An Introduction to Applied Topology

Henry Adams
Colorado State University

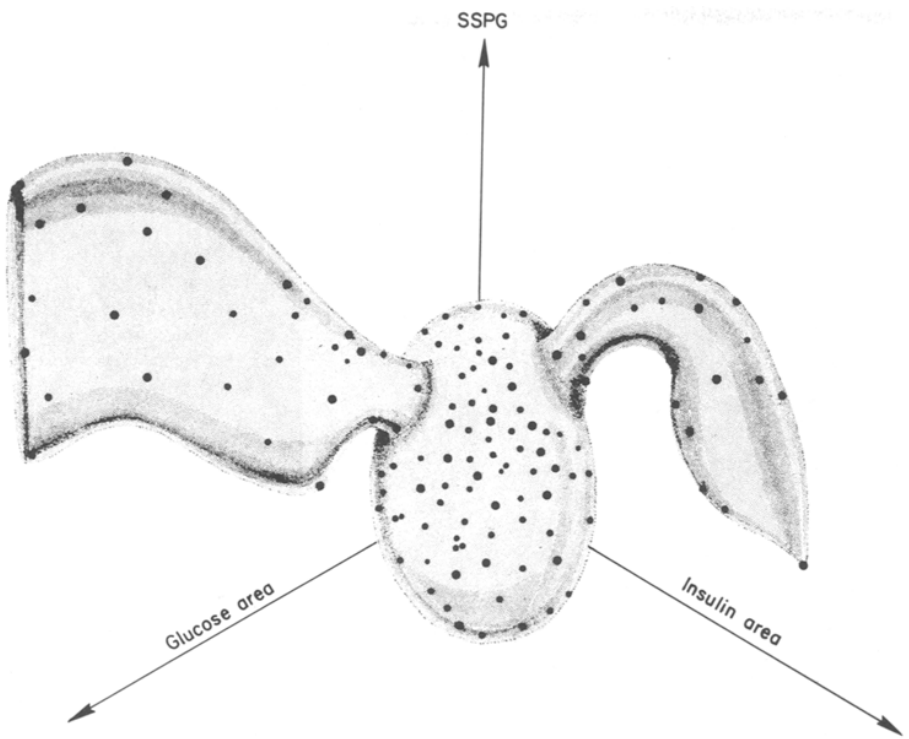
An Introduction to Applied Topology

Thanks to Murray State and the organizers:
Susanne D'Angelo, Dubravko Ivanšić,
Ted Porter, Tim Schroeder,
and all special session organizers

AATRN: 1 or 2 live talks per week
YouTube channel: 2,200 subscribers
20 hours watched per day

Datasets have shapes

Example: Diabetes study
145 points in 5-dimensional space

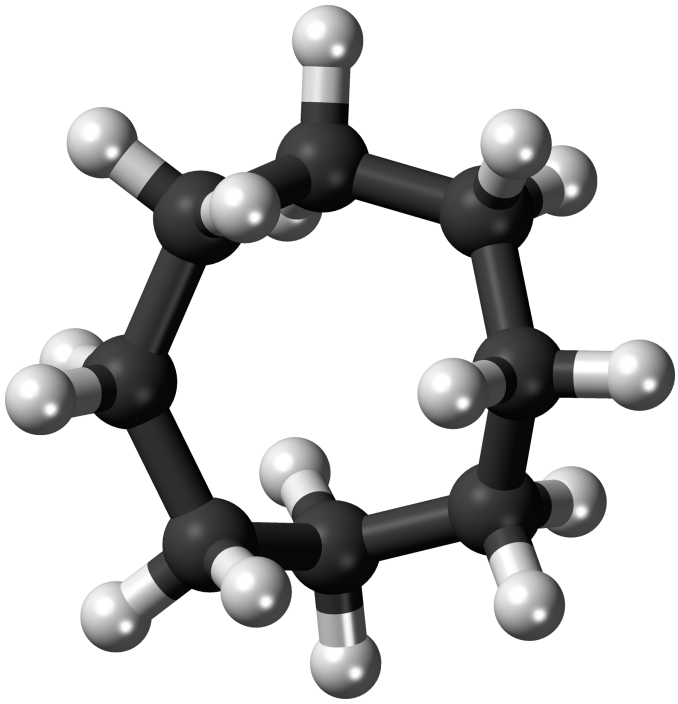


An attempt to define the nature of chemical diabetes using a multidimensional analysis by G. M. Reaven and R. G. Miller, 1979

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space

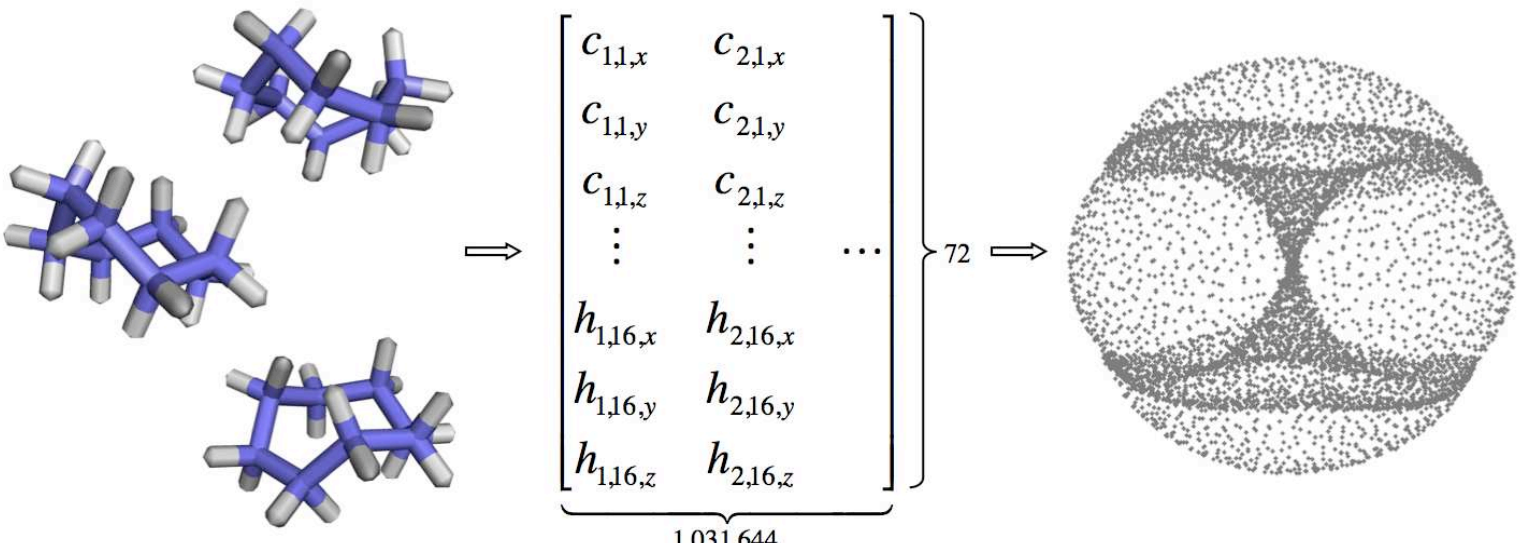


Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space

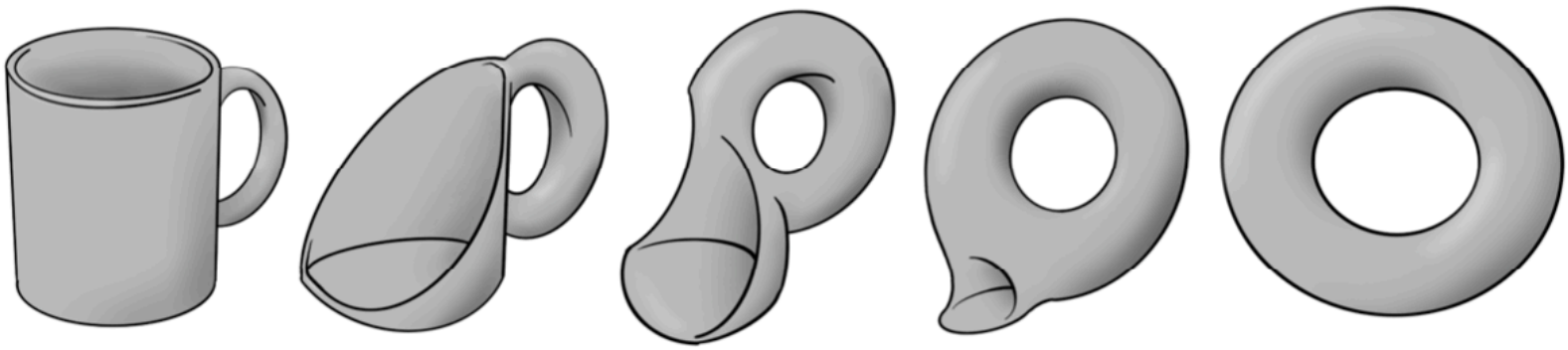


Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Datasets have shapes

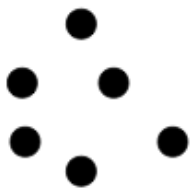
Topology studies shapes

A donut and coffee mug are “homotopy equivalent”, and considered to be the same shape. You can bend and stretch (but not tear) one to get the other.



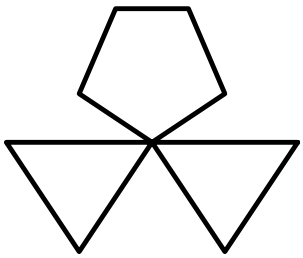
Homology

- i-dimensional homology H_i "counts the number of i-dimensional holes"
- i-dimensional homology H_i actually has the structure of a vector space!



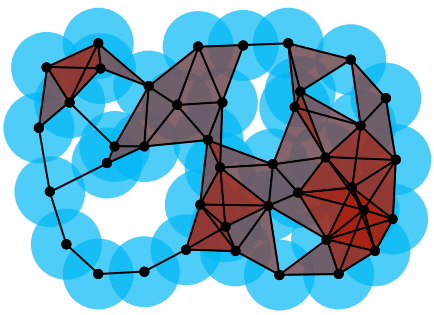
0-dimensional homology H_0: rank 6

1-dimensional homology H_1: rank 0



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 3

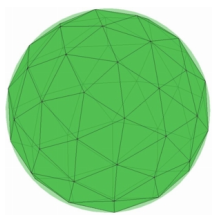


0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 6

Homology

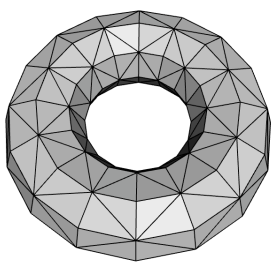
- i-dimensional homology "counts the number of i-dimensional holes"
- i-dimensional homology actually has the structure of a vector space!



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 0

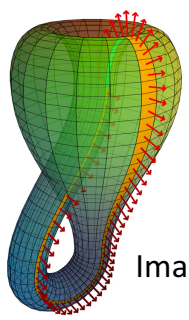
2-dimensional homology H_2: rank 1



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 2

2-dimensional homology H_2: rank 1

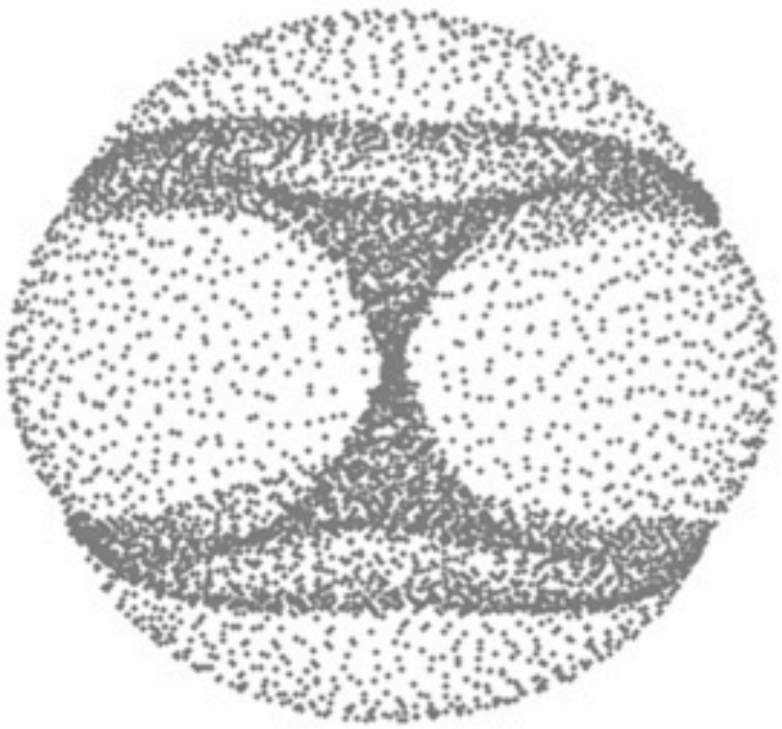


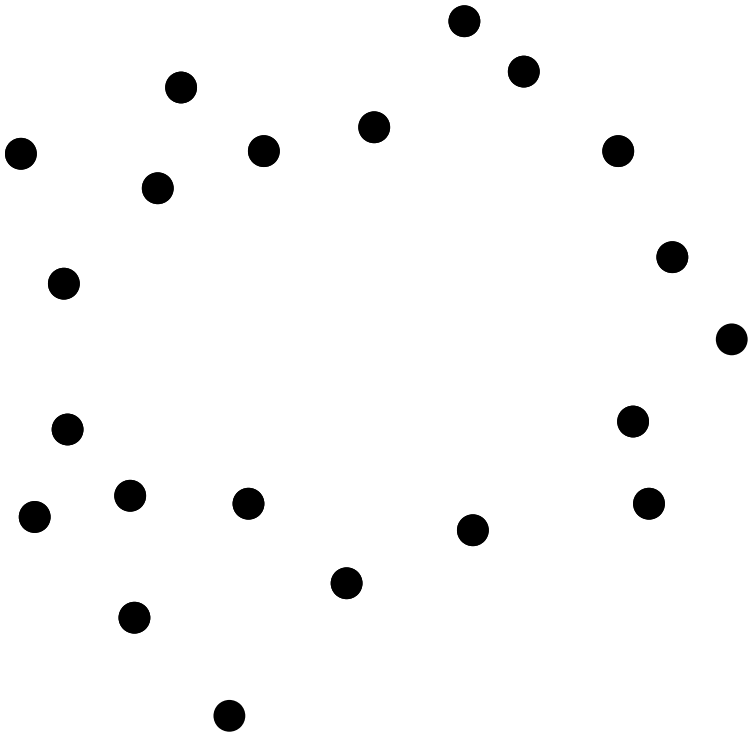
Be careful! (Same as torus over $\mathbb{Z}/2\mathbb{Z}$)

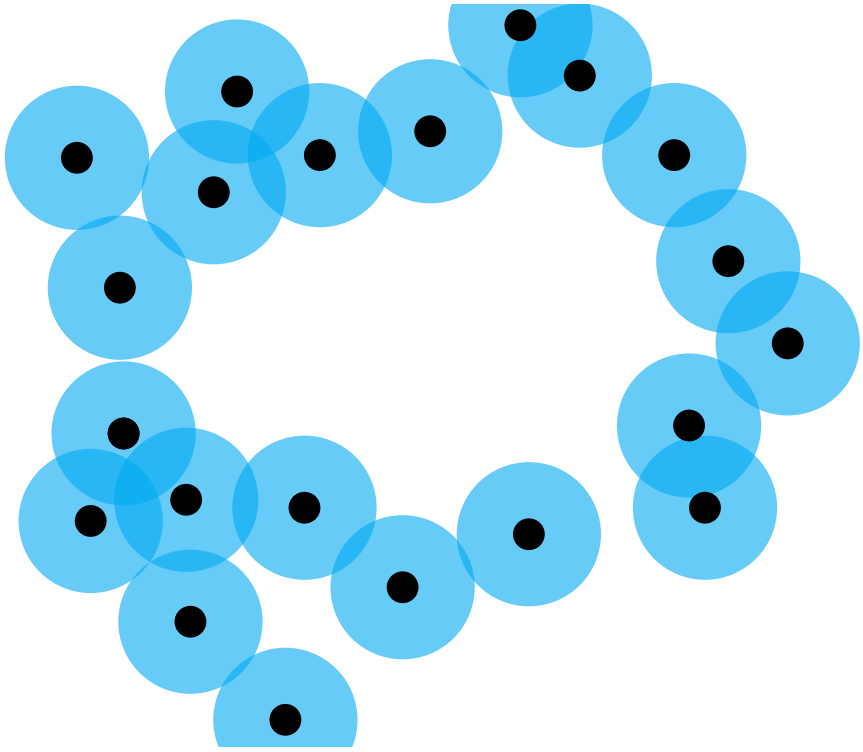
Image credit: https://plus.maths.org/content/imaging-maths-inside-klein-bottle

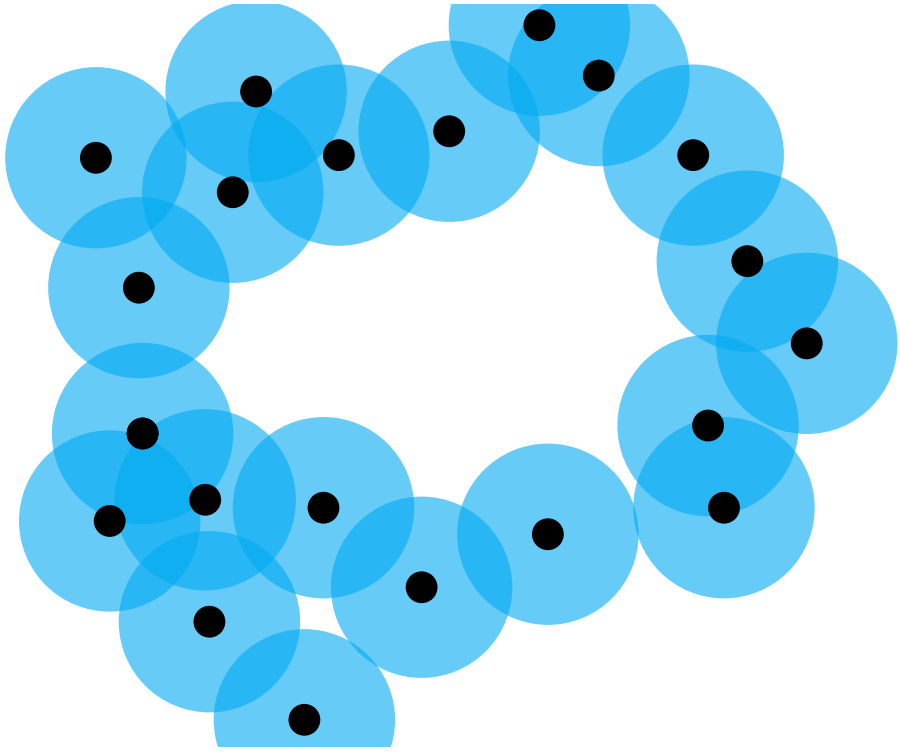
Topology studies shapes

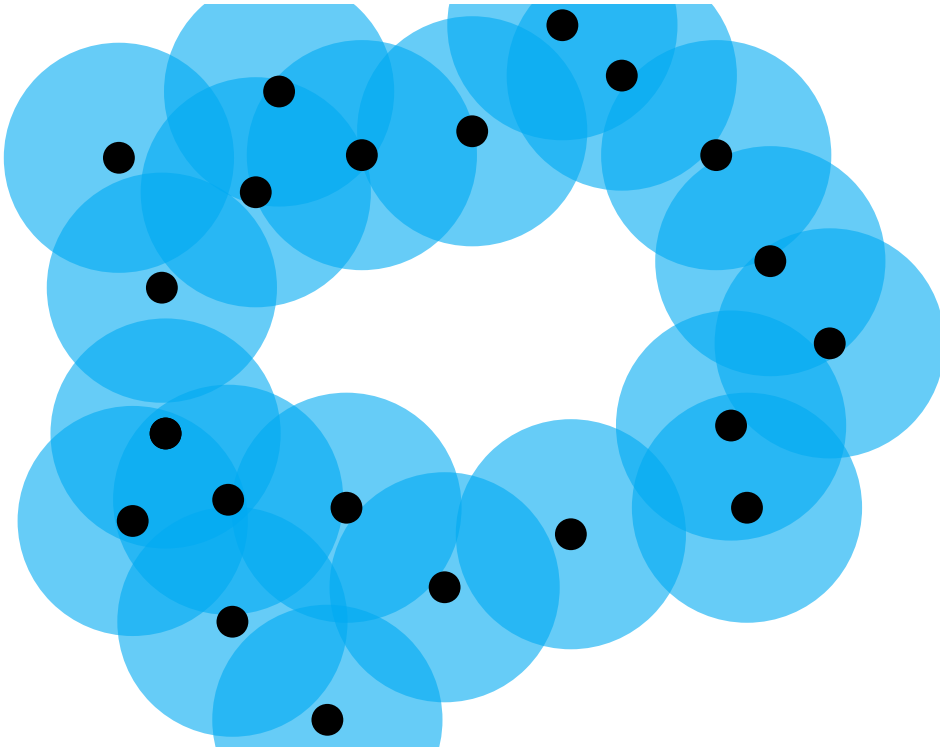
What shape is this?

