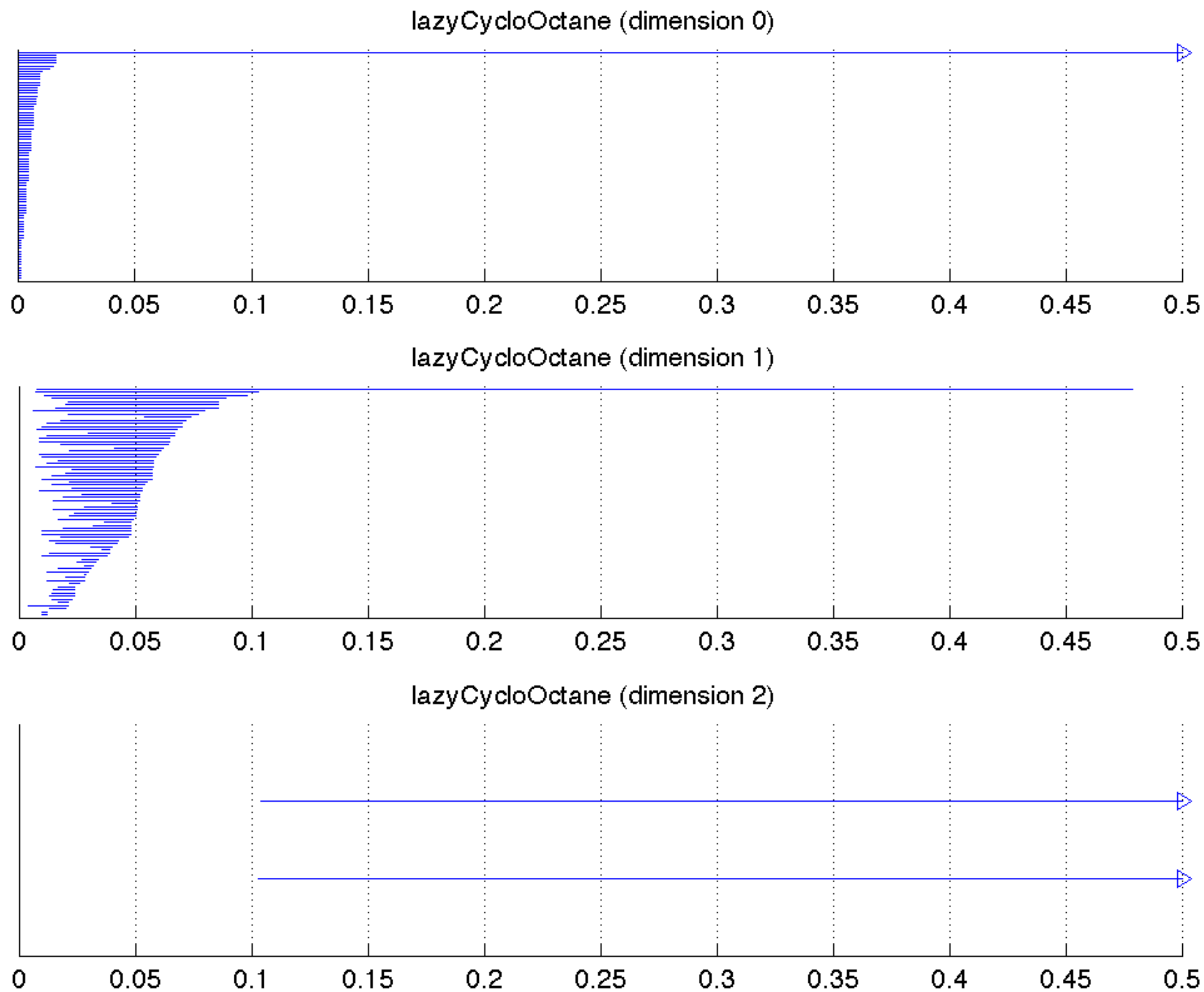
Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space





Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
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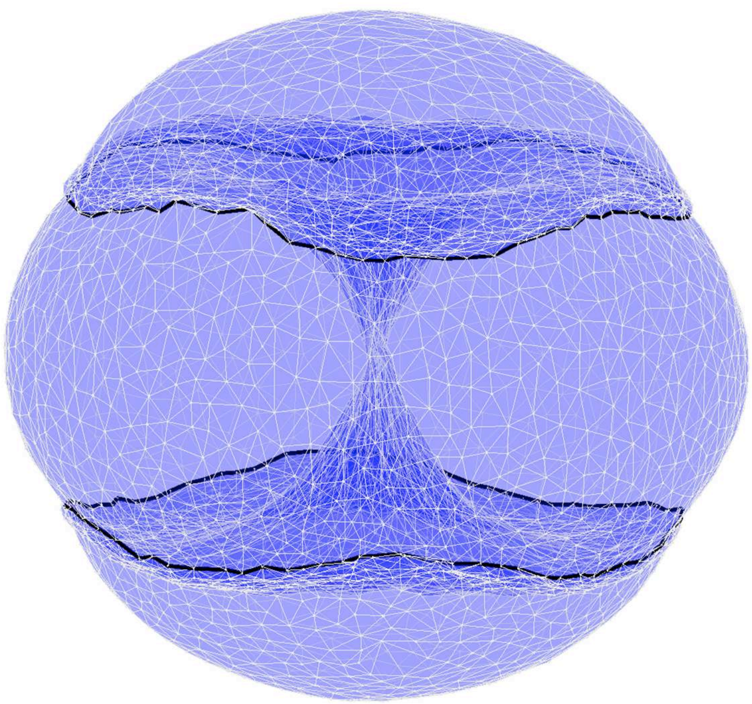