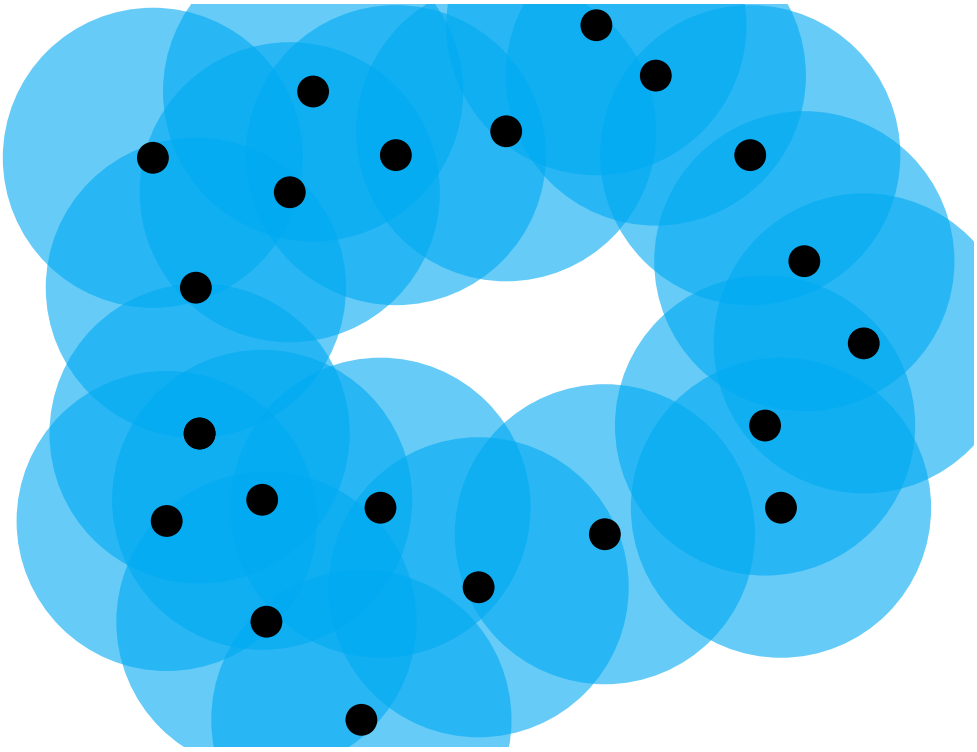


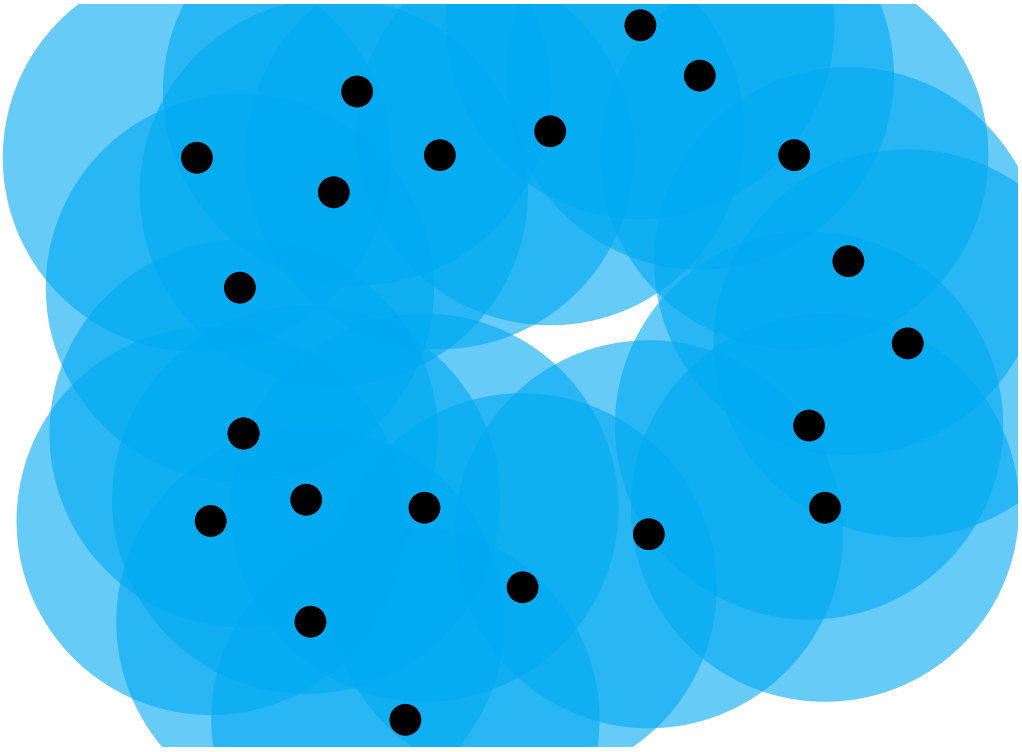


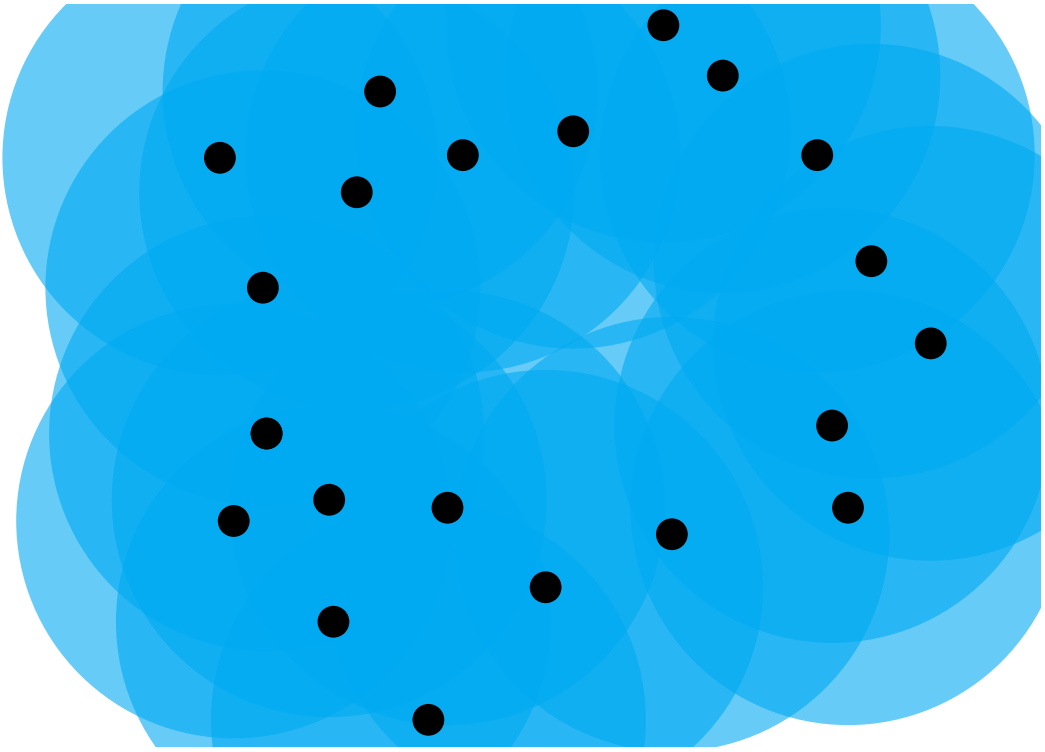


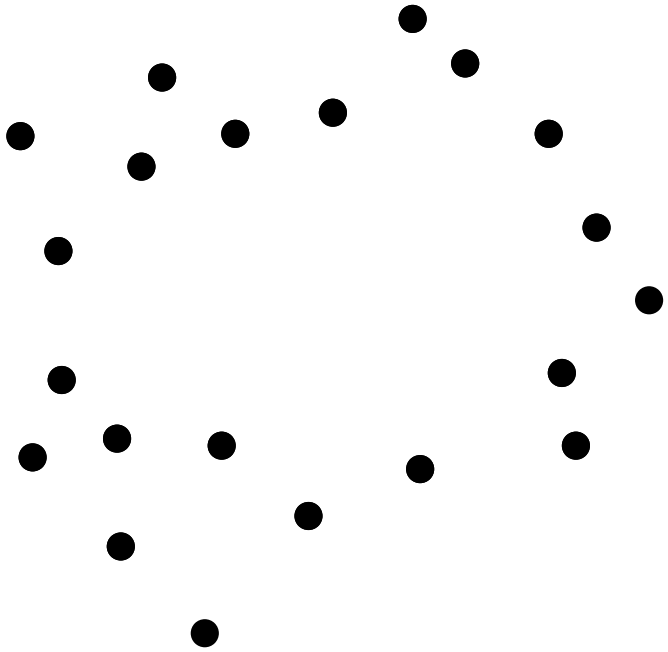








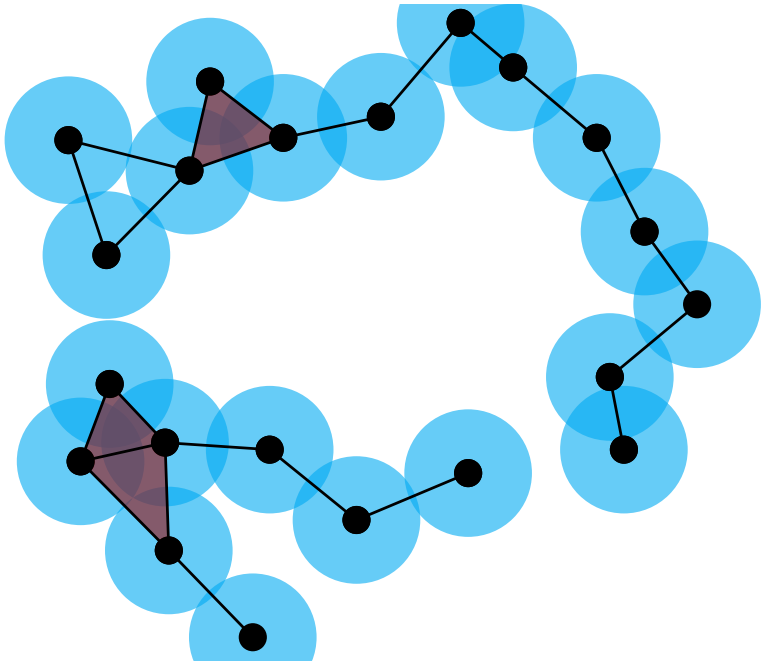




Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\text{C}}\text{ech}(X; r)$ has

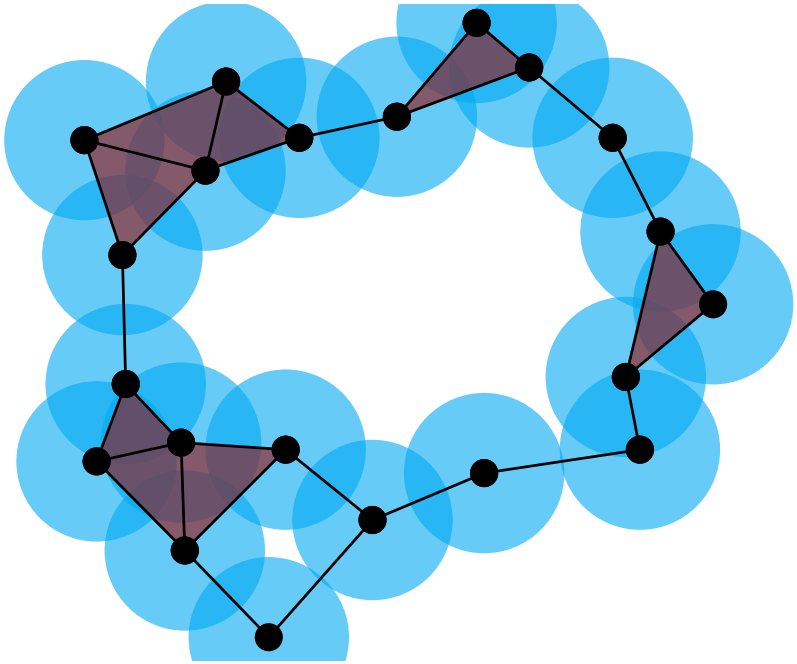
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\text{Čech}(X; r)$ has

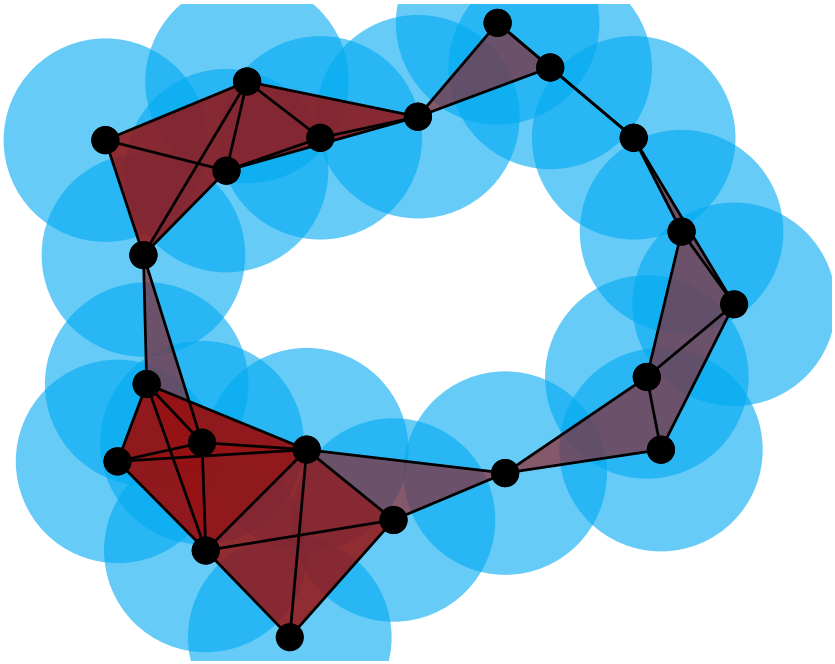
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\text{C}}\text{ech}(X; r)$ has

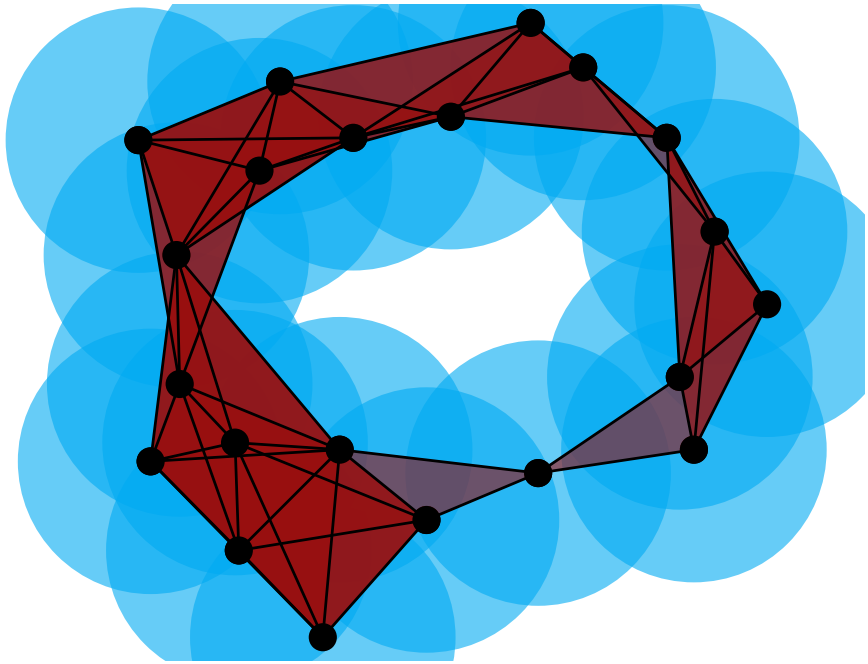
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\text{C}}\text{ech}(X; r)$ has

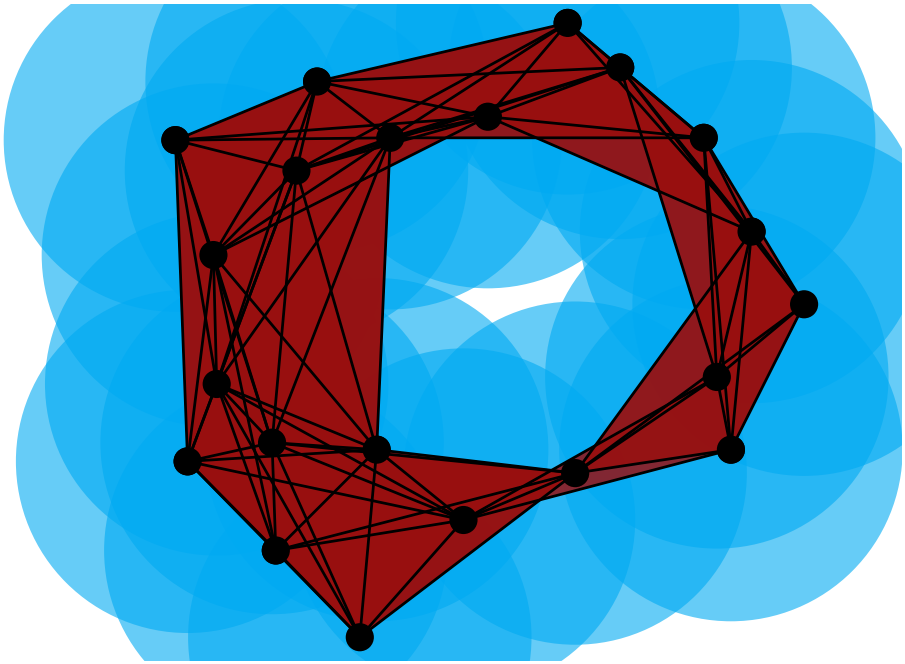
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\text{C}}\text{ech}(X; r)$ has

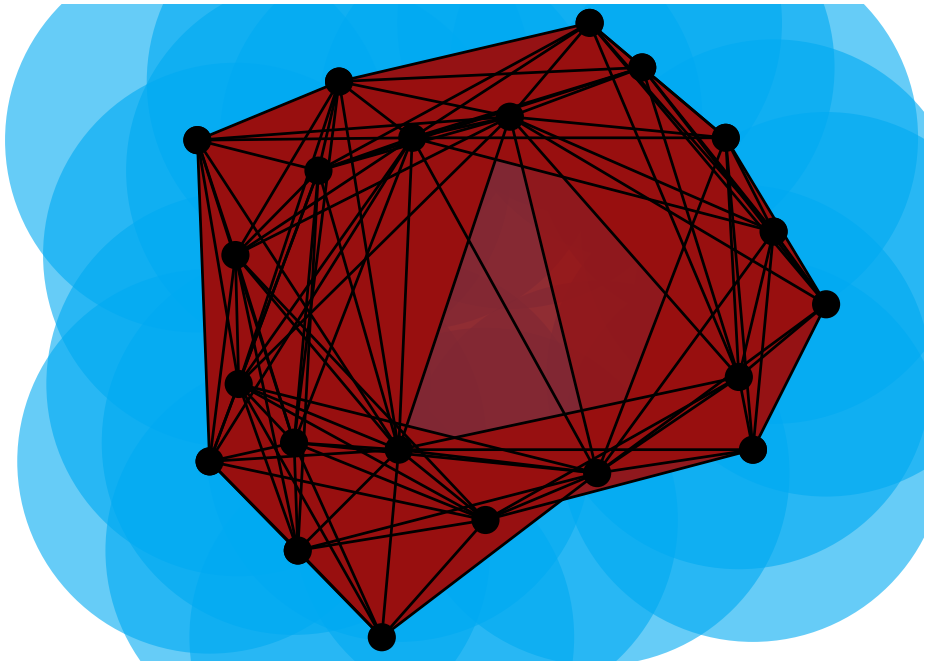
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\text{C}}\text{ech}(X; r)$ has

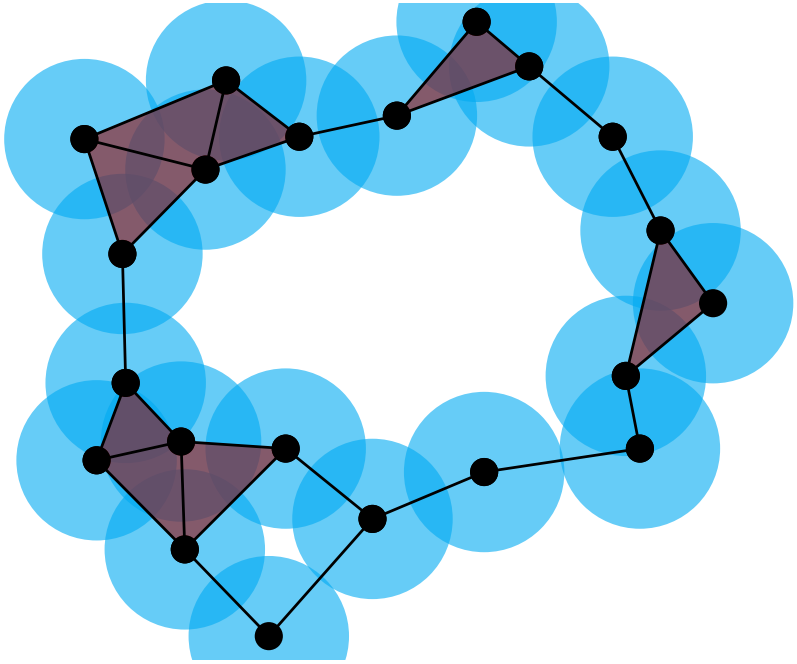
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\text{Čech}(X; r)$ has

- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.

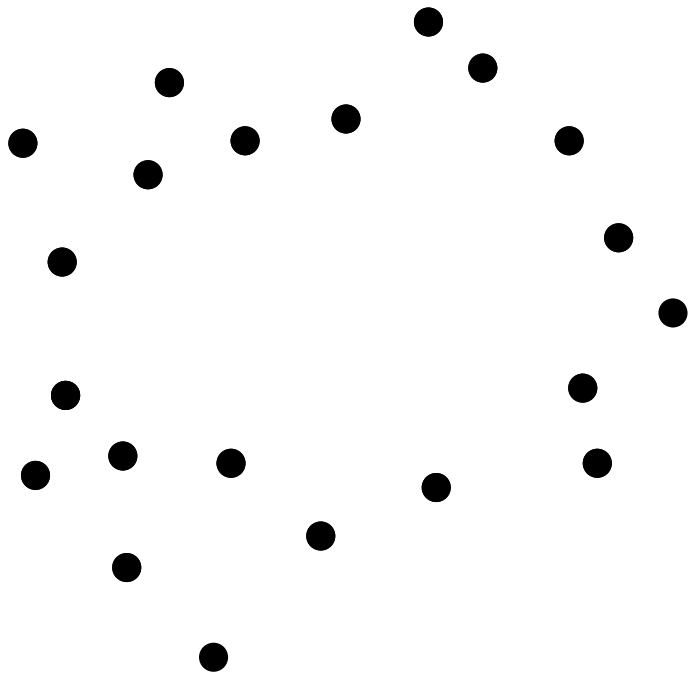


<u>Nerve Lemma.</u> $\check{\mathrm{C}}\mathrm{ech}(X;r) \simeq$ union of balls

Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\mathrm{C}}\mathrm{ech}(X;r)$ has

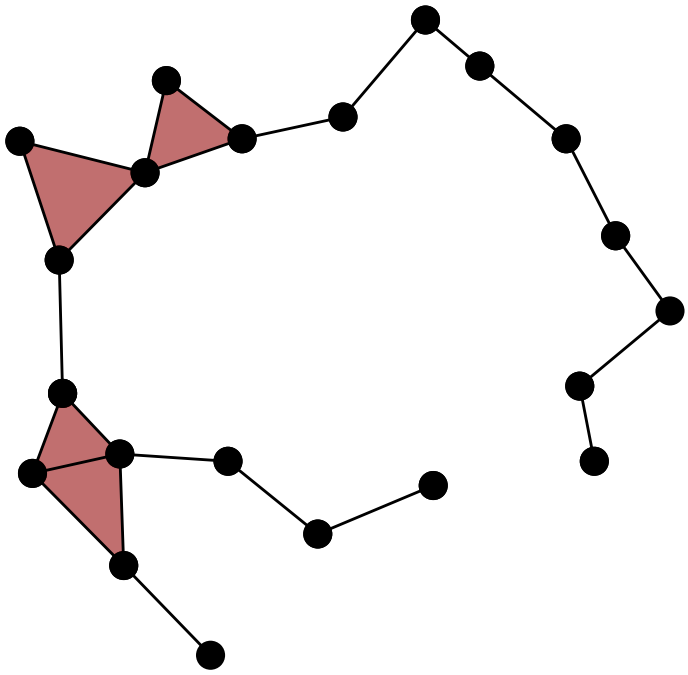
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

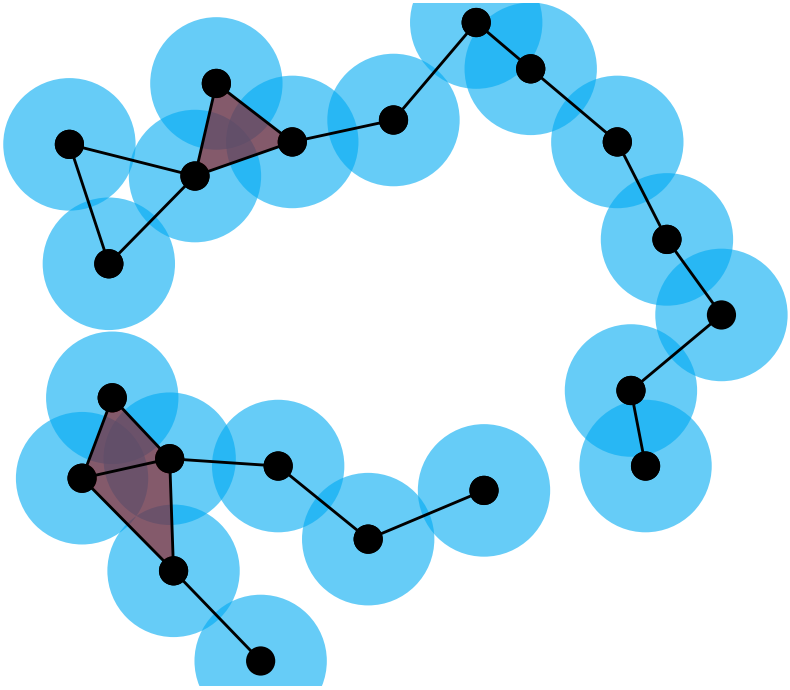
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.



Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

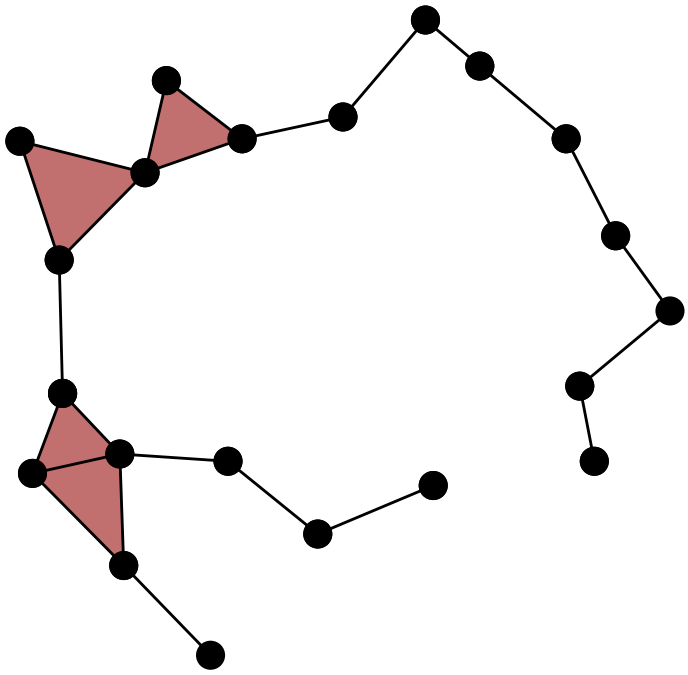
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.



Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

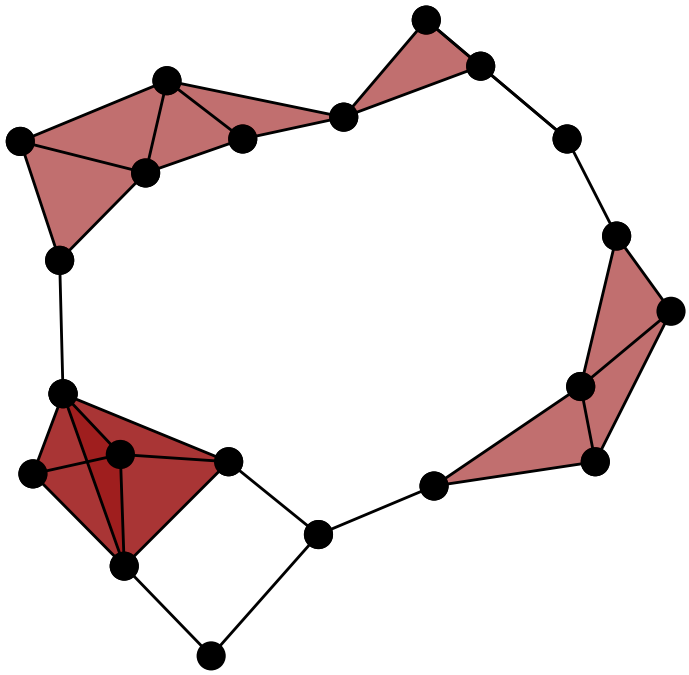
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.



Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

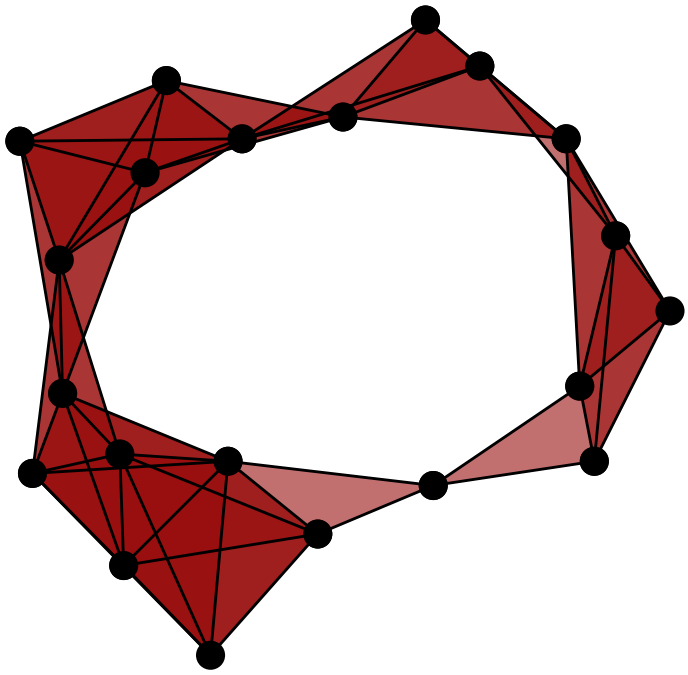
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.



Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

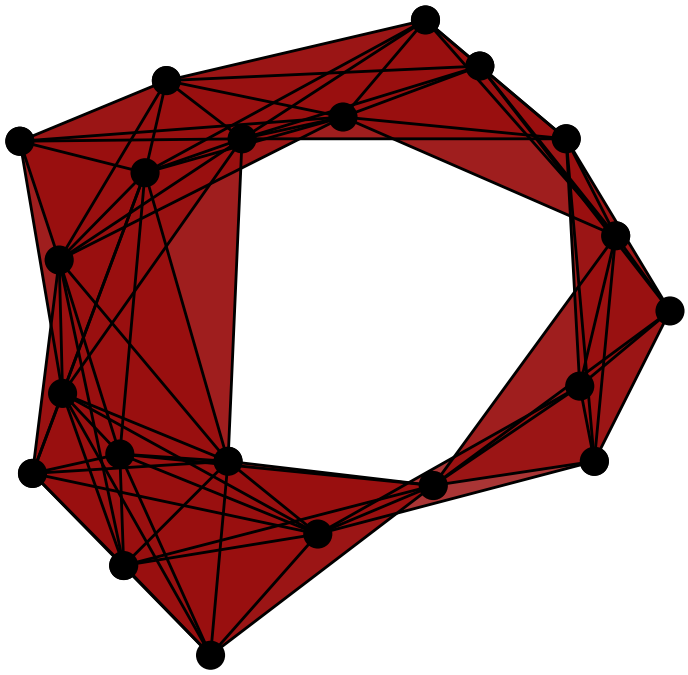
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.



Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

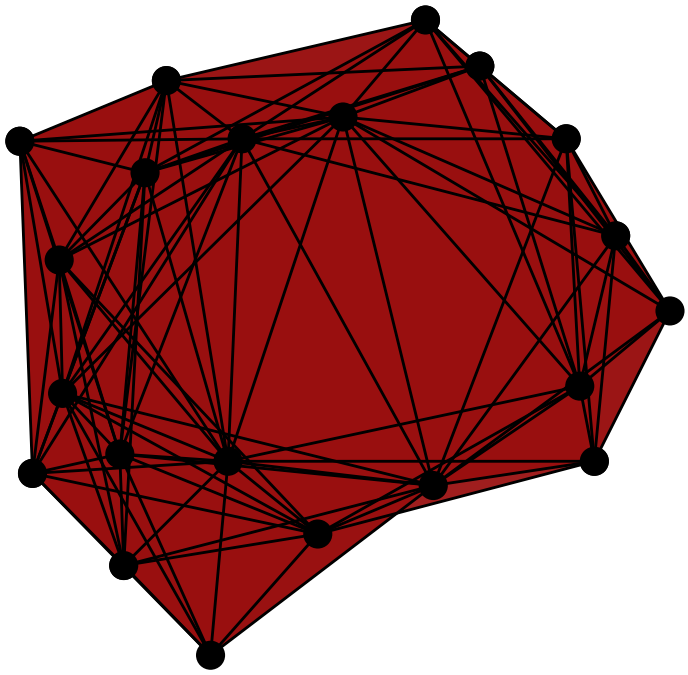
- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.



Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.

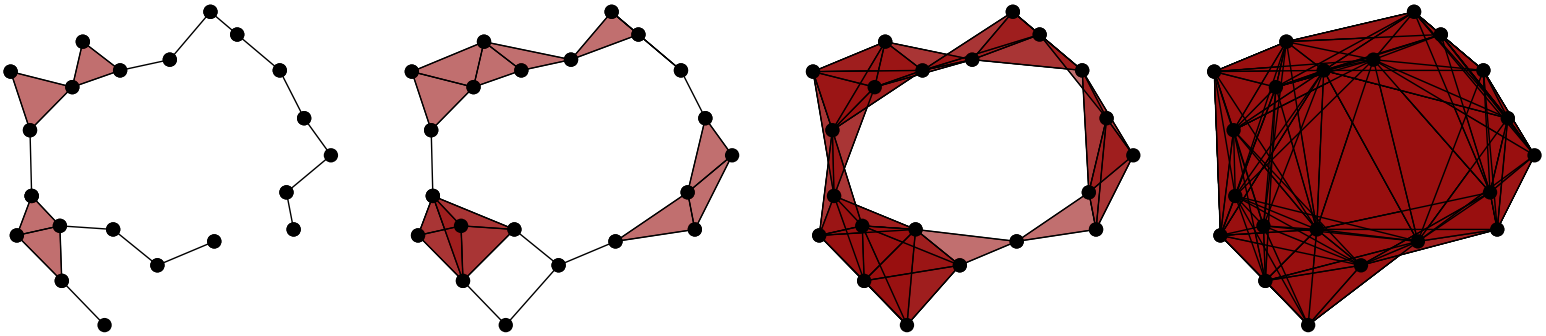


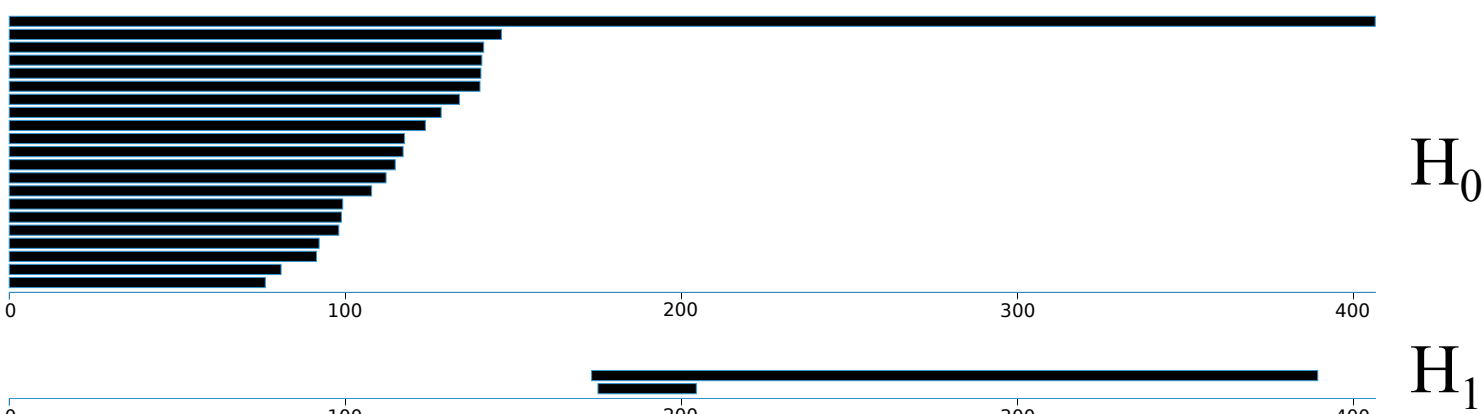
Definition

For a metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $d(x_i, x_j) \leq r$ for all i, j.

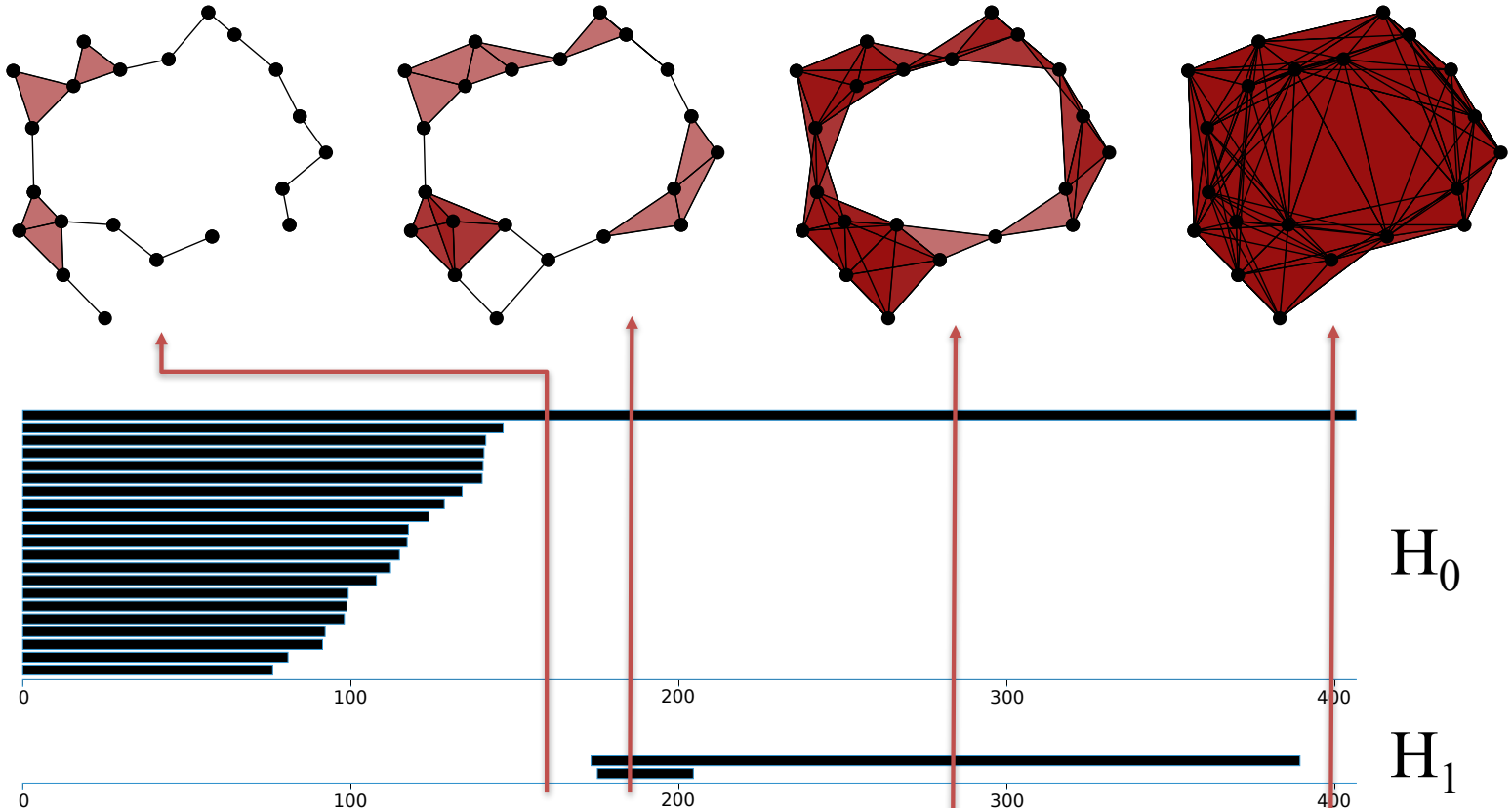
Persistent homology





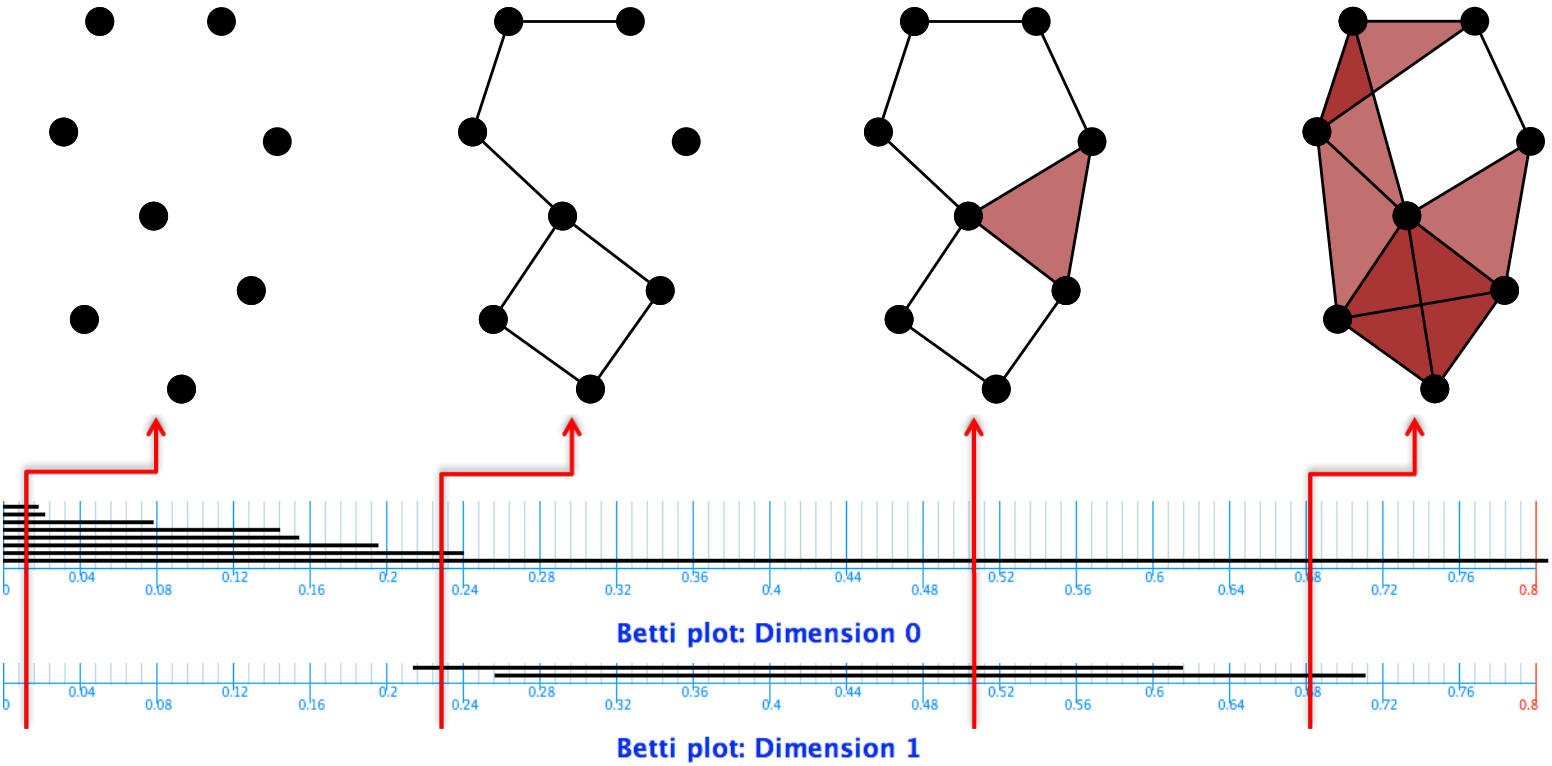
- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

Persistent homology



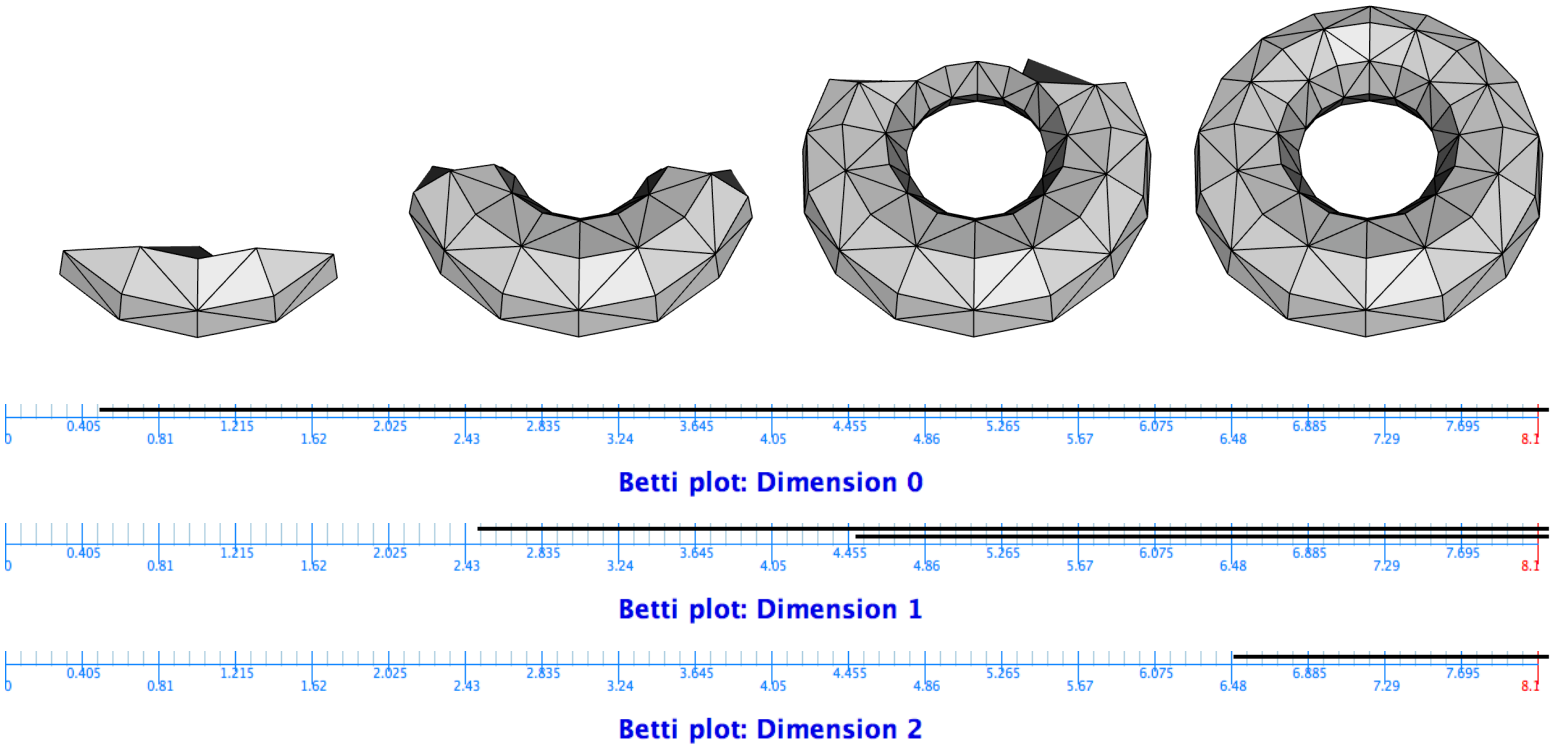
- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