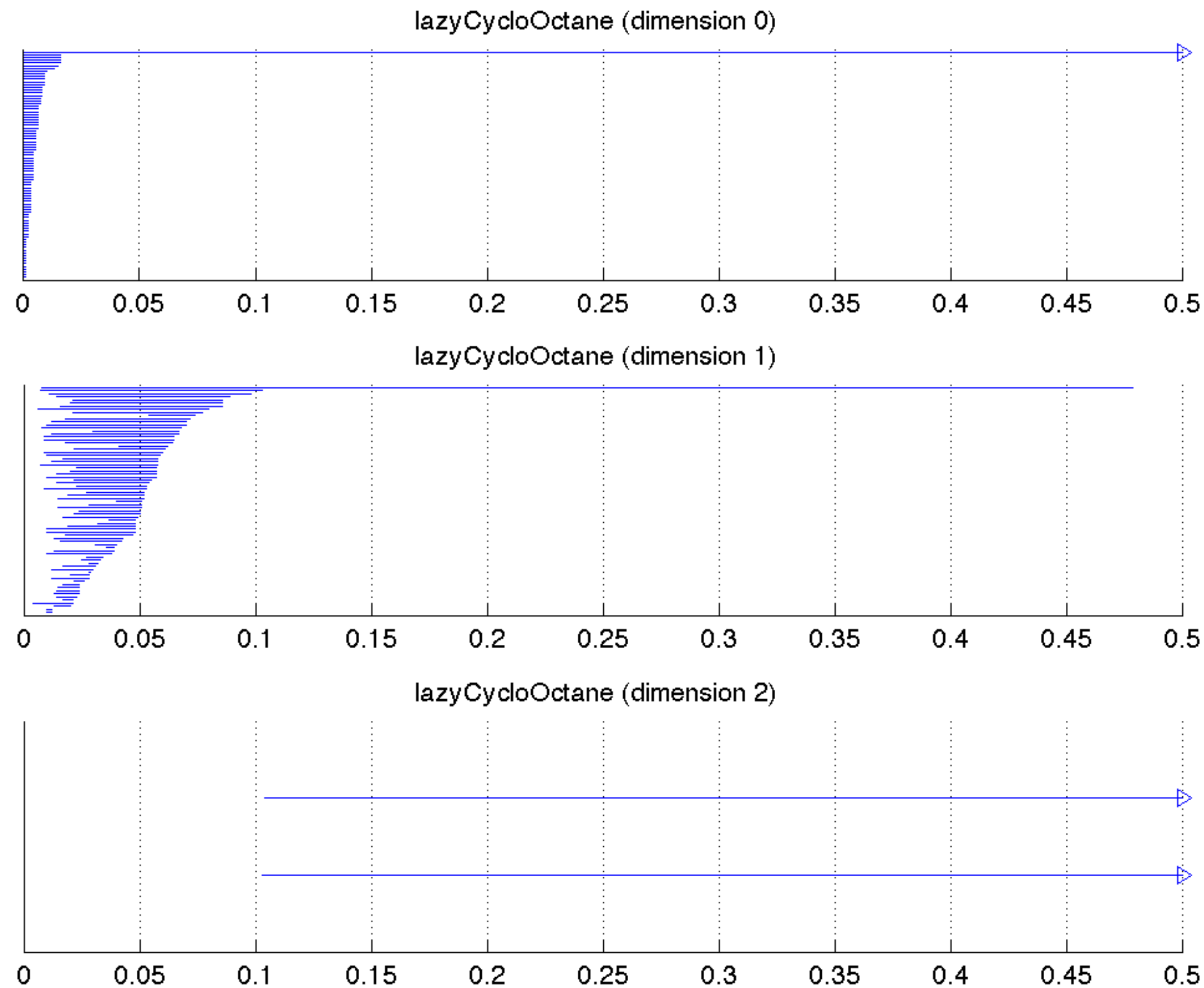

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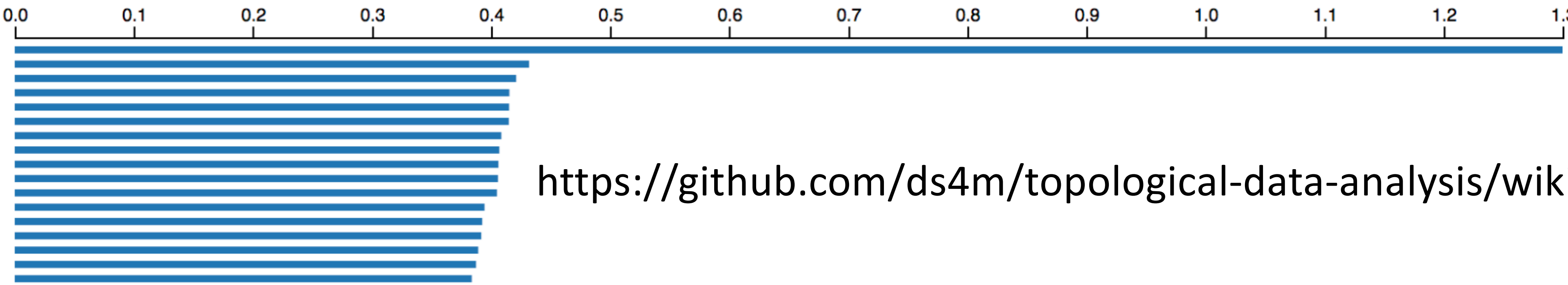
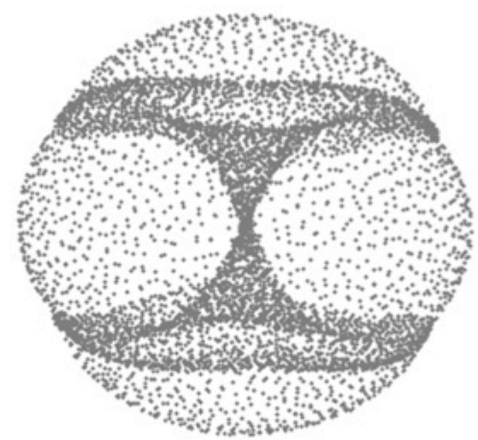
Persistence intervals in dimension 0:

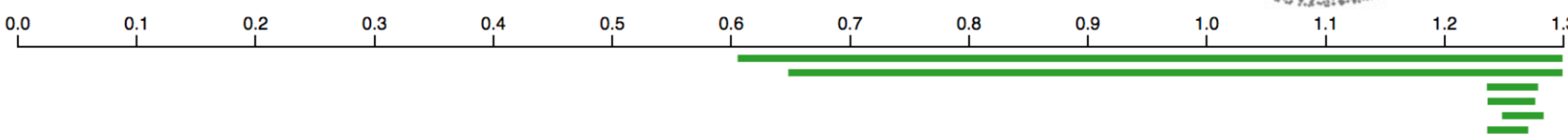


https://github.com/ds4m/topological-data-analysis/wiki

Persistence intervals in dimension 1:



Persistence intervals in dimension 2:



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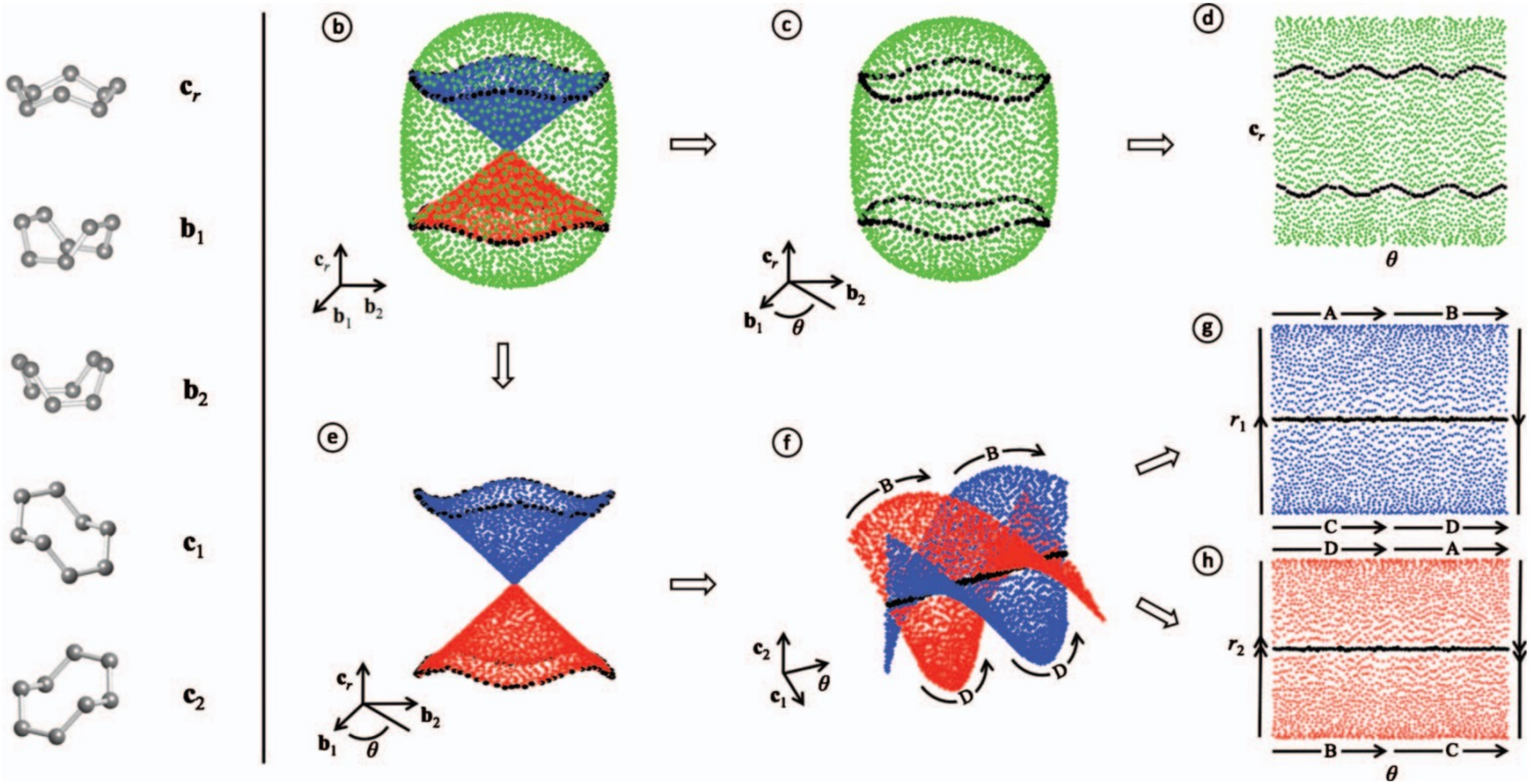

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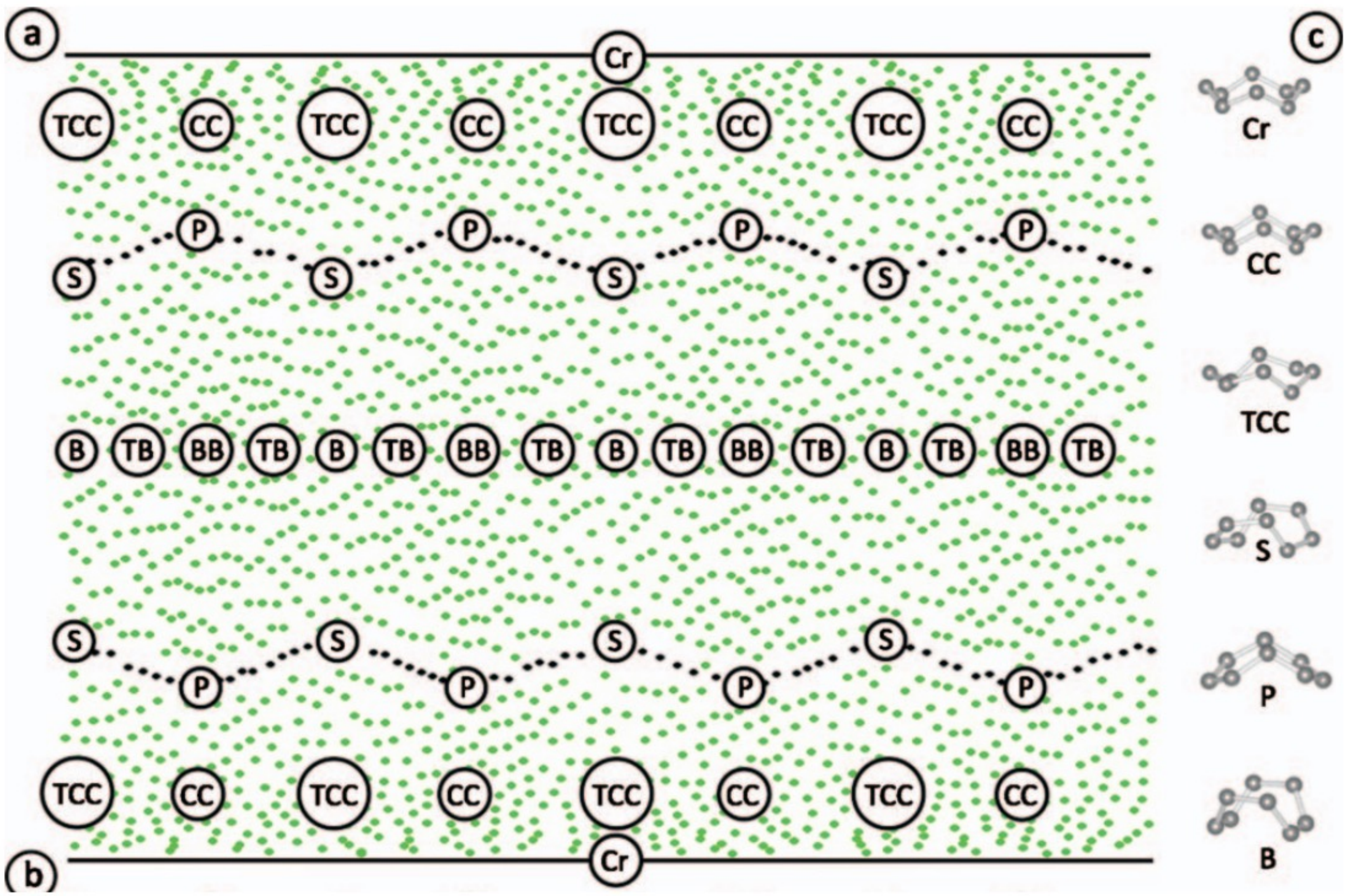