Persistent homology



- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

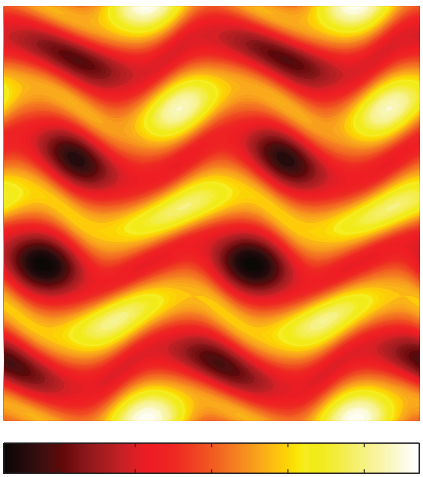
Persistent homology



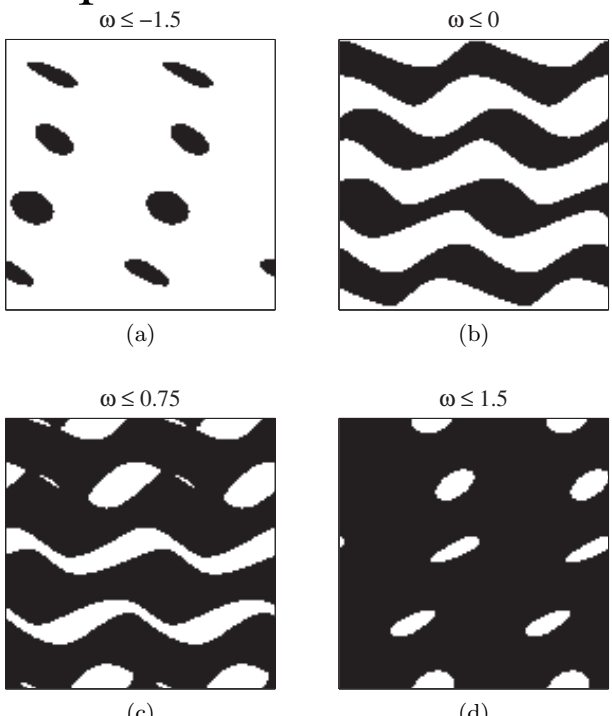
- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

Persistent homology

Sublevelset persistence



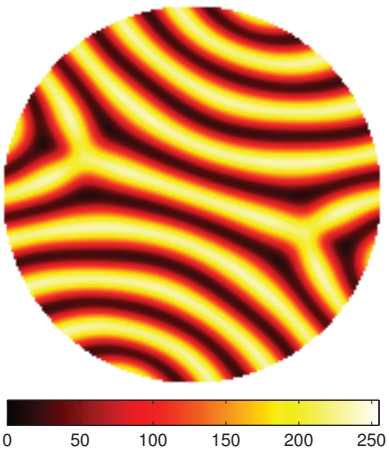
Analysis of Kolmogorov flow and Rayleigh–Bénard convection using persistent homology by Miroslav Kramár, Rachel Levanger, Jeffrey Tithof, Balachandra Suri, Mu Xu, Mark Paul, Michael F Schatz, Konstantin Mischaikow



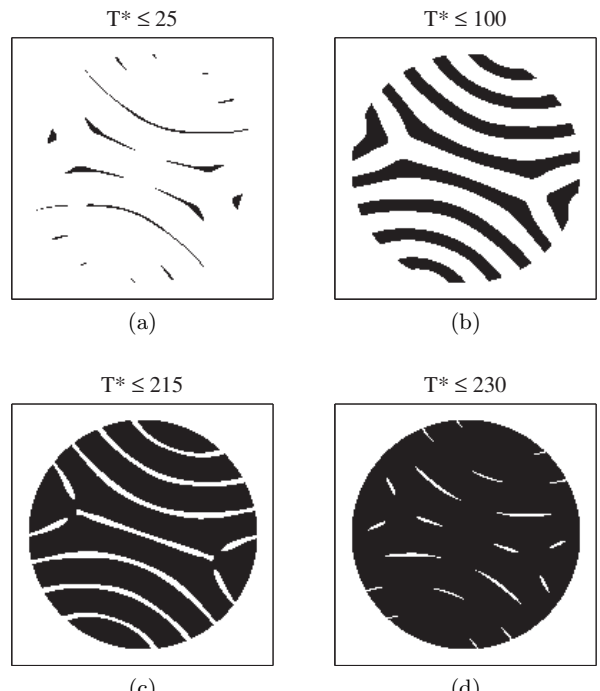
- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

Persistent homology

Sublevelset persistence



Analysis of Kolmogorov flow and Rayleigh–Bénard convection using persistent homology by Miroslav Kramár, Rachel Levanger, Jeffrey Tithof, Balachandra Suri, Mu Xu, Mark Paul, Michael F Schatz, Konstantin Mischaikow



- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

Zigzag persistence

- <u>Definition.</u> A zigzag module over zigzag diagram

$$\bullet \leftrightarrow \bullet \leftrightarrow \ldots \leftrightarrow \bullet \leftrightarrow \bullet$$

has a vector space at each vertex and a linear map at each edge.

- <u>Example.</u>

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$V_1 \rightarrow V_2 \leftarrow V_3 \rightarrow V_4$$

Zigzag Persistence by G. Carlsson and V. de Silva

Zigzag persistence

- <u>Definition.</u> A zigzag module over zigzag diagram

$$\bullet \leftrightarrow \bullet \leftrightarrow \ldots \leftrightarrow \bullet \leftrightarrow \bullet$$

has a vector space at each vertex and a linear map at each edge.

- <u>Example.</u>

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$\begin{array}{c} V_1 \longrightarrow V_2 \longleftarrow V_3 \longrightarrow V_4 \\ \oplus \\ U_1 \longrightarrow U_2 \longleftarrow U_3 \longrightarrow U_4 \end{array}$$

Zigzag Persistence by G. Carlsson and V. de Silva

Zigzag persistence

- <u>Definition.</u> A zigzag module over zigzag diagram

$$\bullet \leftrightarrow \bullet \leftrightarrow \ldots \leftrightarrow \bullet \leftrightarrow \bullet$$

has a vector space at each vertex and a linear map at each edge.

- <u>Example.</u>

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$V_1 \longrightarrow V_2 \longleftarrow V_3 \longrightarrow V_4$$

- <u>Theorem (Gabriel).</u> Indecomposables classified by intervals.

$$0 \leftrightarrow \ldots \leftrightarrow 0 \leftrightarrow \overline{k \overset{id}{\leftrightarrow} \ldots \overset{id}{\leftrightarrow} k} \leftrightarrow 0 \leftrightarrow \ldots \leftrightarrow 0$$

Zigzag Persistence by G. Carlsson and V. de Silva

Zigzag persistence

- <u>Definition.</u> A zigzag module over zigzag diagram

$$\bullet \leftrightarrow \bullet \leftrightarrow \ldots \leftrightarrow \bullet \leftrightarrow \bullet$$

has a vector space at each vertex and a linear map at each edge.

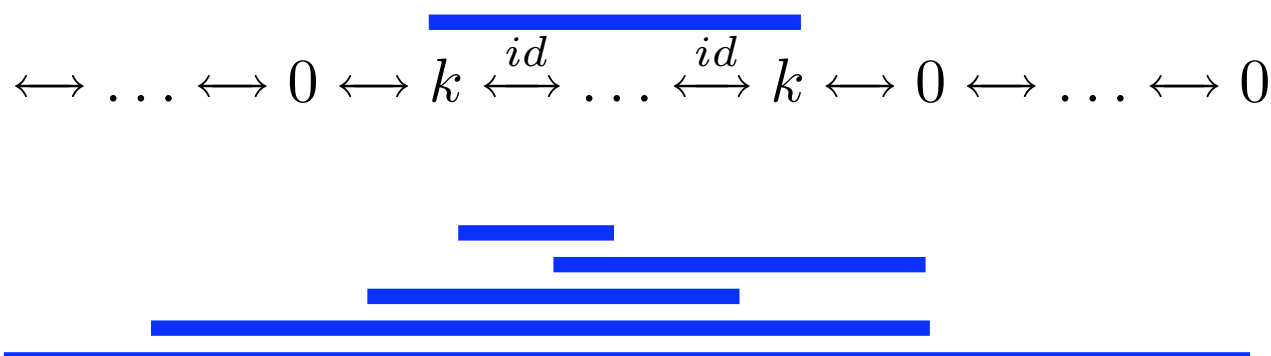
- <u>Example.</u>

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

$$V_1 \longrightarrow V_2 \longleftarrow V_3 \longrightarrow V_4$$

- <u>Theorem (Gabriel).</u> Indecomposables classified by intervals.

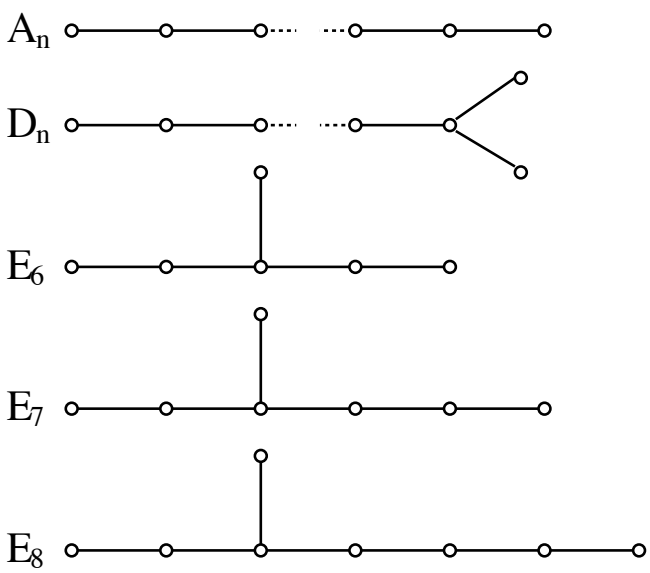
$$0 \leftrightarrow \ldots \leftrightarrow 0 \leftrightarrow k \overset{id}{\leftrightarrow} \ldots \overset{id}{\leftrightarrow} k \leftrightarrow 0 \leftrightarrow \ldots \leftrightarrow 0$$



Zigzag Persistence by G. Carlsson and V. de Silva

Zigzag persistence

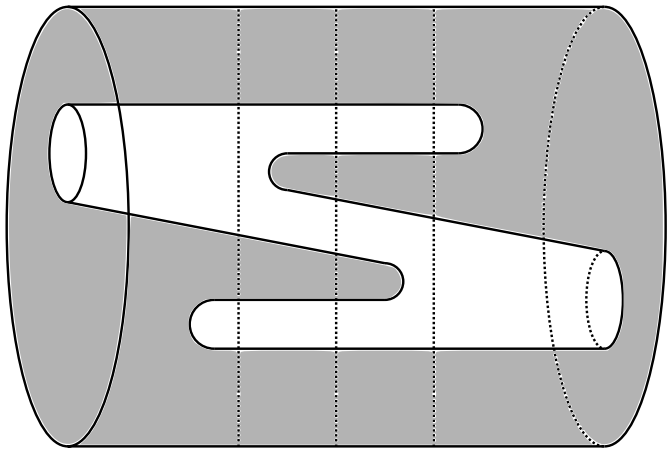
- <u>Theorem (Gabriel).</u>
  Diagram D has a finite number of indecomposables $\Leftrightarrow$ it's a union of certain Dynkin diagrams.