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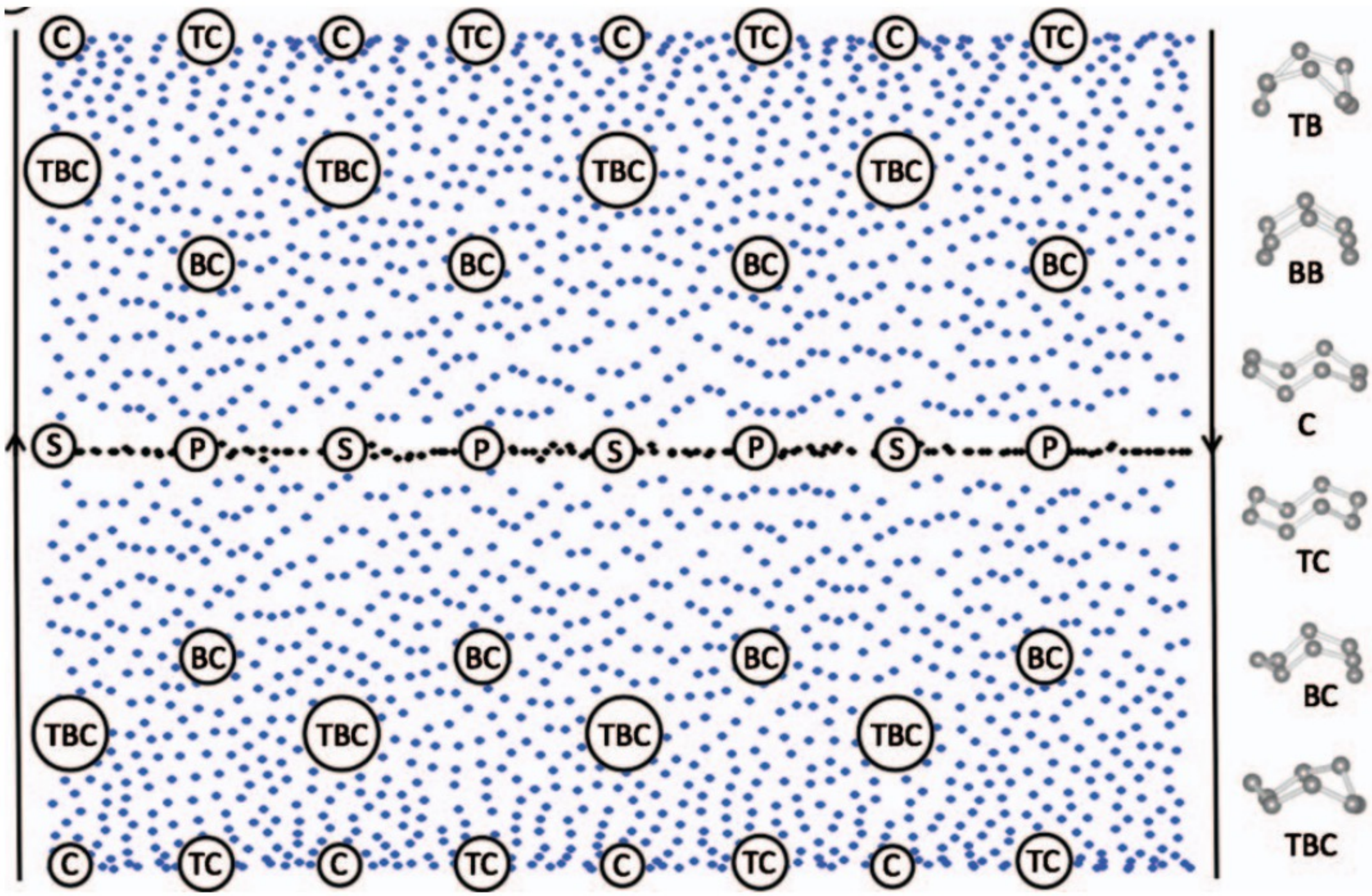

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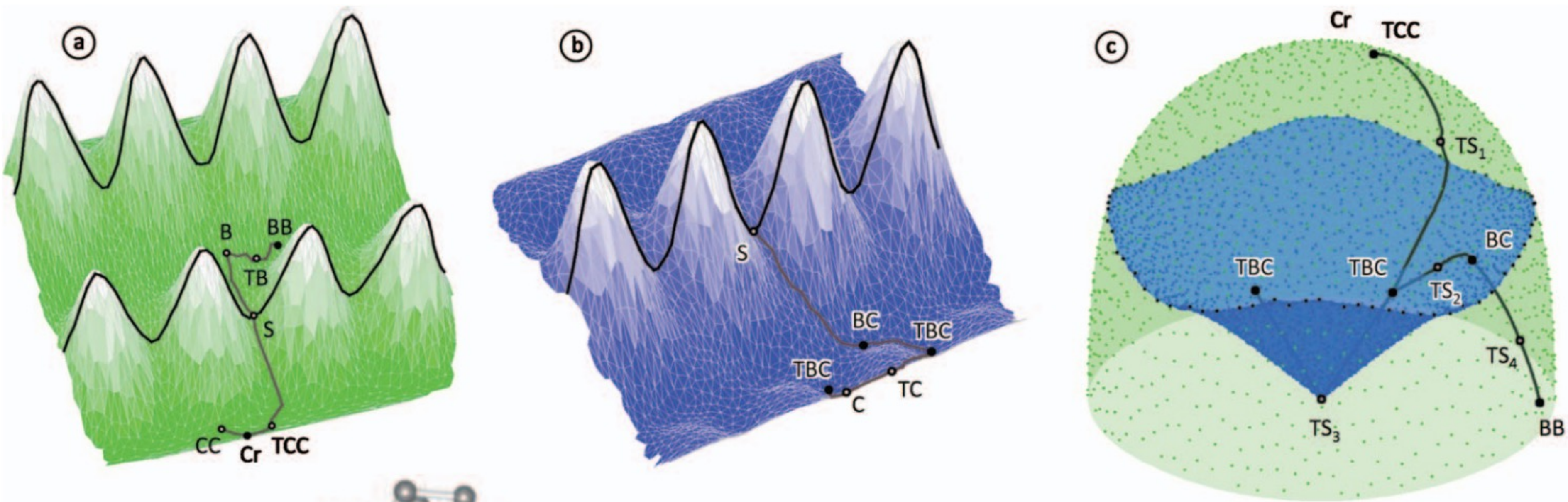

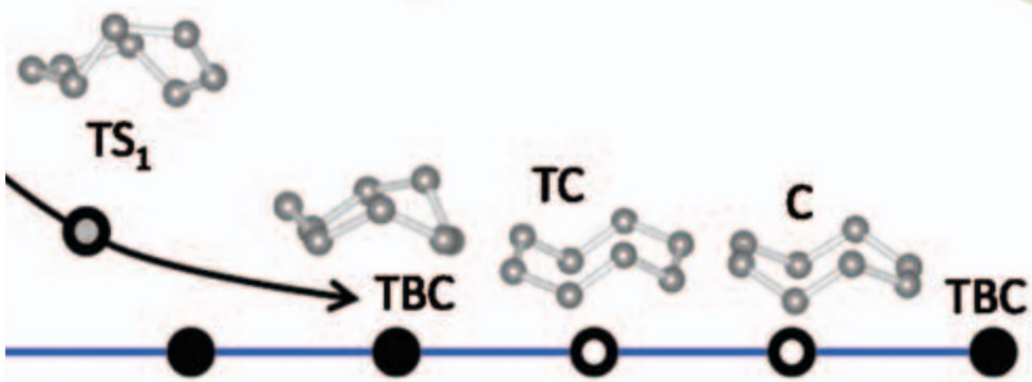

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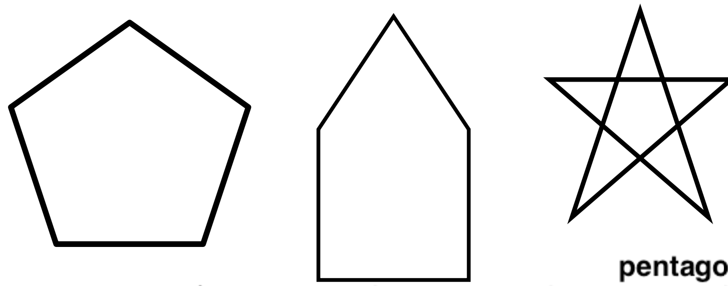
Persistent homology applied to data