Zigzag Persistence by G. Carlsson and V. de Silva

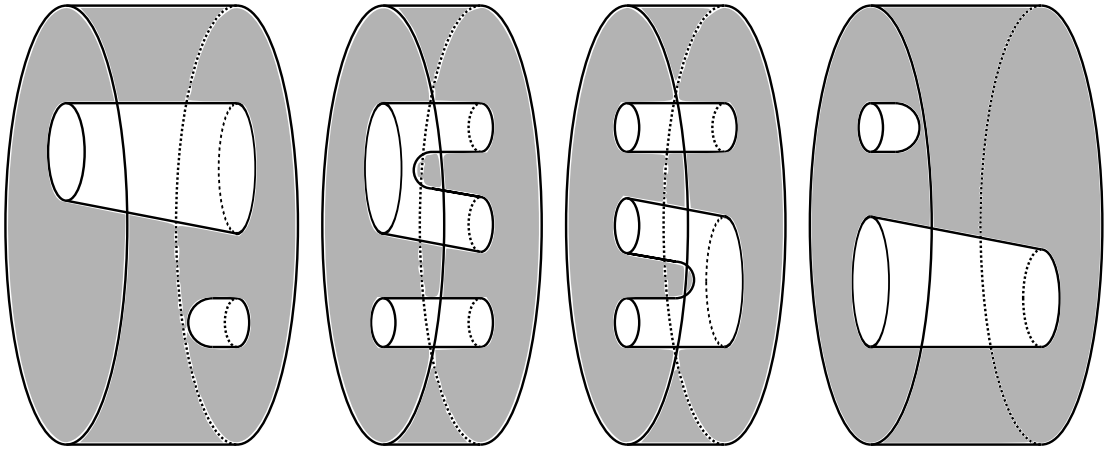
Zigzag persistence

- Form zigzag module for $X \to I$



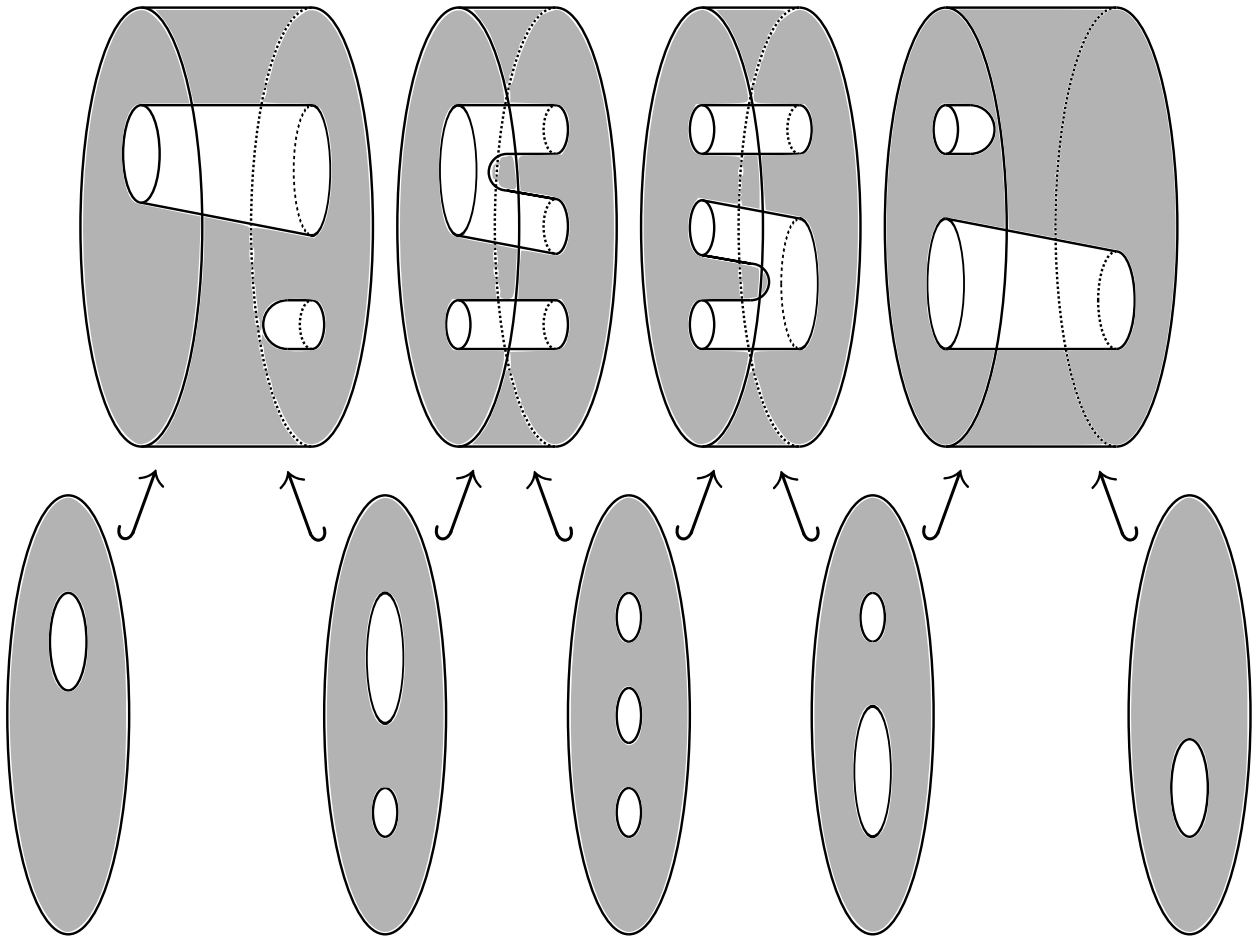
Zigzag persistence

- Form zigzag module for $X \to I$



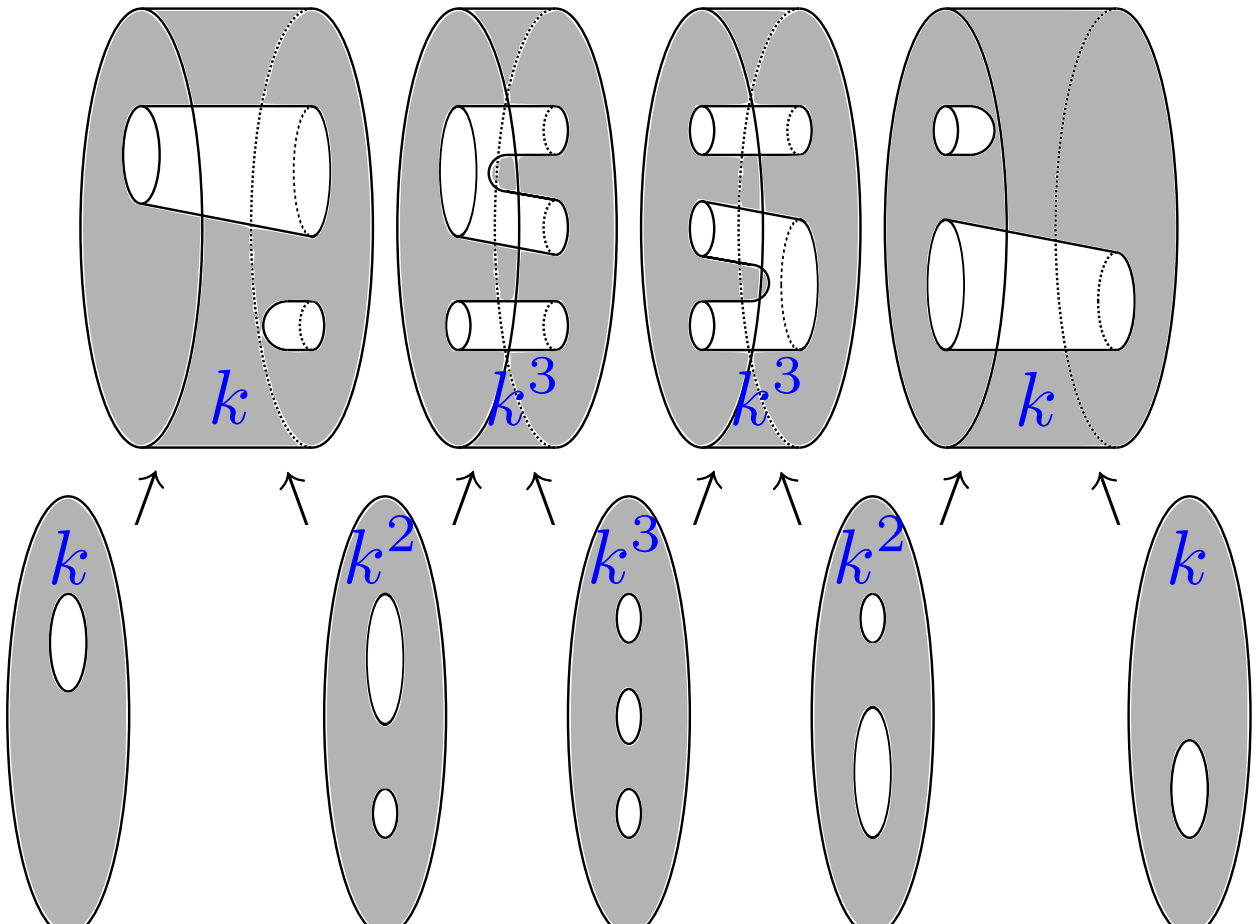
Zigzag persistence

- Form zigzag module for $X \to I$



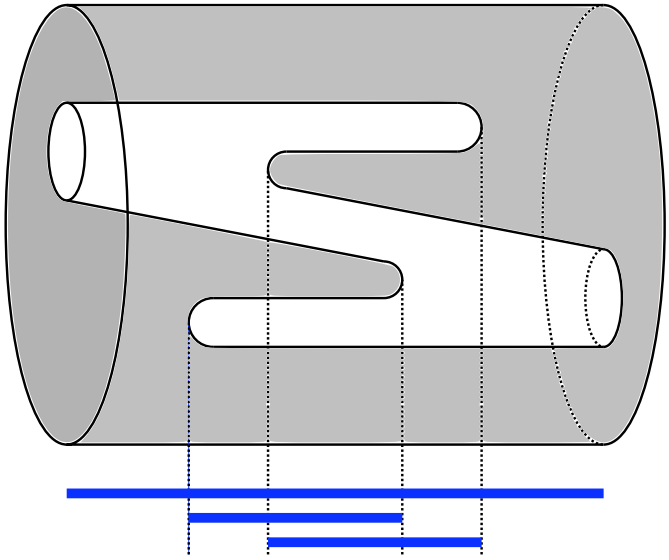
Zigzag persistence

- Form zigzag module for $X \to I$



Zigzag persistence

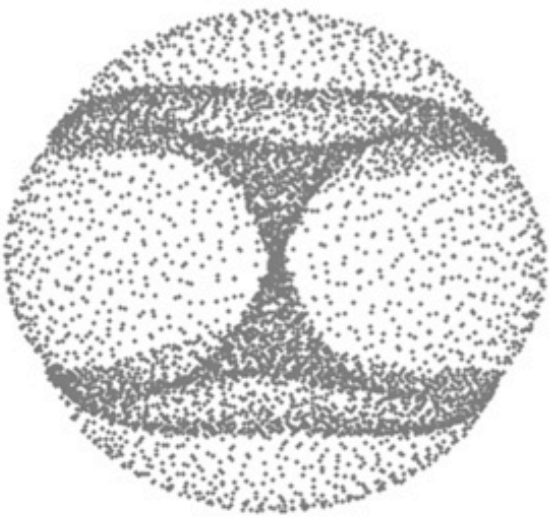
- Form zigzag module for $X \to I$

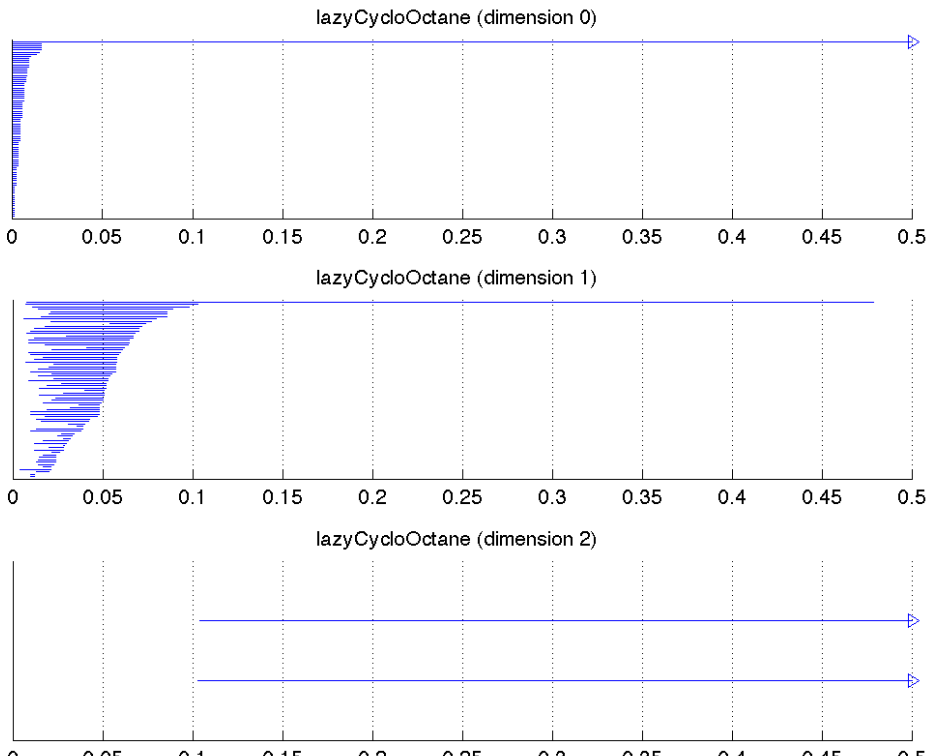


Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space



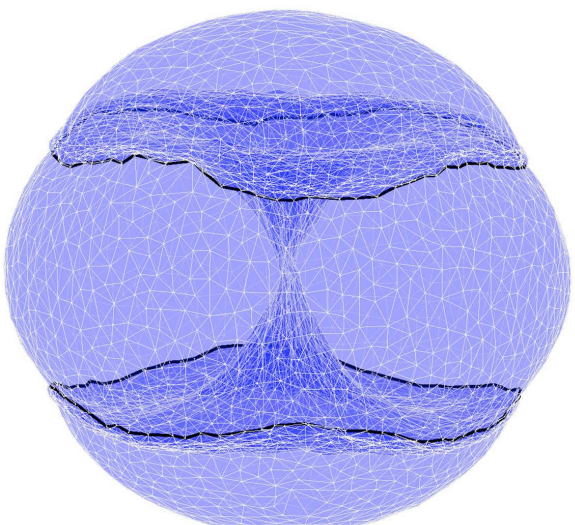


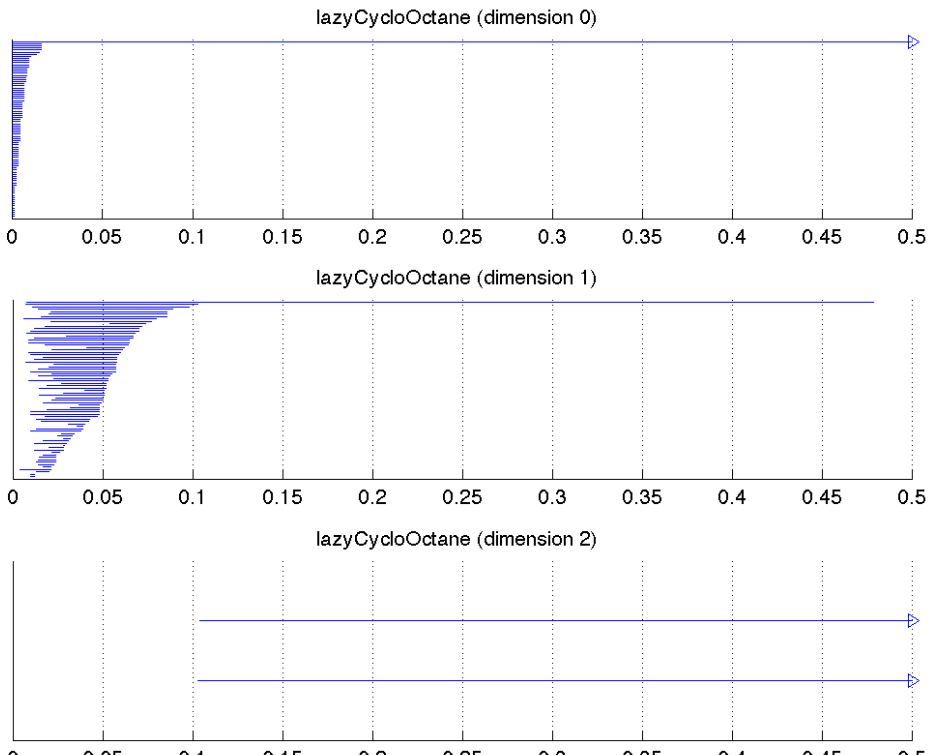
Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

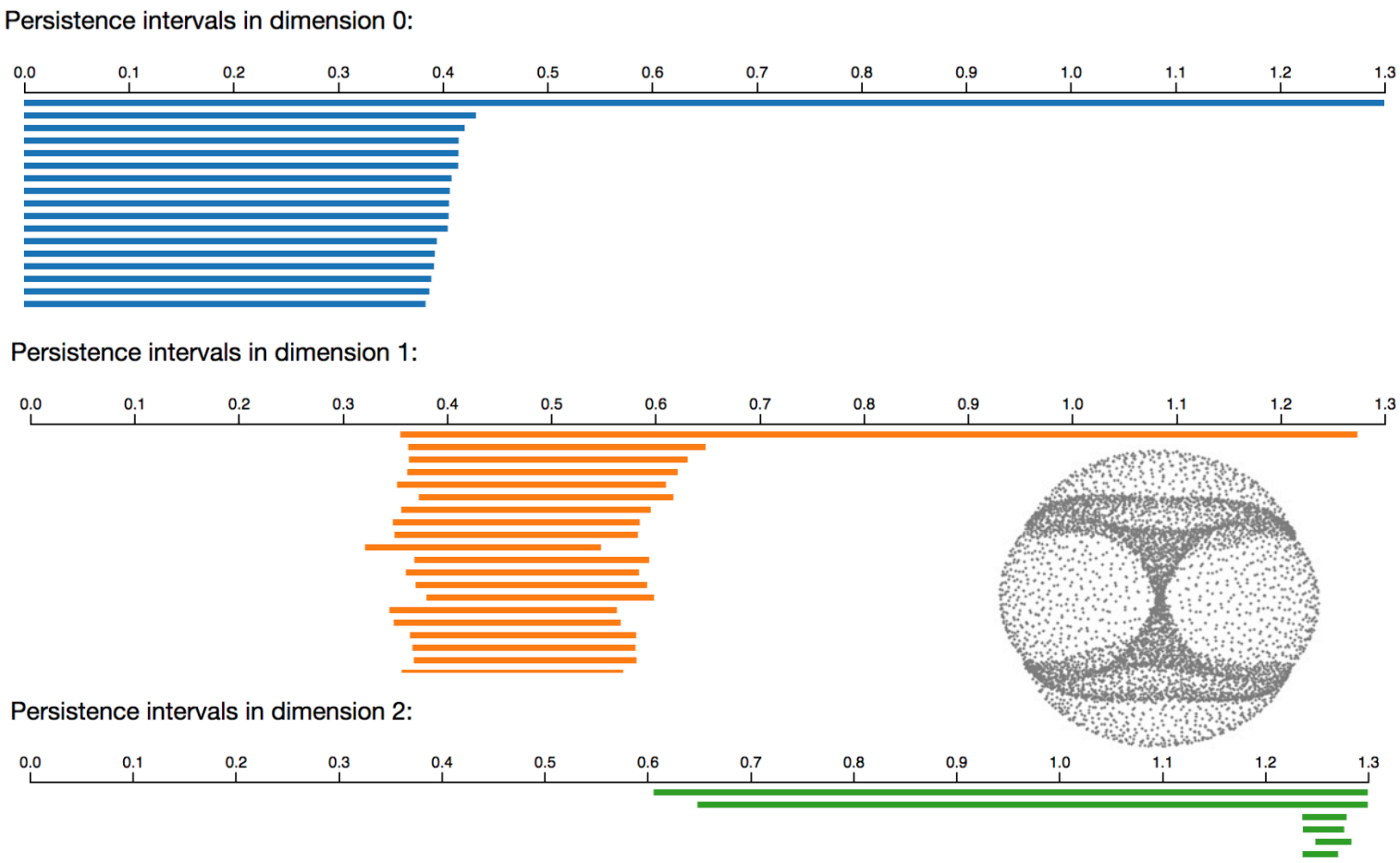
1,000,000+ points in 24-dimensional space





Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

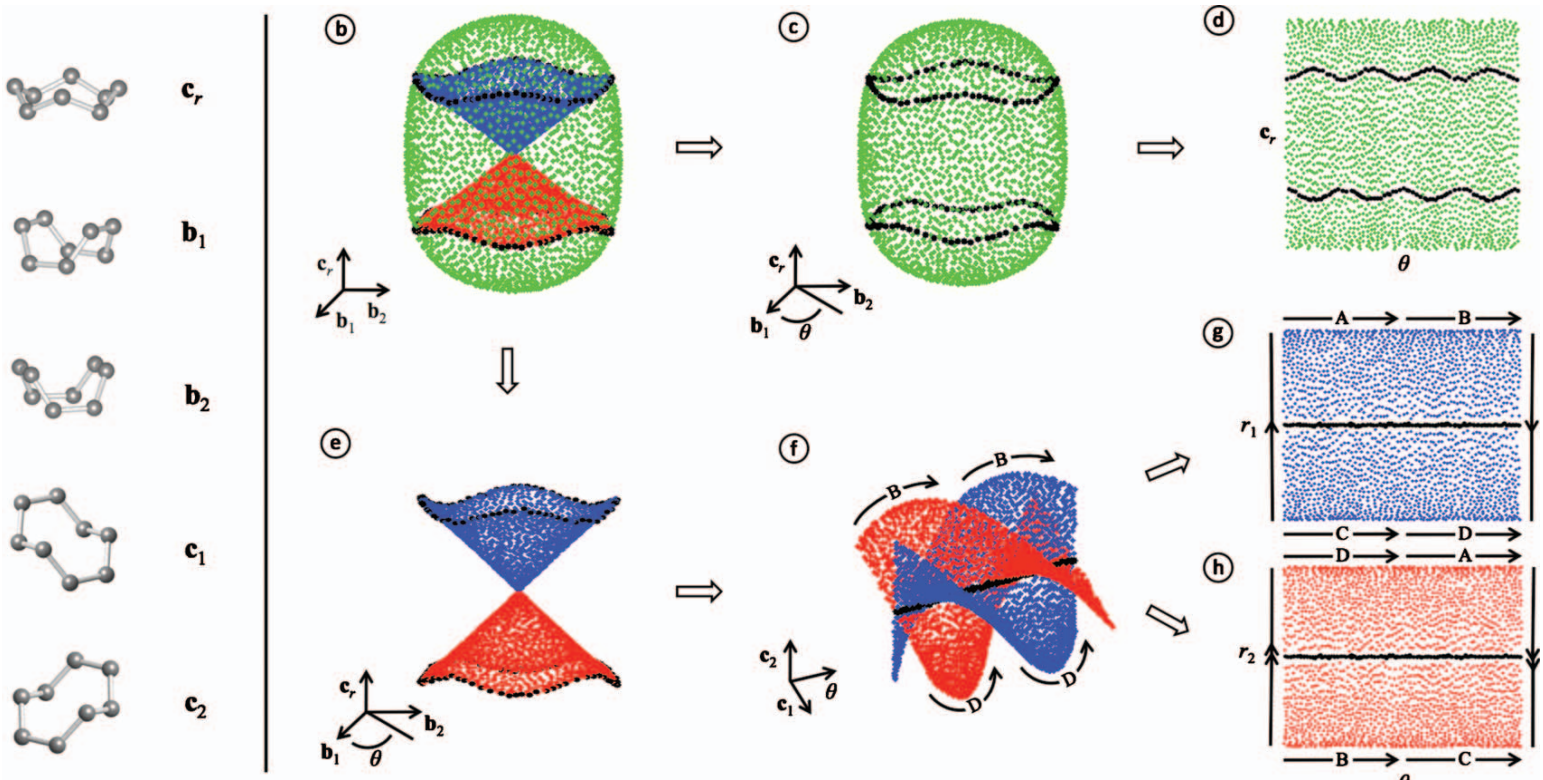


Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space

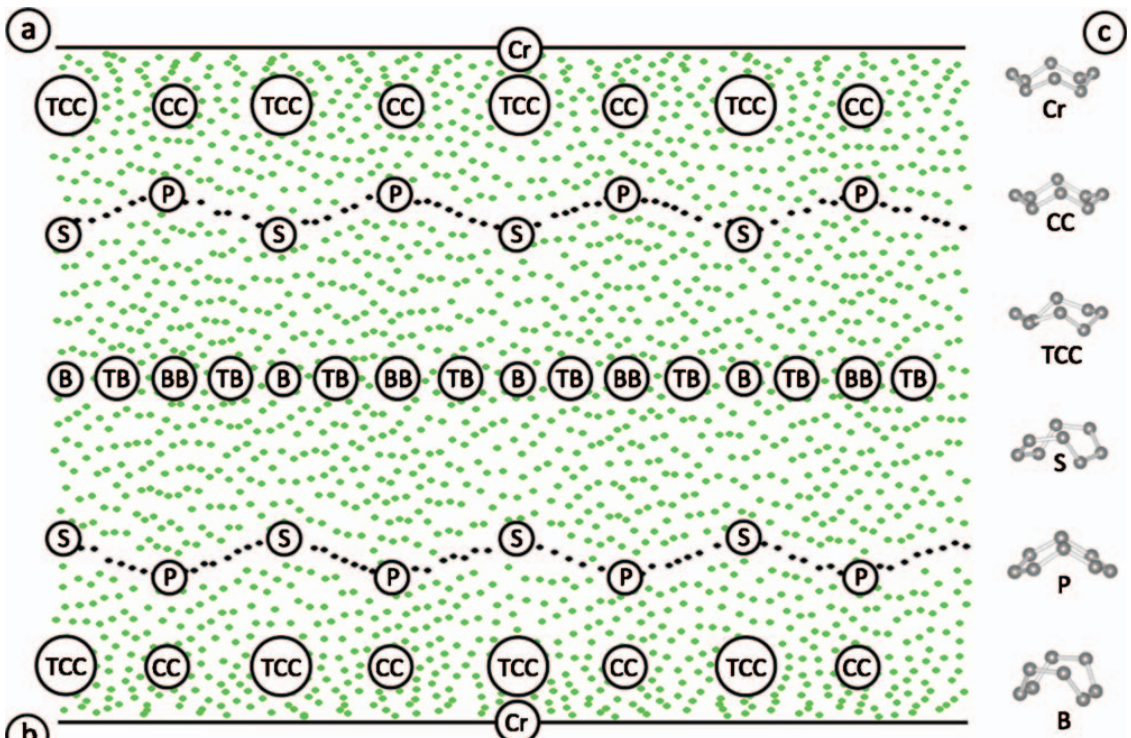


Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space

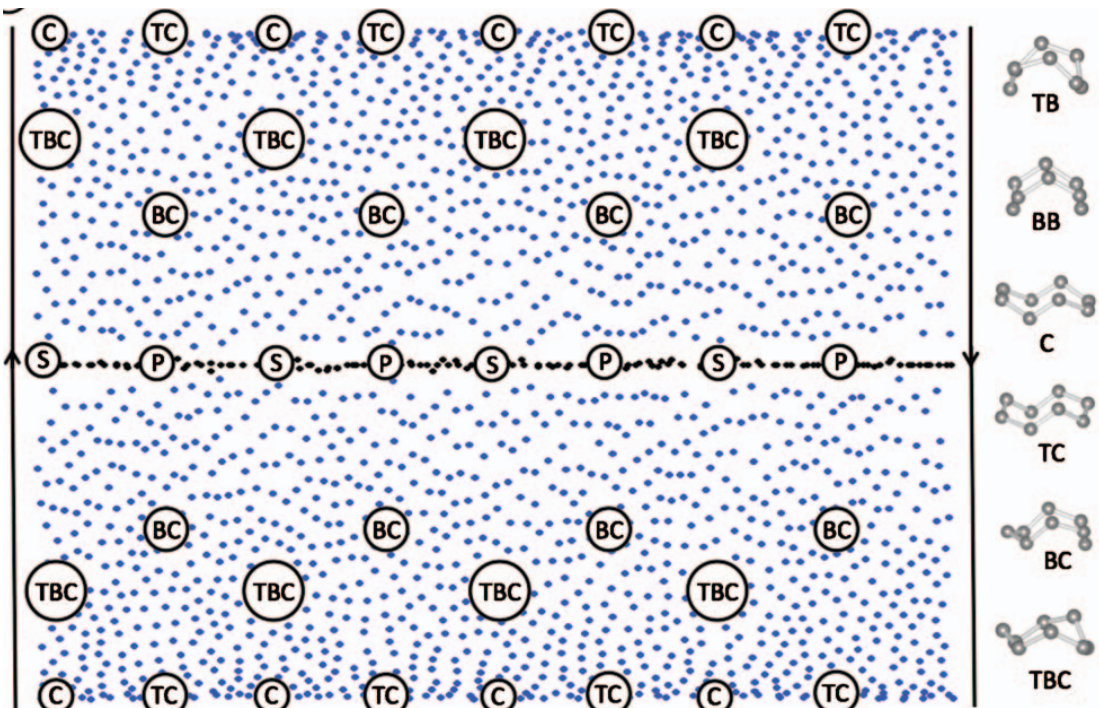


Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space

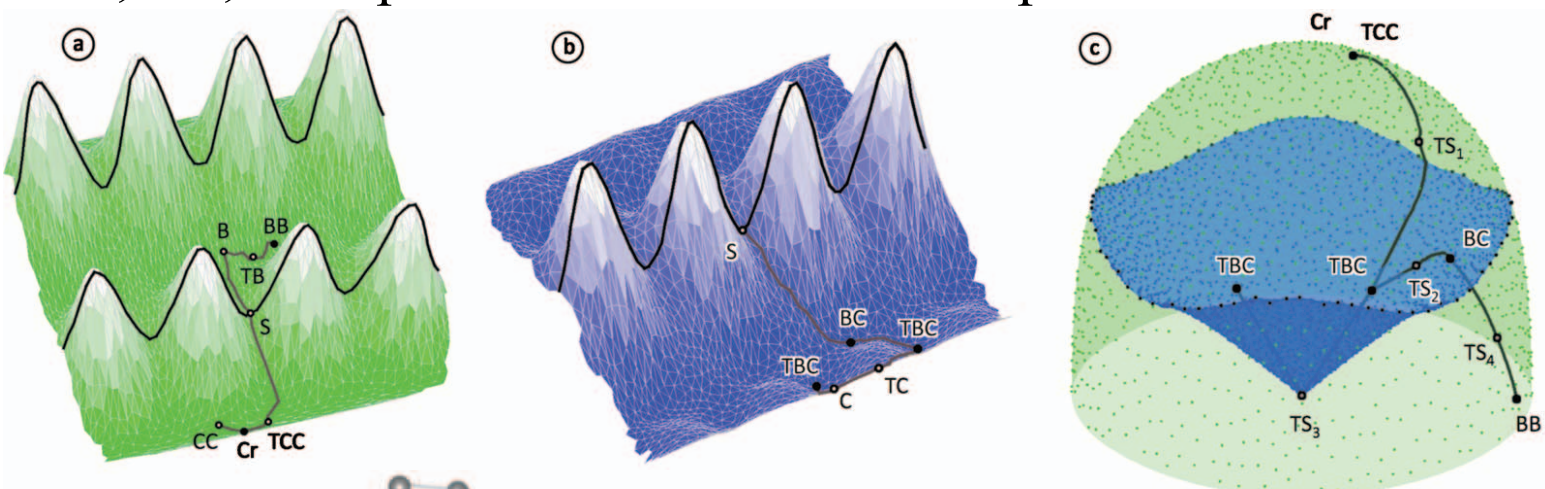


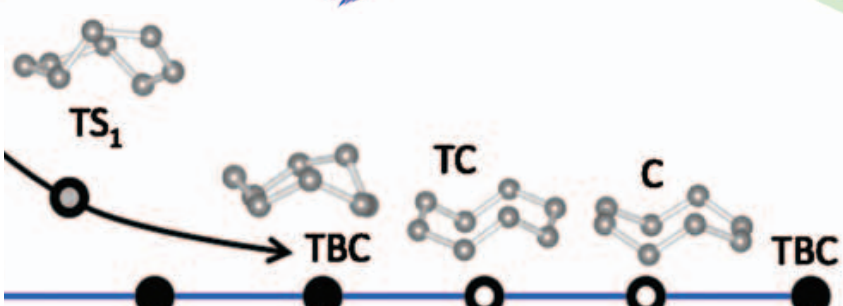
Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space