Example: Equilateral pentagons in the plane



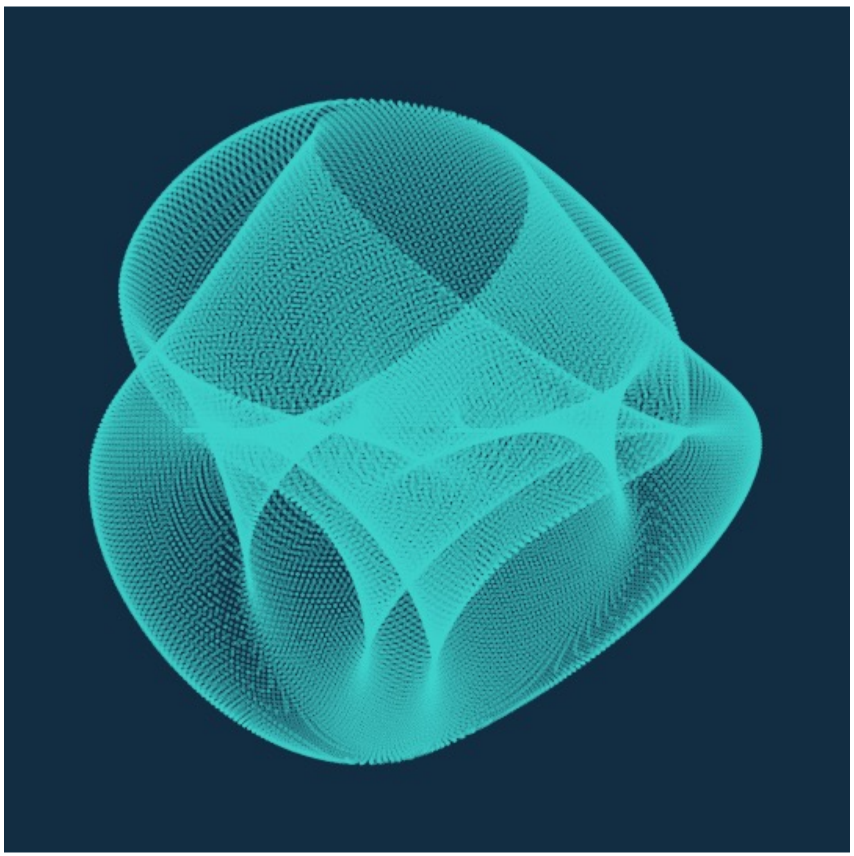
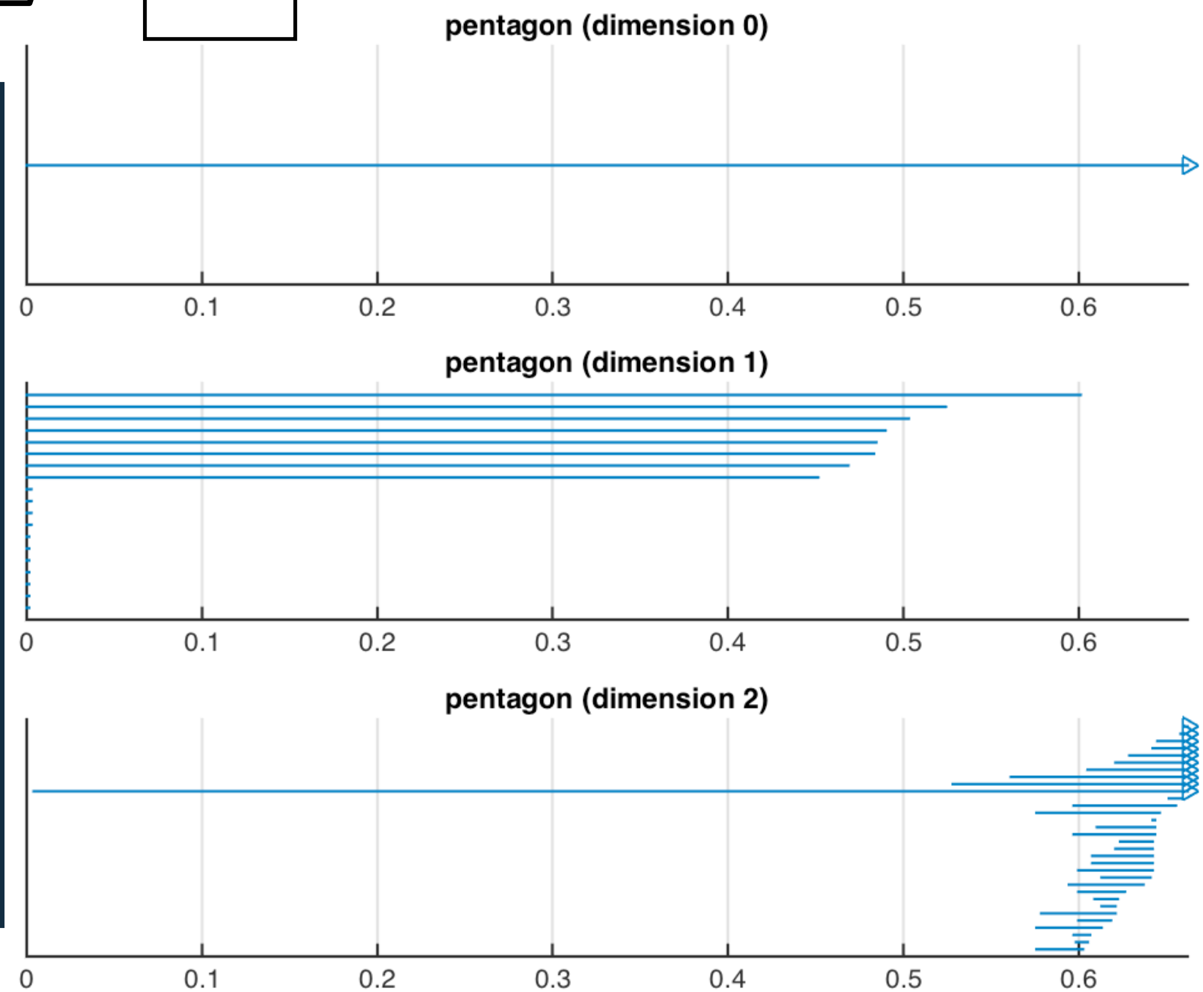
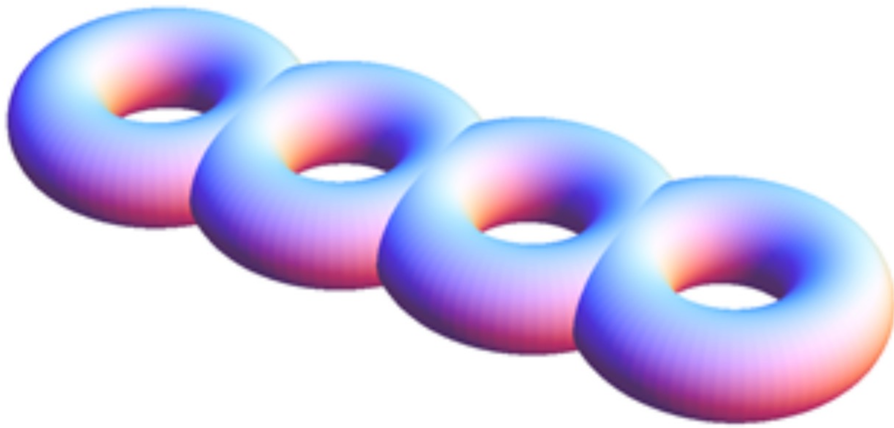
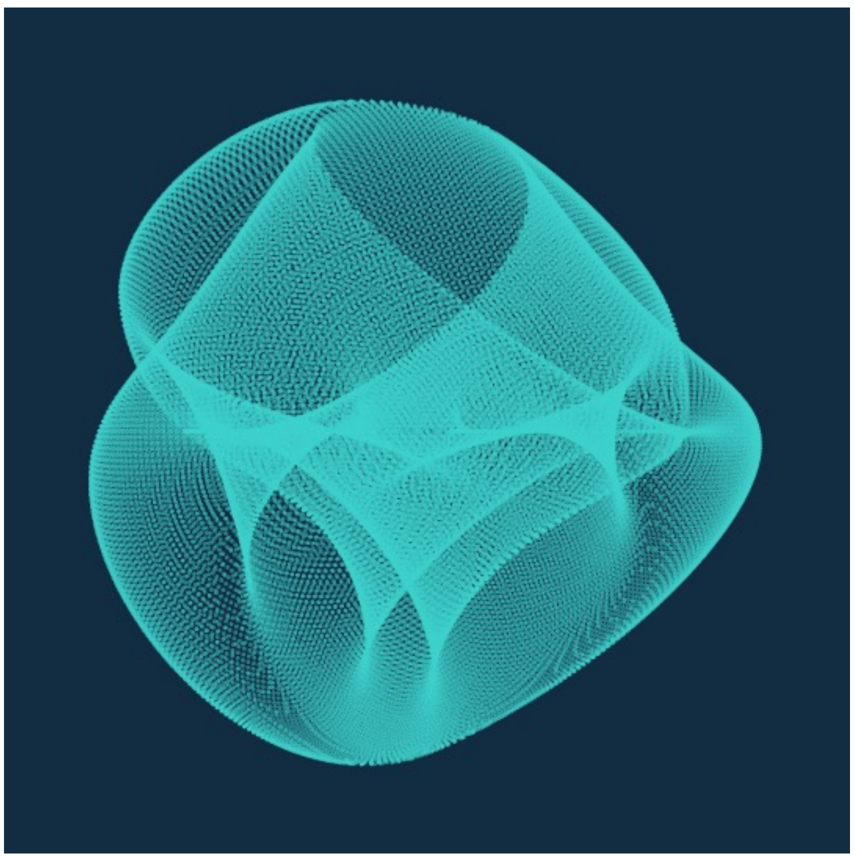