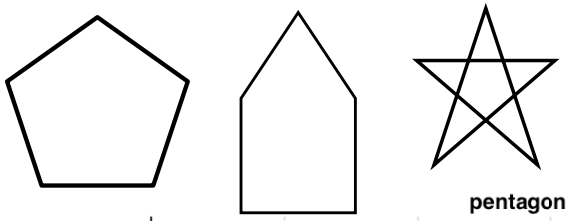




Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

Example: Equilateral pentagons in the plane



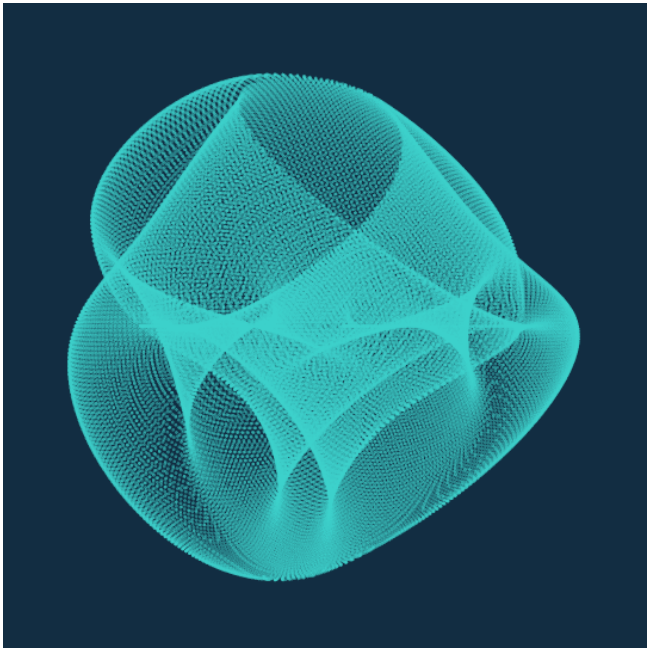
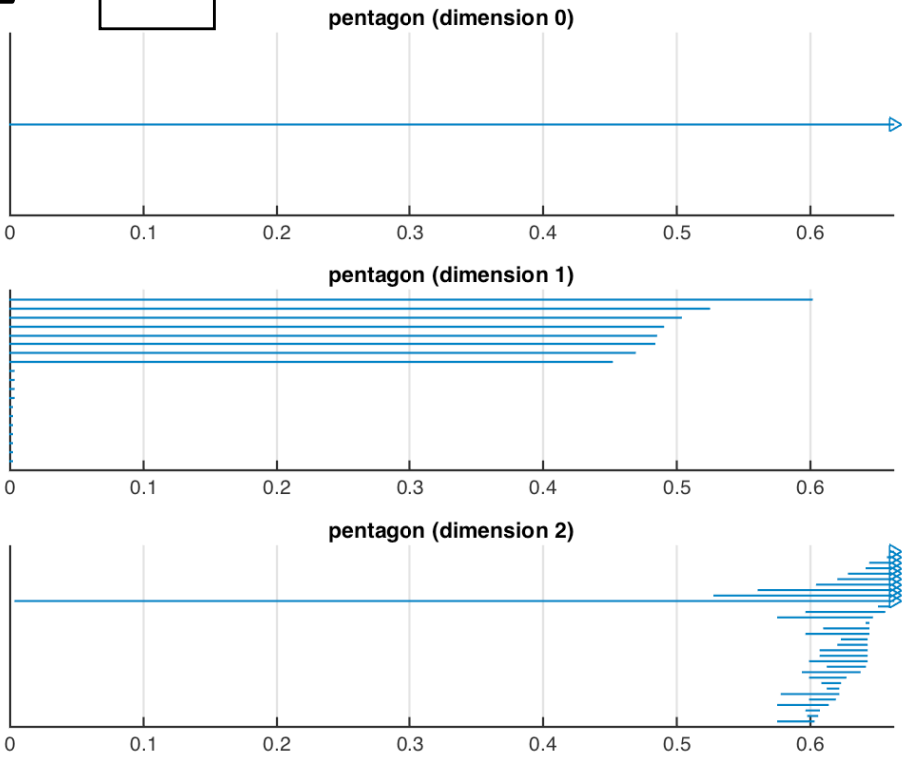
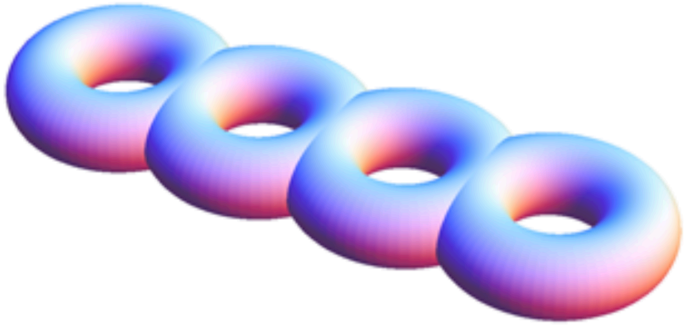


Image credit: Clayton Shonkwiler



Persistent homology applied to data



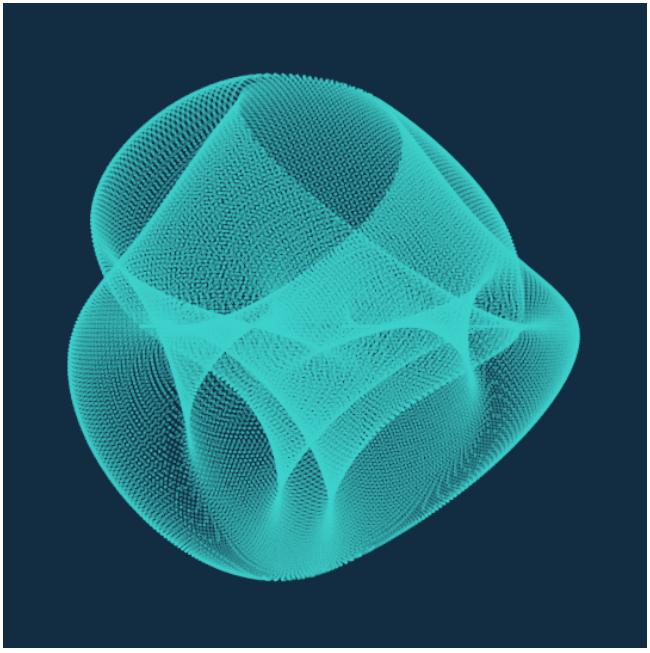
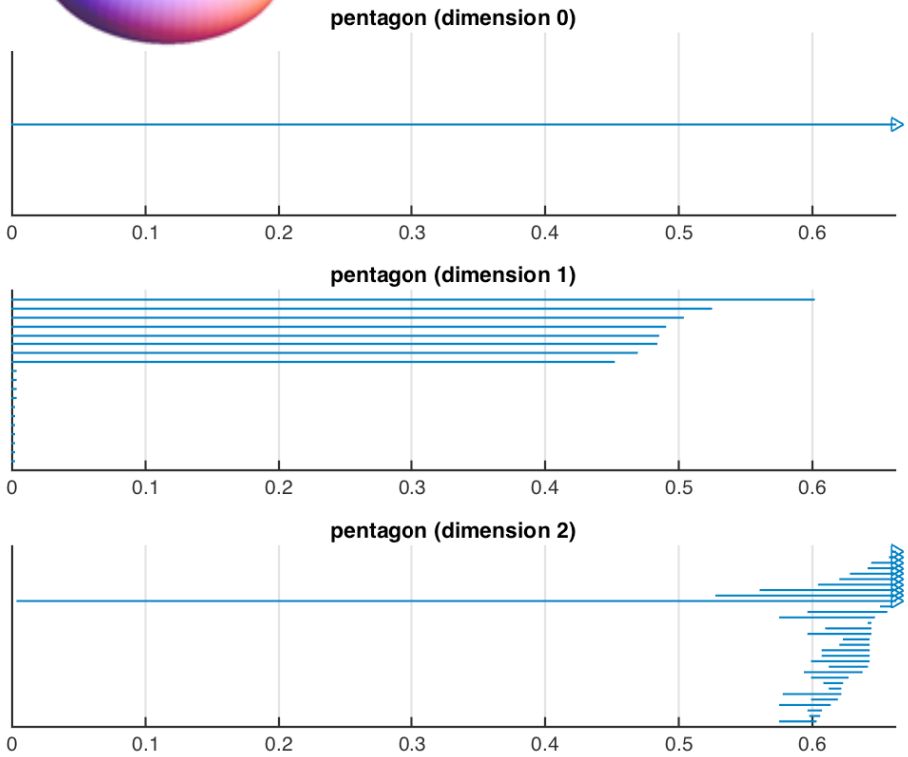


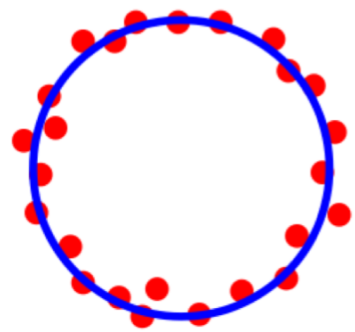
Image credit: Clayton Shonkwiler



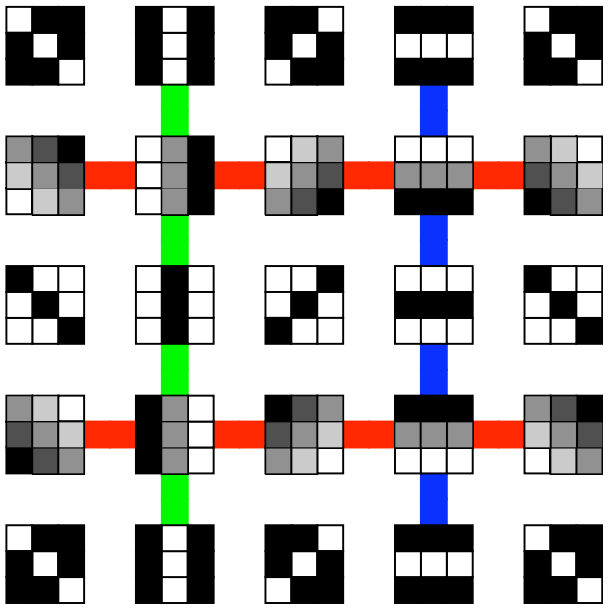
Persistent homology applied to data

- <u>Stability Theorem.</u>
  If X and Y are metric spaces, then

$$d_b(\mathrm{PH}(\check{\mathrm{C}}\mathrm{ech}(X)), \mathrm{PH}(\check{\mathrm{C}}\mathrm{ech}(Y))) \leq 2d_{\mathrm{GH}}(X, Y)$$



Topology applied to image data



Persistent homology applied to data

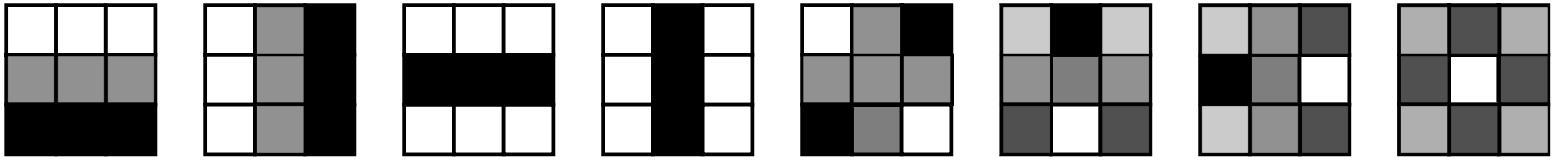
The receptive fields of cells in our primary visual cortex (V1) are related to the statistics natural images.

Independent component filters of natural images compared with simple cells in primary visual cortex by JH van Hateren and A van der Schaaf, 1997

Persistent homology applied to data

3x3 *high-contrast* patches from images

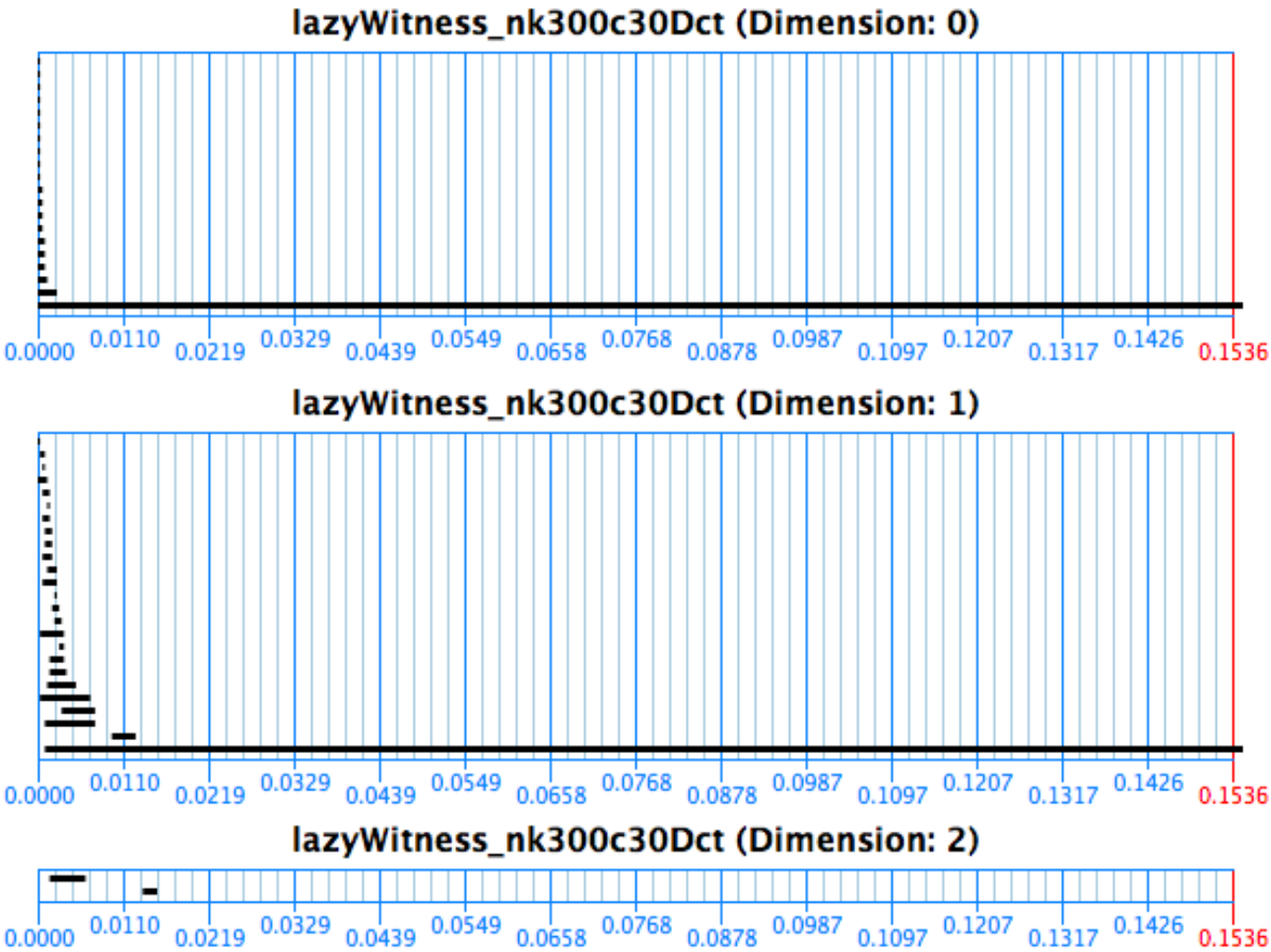
Points in 9-dimensional space, normalized to have average color gray and contrast norm one (on 7-sphere).



On the Local Behavior of Spaces of Natural Images by Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian, 2008.

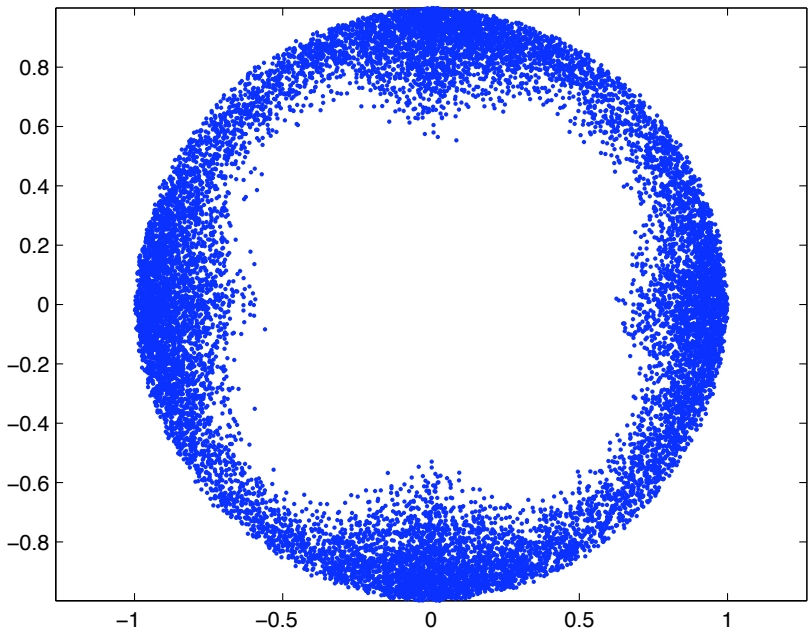
Persistent homology applied to data

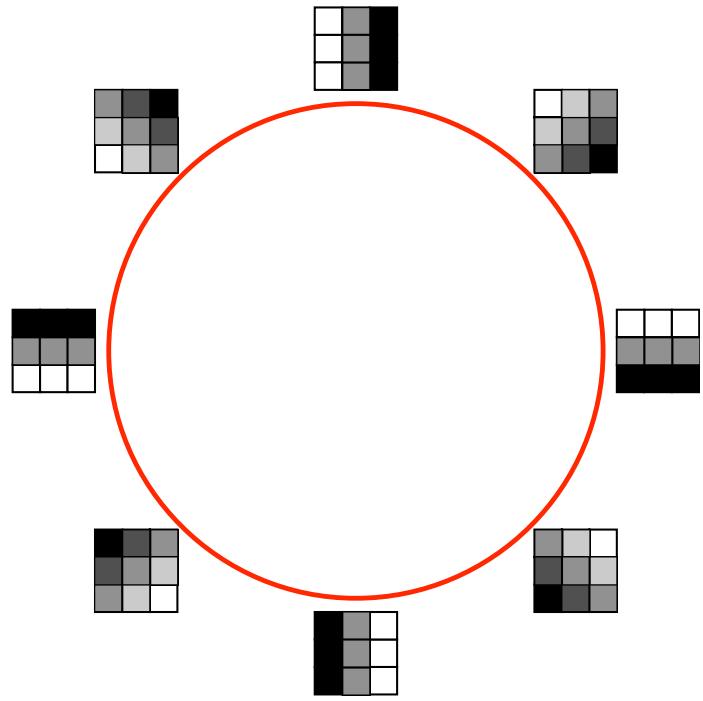
1. Densest patches according to a global estimate



Persistent homology applied to data

1. Densest patches according to a global estimate

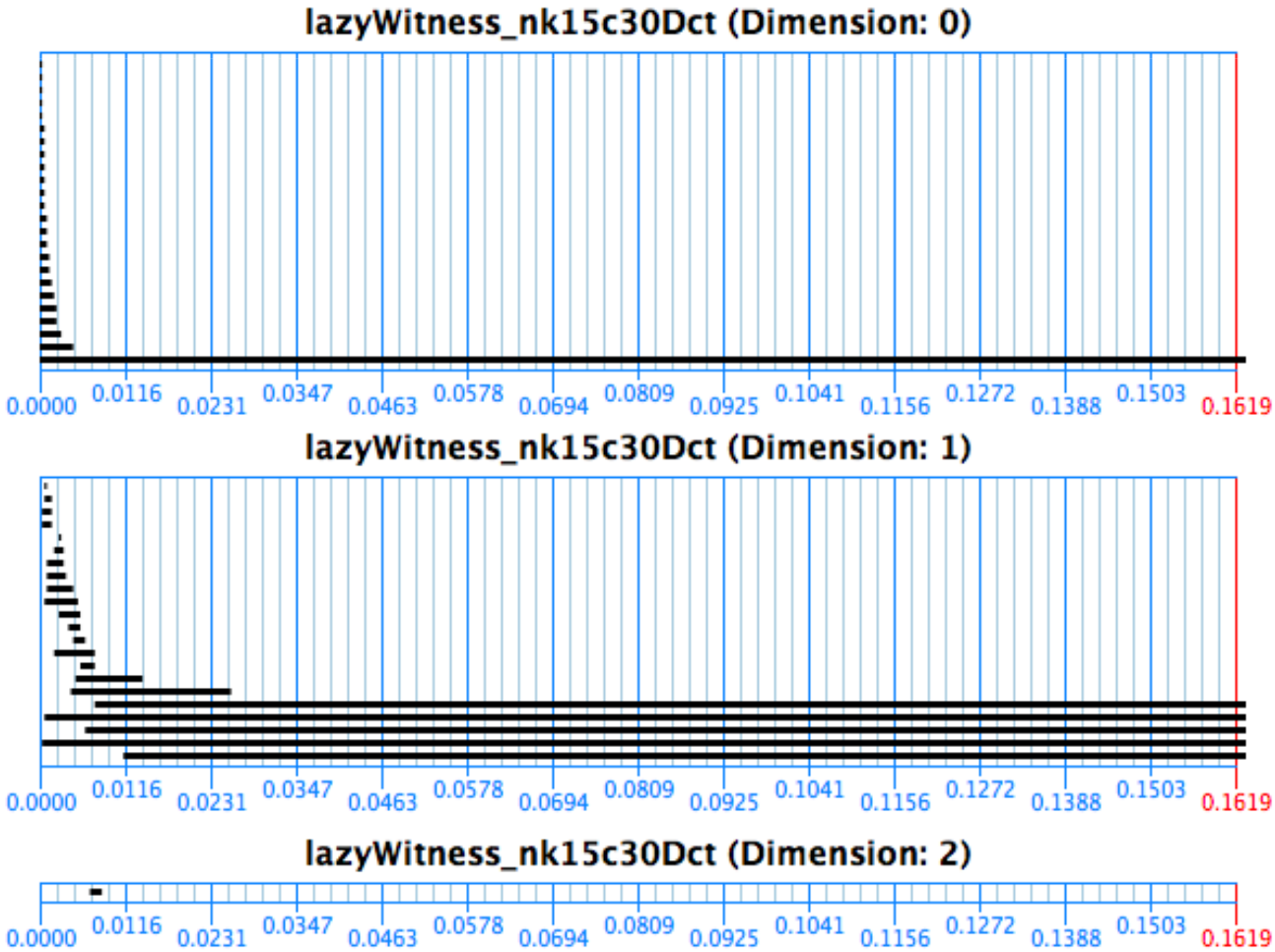




Interpretation: nature prefers linearity

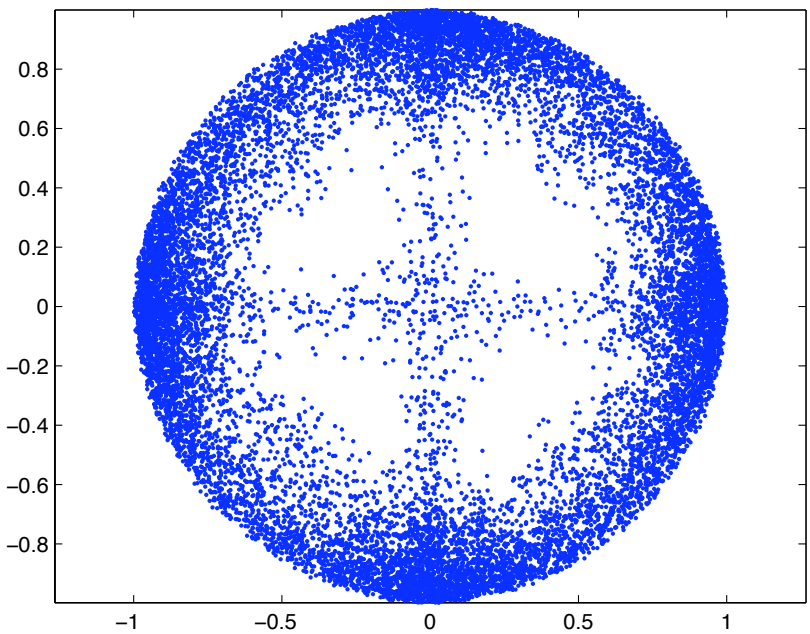
Persistent homology applied to data

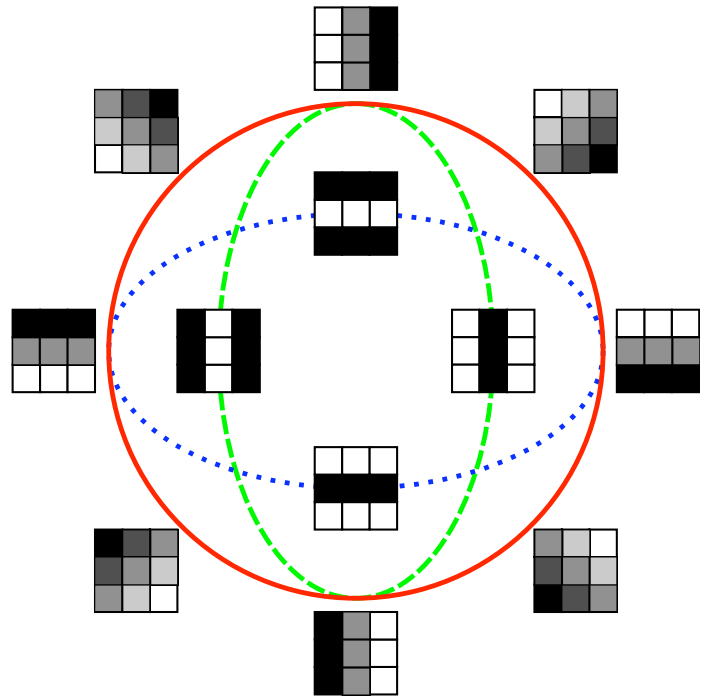
2. Densest patches according to an intermediate estimate