Image credit: Clayton Shonkwiler



Persistent homology applied to data





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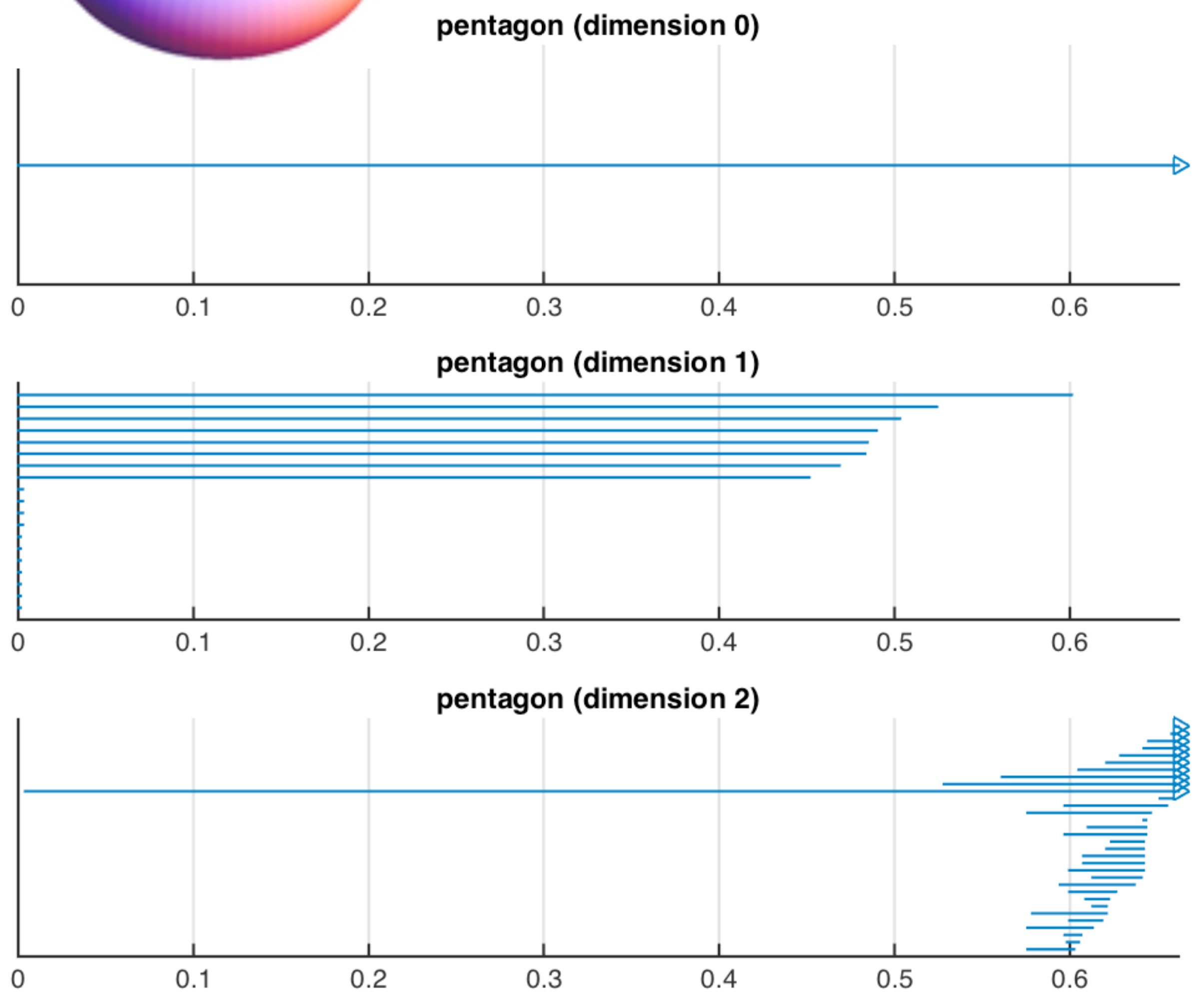
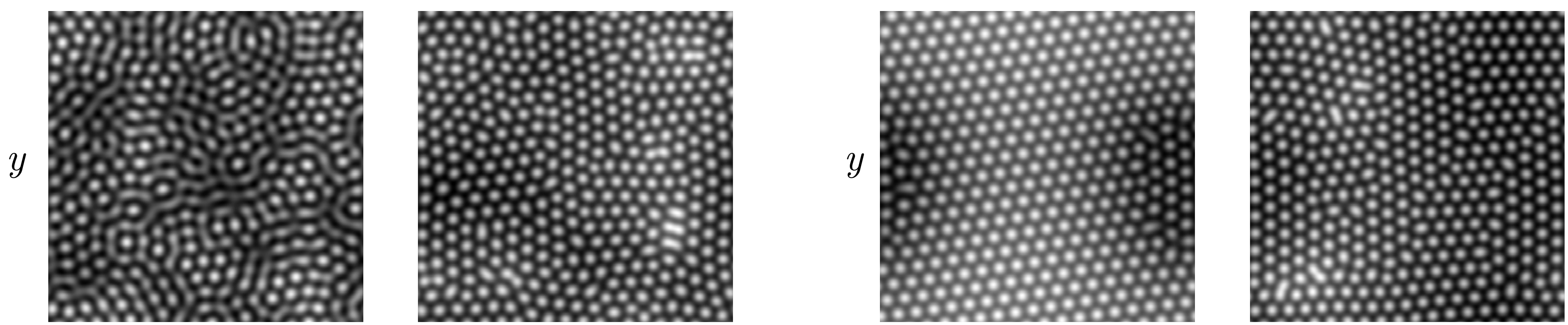


Why is applied topology popular when few datasets have Klein bottles?