Persistent homology applied to data

2. Densest patches according to an intermediate estimate

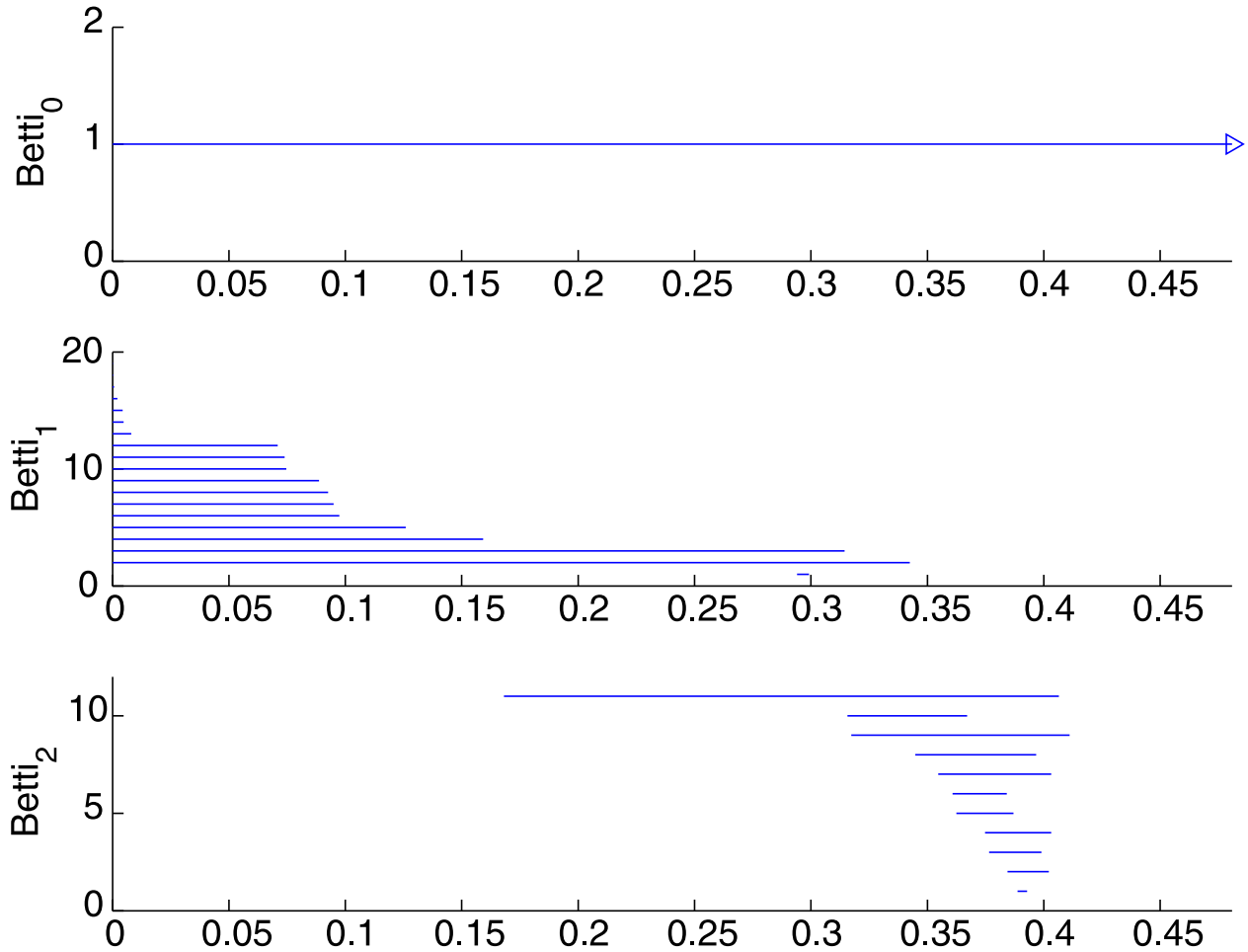




Interpretation: nature prefers horizontal and vertical directions

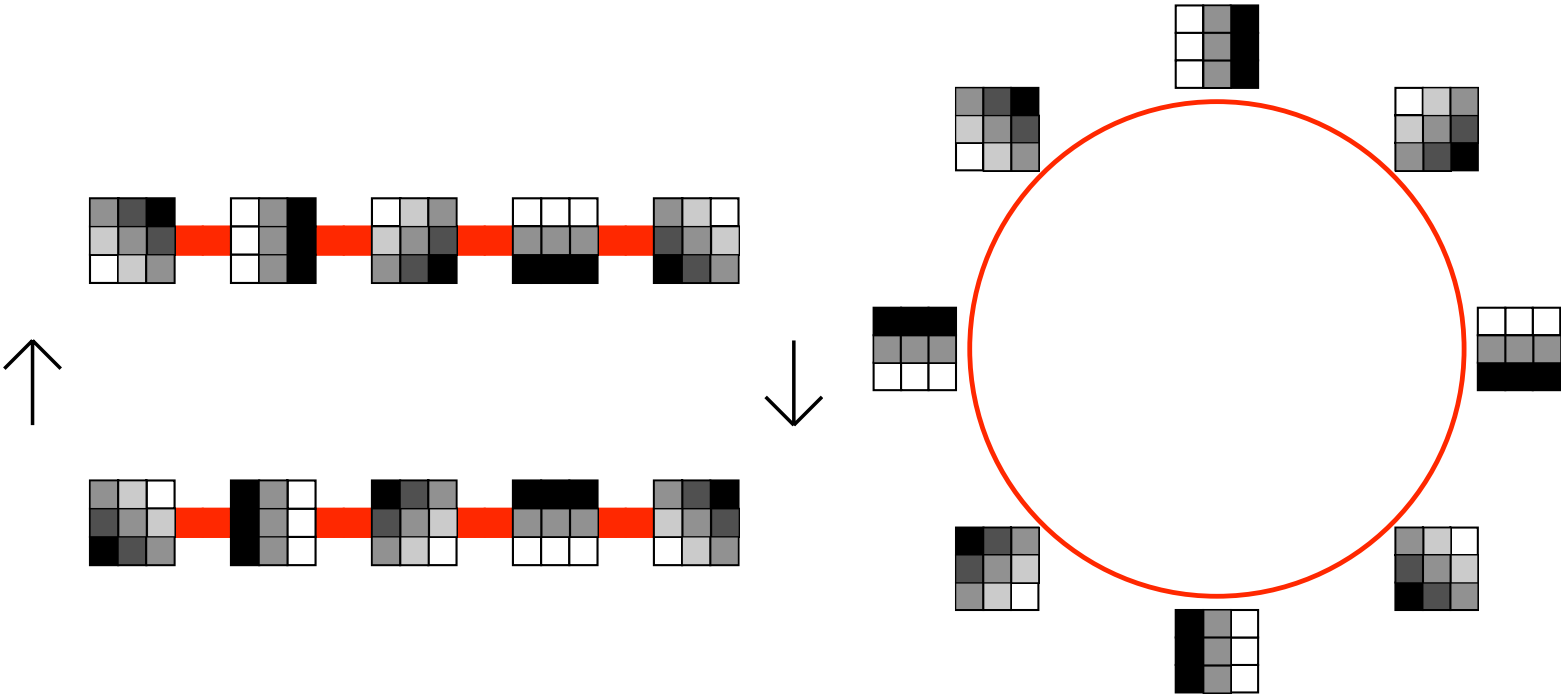
Persistent homology applied to data

3. Densest patches according to a local estimate



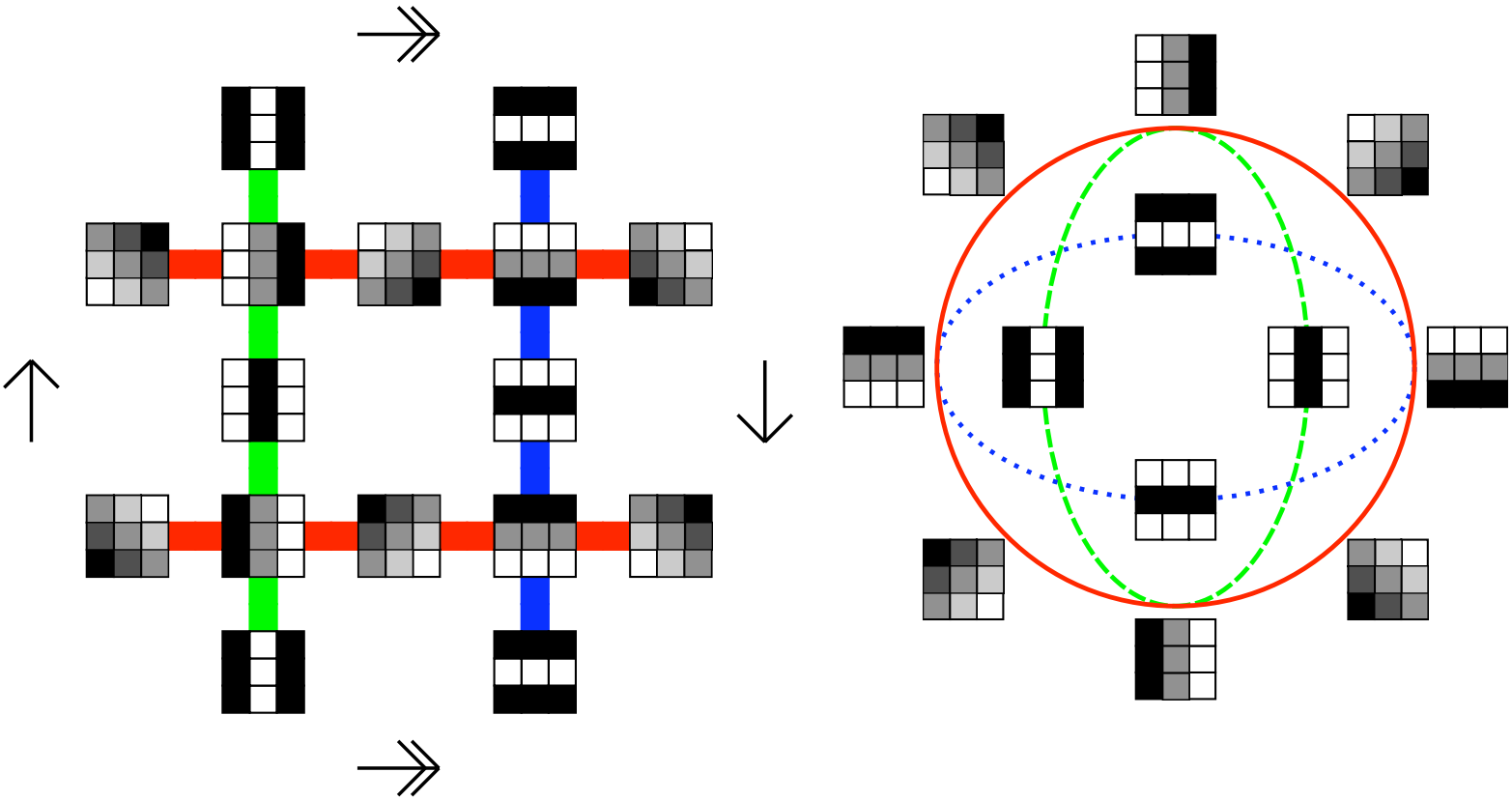
Persistent homology applied to data

3. Densest patches according to a local estimate



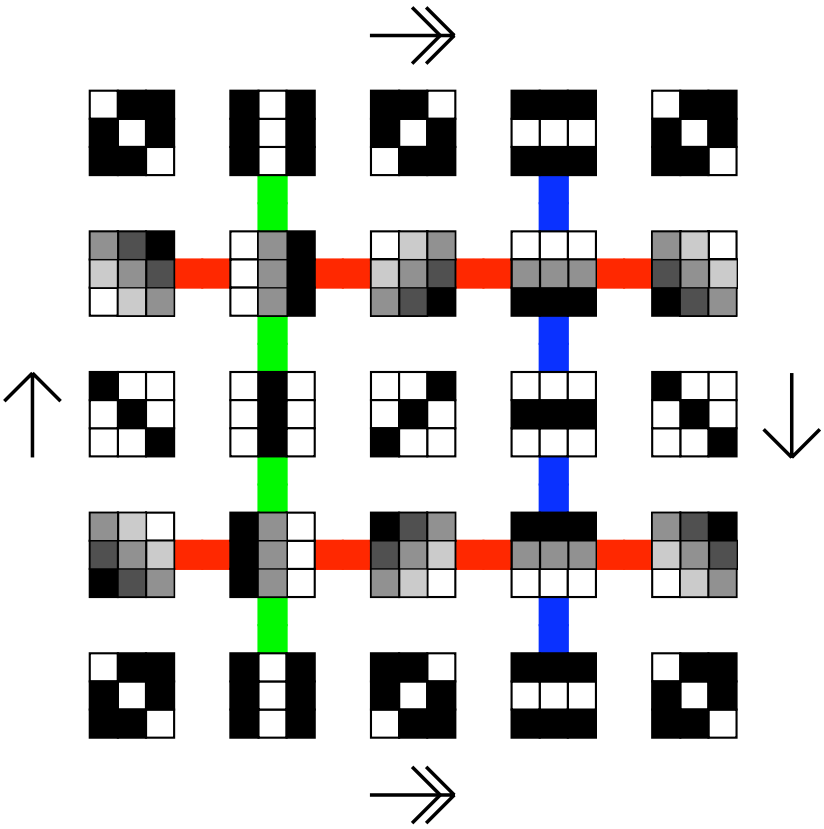
Persistent homology applied to data

3. Densest patches according to a local estimate



Persistent homology applied to data

3. Densest patches according to a local estimate



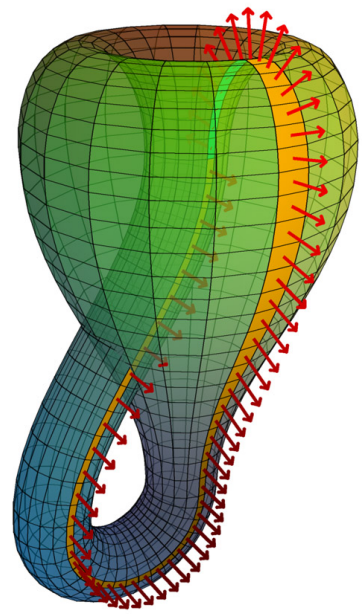


Image credit: https://plus.maths.org/content/imaging-maths-inside-klein-bottle

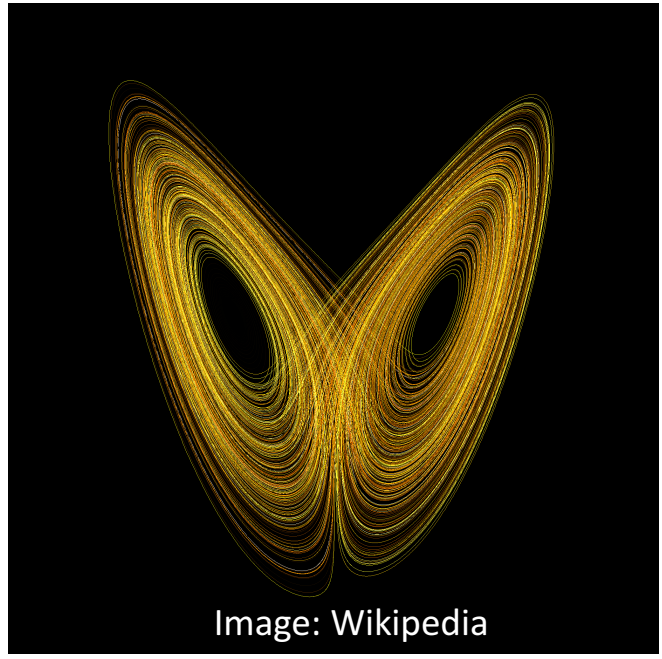
Interpretation: nature prefers linear and quadratic patches at all angles

Takens' Theorem

Roughly speaking: Let M be a d-dimensional compact manifold, let $\phi\colon M \to M$ be a flow, and let $f\colon M \to \mathbb{R}$ be a measurement. Then *generically*,

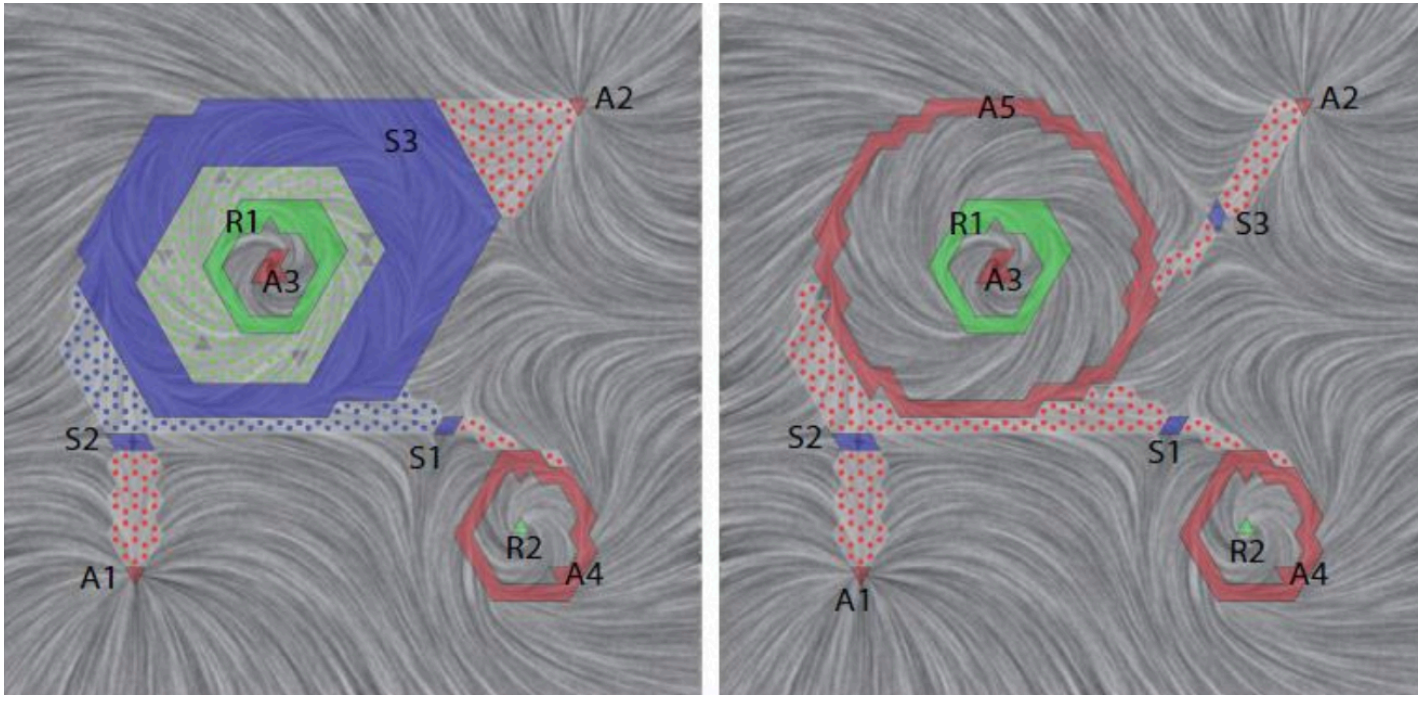
$$m \mapsto \big(f(m), f(\phi(m), f(\phi^2(m), \ldots, f(\phi^{2d}(m))\big)$$

is an embedding $M \hookrightarrow \mathbb{R}^{2d+1}$.



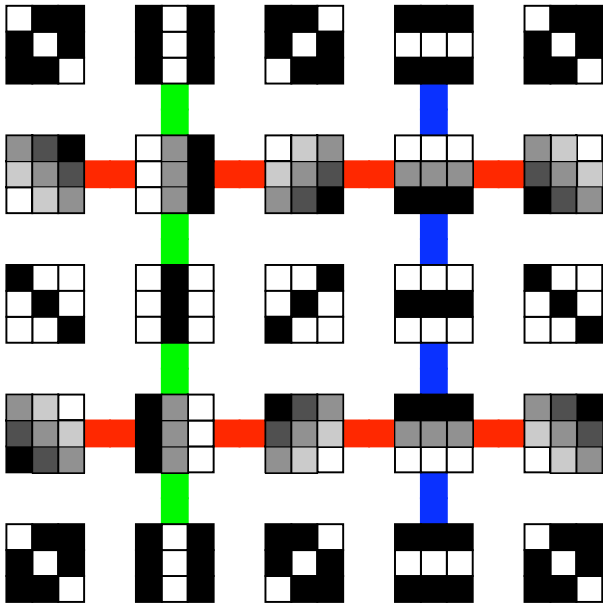
Detecting strange attractors in turbulence by Floris Takens, 1982

Conley index theory



Conclusions ... so far!