- Many datasets have clusters & flares (as in the diabetes example)
- Motivates interesting questions in many pure disciplines: mathematics, computer science (computational geometry), statistics
- Interest from domain experts in biology, neuroscience, computer vision, dynamical systems, sensor networks, ...
- Materials science, pattern formation
- Machine learning: small features matter
- Agent-based modeling (swarming)

Possible answer: Persistent homology measures both the local geometry and the global topology of a dataset.



Measures of Order for nearly hexagonal lattices by Francis Motta, Rachel Neville, Patrick Shipman, Daniel Pearson, and Mark Bradley, 2018.

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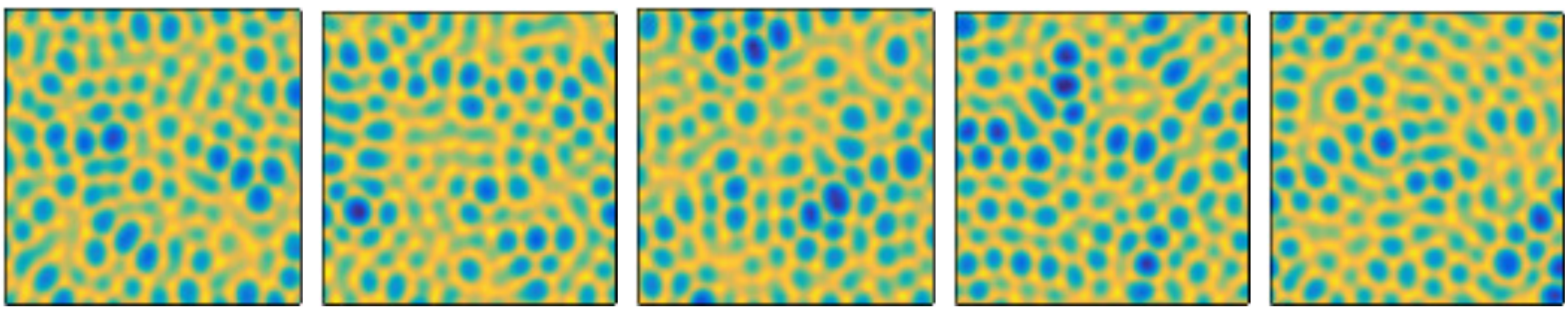

Answer: (from left) $r = 1.75, 2, 1.75, 2, 2.$

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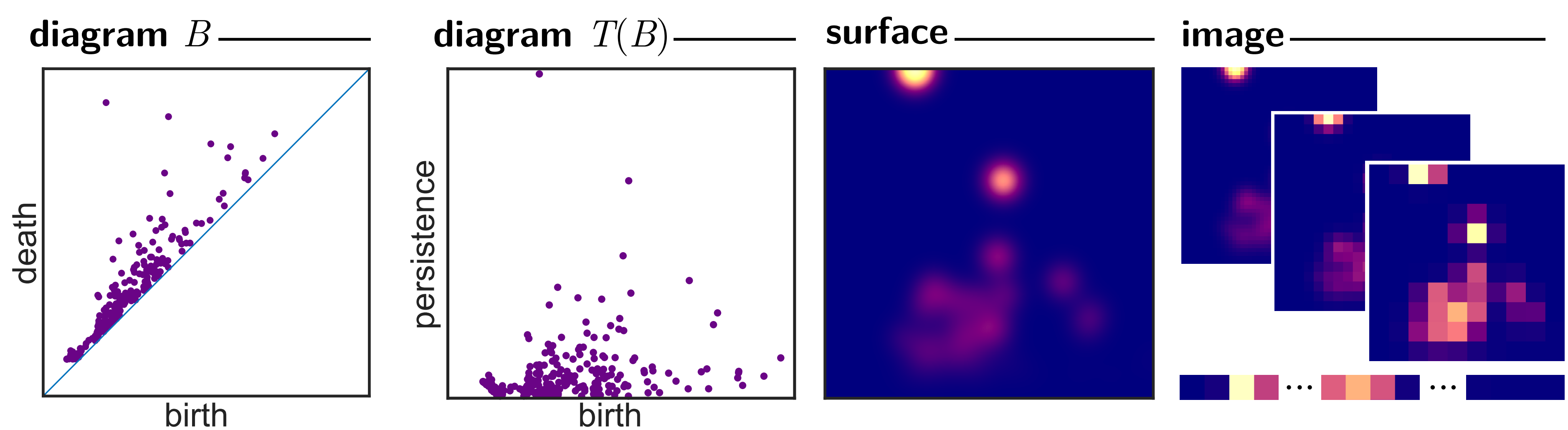
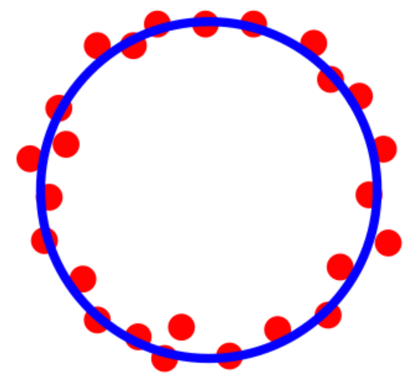


Persistent homology applied to data

- Stability Theorem.

  If X and Y are metric spaces, then

$$d_b(\mathrm{PH}(\check{\mathrm{C}}\mathrm{ech}(X)), \mathrm{PH}(\check{\mathrm{C}}\mathrm{ech}(Y))) \leq 2d_{\mathrm{GH}}(X, Y)$$



An Introduction to Applied Topology

Henry Adams
University of Florida