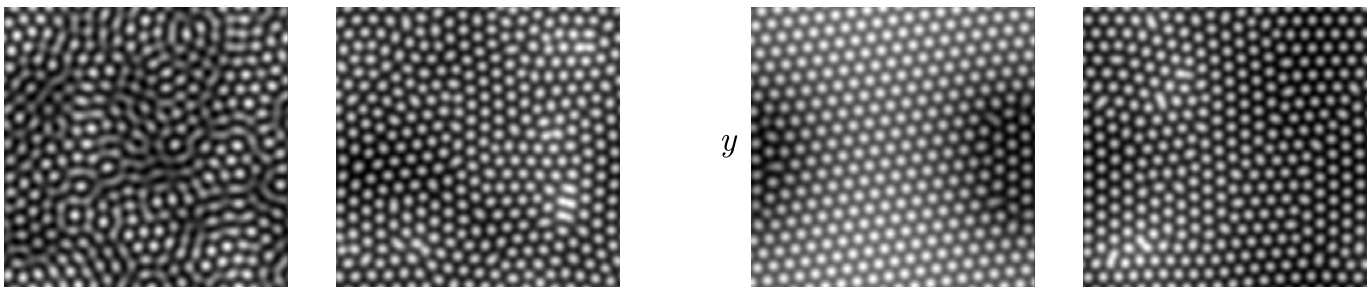
- Datasets have shape, which are reflective of patterns within.
- Persistent homology is a way to measure some of the local geometry and global topology of a dataset.



Why is applied topology popular when few datasets have Klein bottles?

- Many datasets have clusters & flares (as in the diabetes example)
- Motivates interesting questions in many pure disciplines: mathematics, computer science (computational geometry), statistics
- Interest from domain experts in biology, neuroscience, computer vision, dynamical systems, sensor networks, ...
- Materials science, pattern formation
- Machine learning: small features matter
- Agent-based modeling (swarming)

Possible answer: Persistent homology measures both the local geometry and the global topology of a dataset.

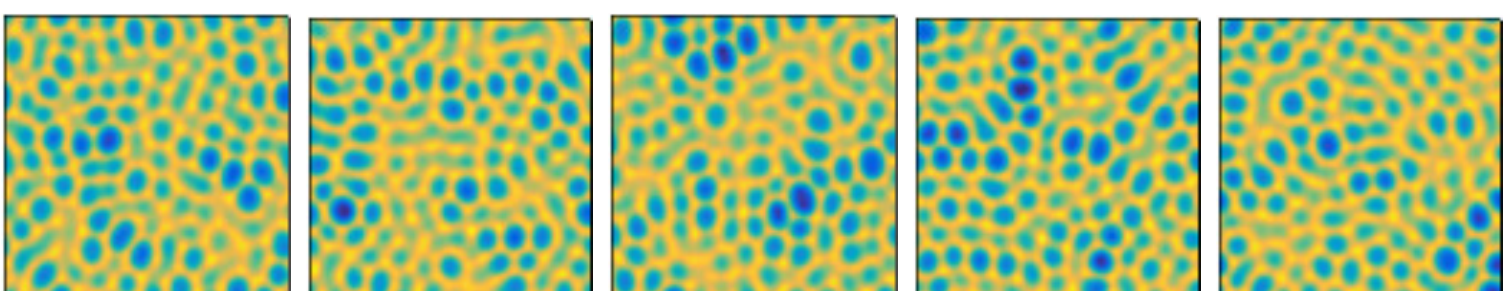
y 

Measures of Order for nearly hexagonal lattices by Francis Motta, Rachel Neville, Patrick Shipman, Daniel Pearson, and Mark Bradley, 2018.

Why is applied topology popular when few datasets have Klein bottles?

- Many datasets have clusters & flares (as in the diabetes example)
- Motivates interesting questions in many pure disciplines: mathematics, computer science (computational geometry), statistics
- Interest from domain experts in biology, neuroscience, computer vision, dynamical systems, sensor networks, ...
- Materials science, pattern formation
- Machine learning: small features matter
- Agent-based modeling (swarming)

Possible answer: Persistent homology measures both the local geometry and the global topology of a dataset.

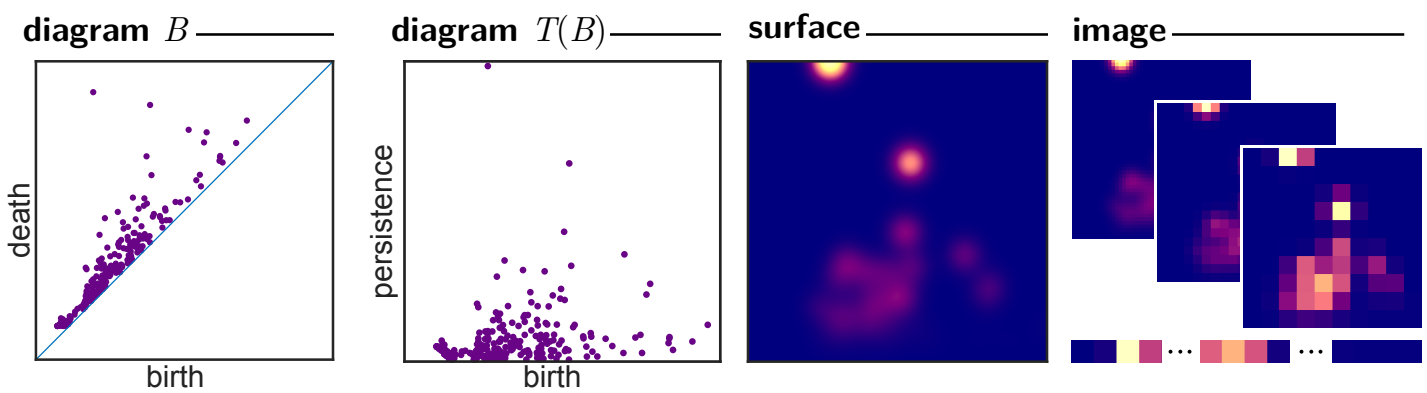


Answer: (from left) $r = 1.75, 2, 1.75, 2, 2.$

Why is applied topology popular when few datasets have Klein bottles?

- Many datasets have clusters & flares (as in the diabetes example)
- Motivates interesting questions in many pure disciplines: mathematics, computer science (computational geometry), statistics
- Interest from domain experts in biology, neuroscience, computer vision, dynamical systems, sensor networks, ...
- Materials science, pattern formation
- Machine learning: small features matter
- Agent-based modeling (swarming)

Possible answer: Persistent homology measures both the local geometry and the global topology of a dataset.

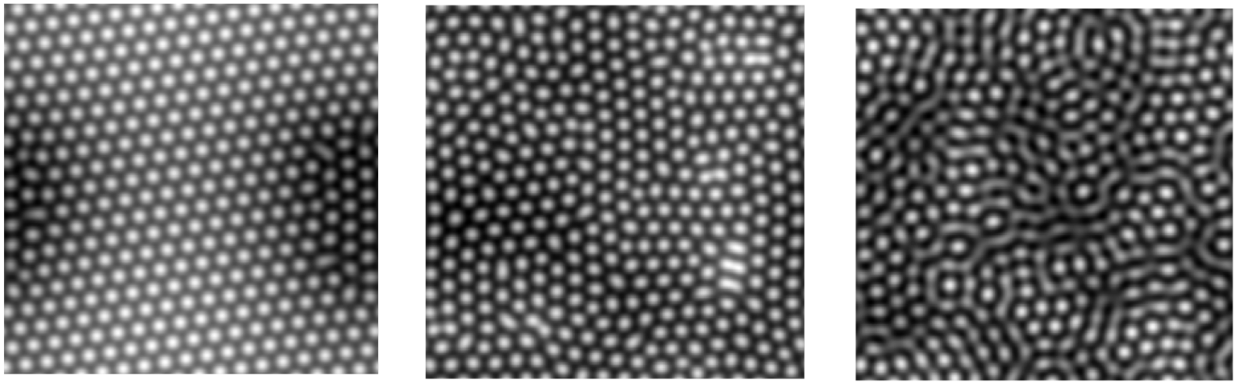


Topology applied to machine learning: From global to local

Survey paper joint with Michael Moy

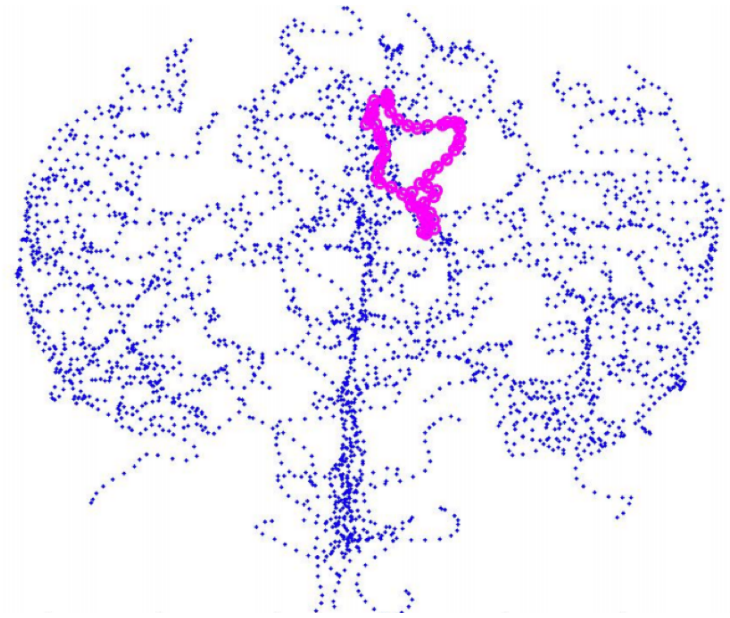
Persistent homology measures both the global topology and the local geometry of a dataset.

Local geometry



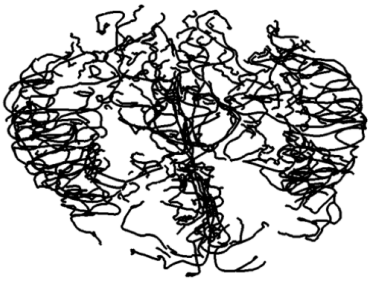
Measures of order for nearly hexagonal lattices
Motta, Neville, Shipman, Pearson, Bradley, 2018

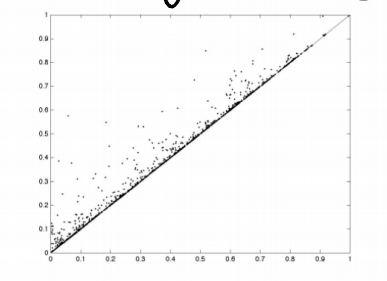
Local geometry

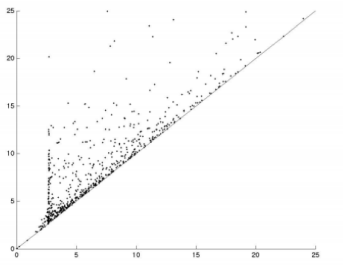


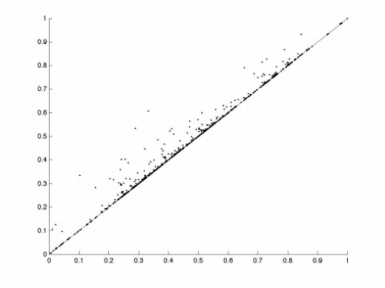
Persistent homology analysis of brain artery trees
Bendich, Marron, Miller, Pieloch, Skwerer, 2014

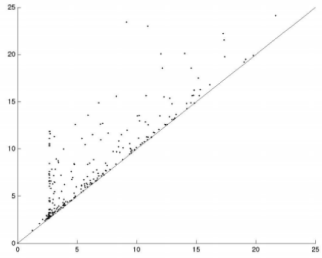
Local geometry





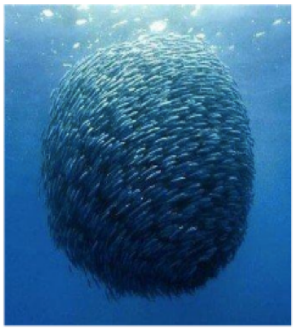






Persistent homology analysis of brain artery trees
Bendich, Marron, Miller, Pieloch, Skwerer, 2014

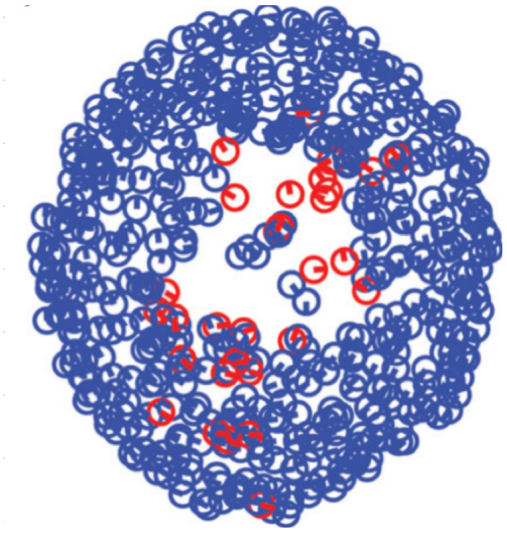
Local geometry



Collective motion, self-organization

Topological data analysis of biological aggregation models
Topaz, Ziegelmeier, Halverson, 2015

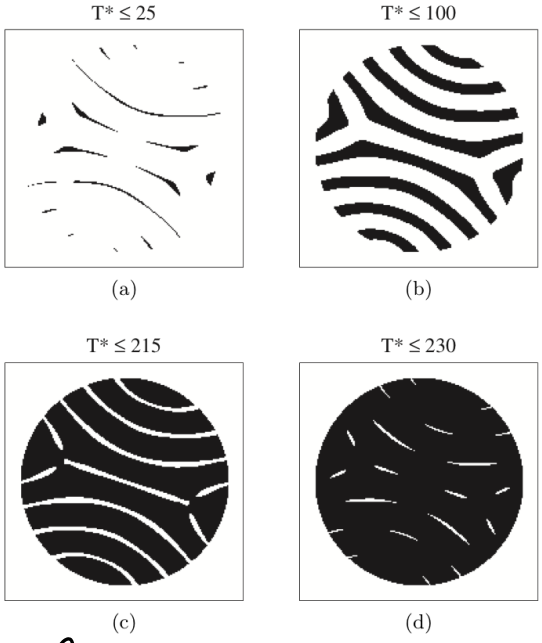
Local geometry



Collective motion, self-organization

Topological data analysis of biological aggregation models
Topaz, Ziegelmeier, Halverson, 2015

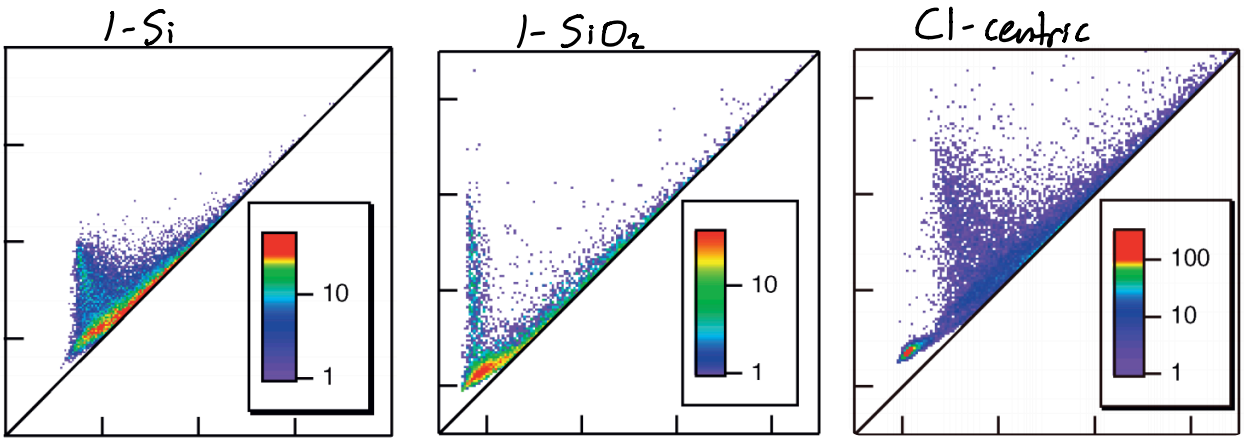
Local geometry



Analysis of Kolmogorov flow and Rayleigh-Bénard convection using persistent homology

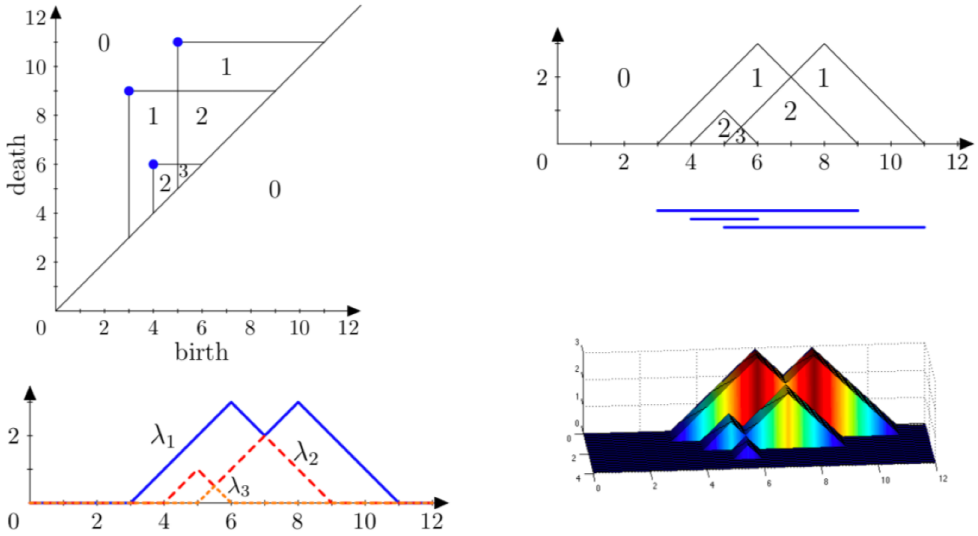
Kramár, Levanger, Tithof, Suri, Xu, Paul, Schatz, Mischaikow, 2016

Local geometry



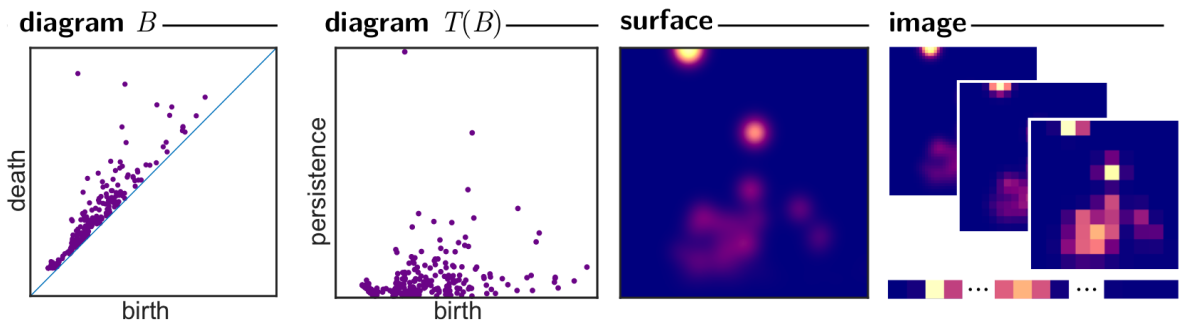
Understanding diffusion patterns of glassy, liquid and amorphous materials via persistent homology analysis
Onodera, Kohara, Tahara, Masuno, Inoue, Shiga, Hirata, Tsuchiya, Hiraoka, Obayashi, Ohara, Mizuno, Sakata, 2019

Local geometry



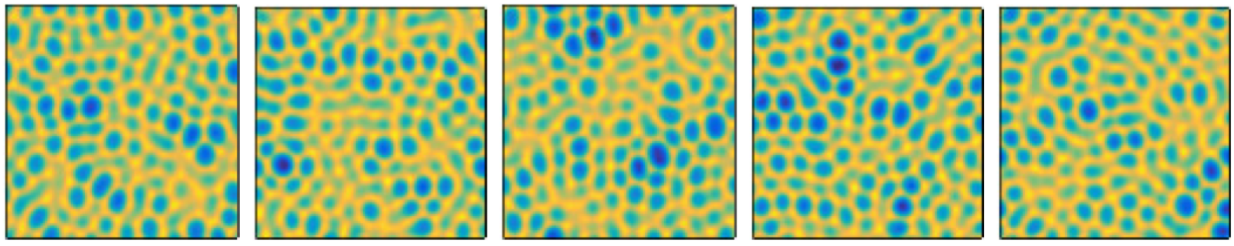
Statistical topological data analysis using persistence landscapes
Bubenik, 2015

Local geometry



Persistence images: A stable vector representation of persistent homology. Adams, Chepushtanova, Emerson, Hanson, Kirby, Motta, Neville, Peterson, Shipman, Ziegelmeier, 2017

Local geometry

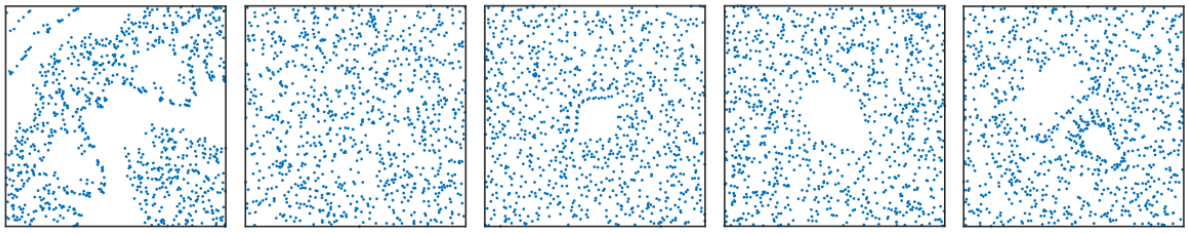


Answer: (from left) $r = 1.75, 2, 1.75, 2, 2.$

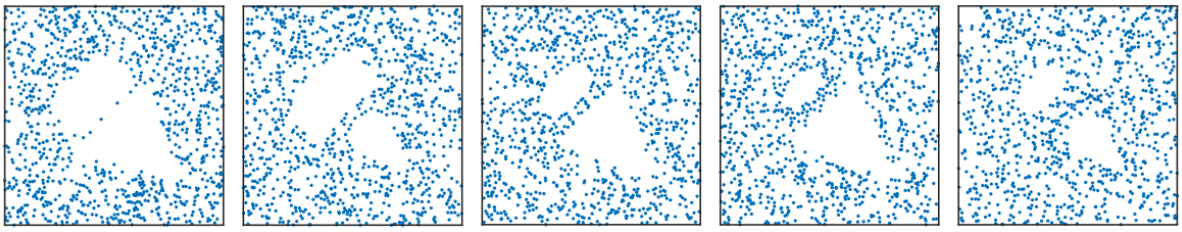
Persistence images: A stable vector representation of persistent homology. Adams, Chepushtanova, Emerson, Hanson, Kirby, Motta, Neville, Peterson, Shipman, Ziegelmeier, 2017

Local geometry

Different parameters:

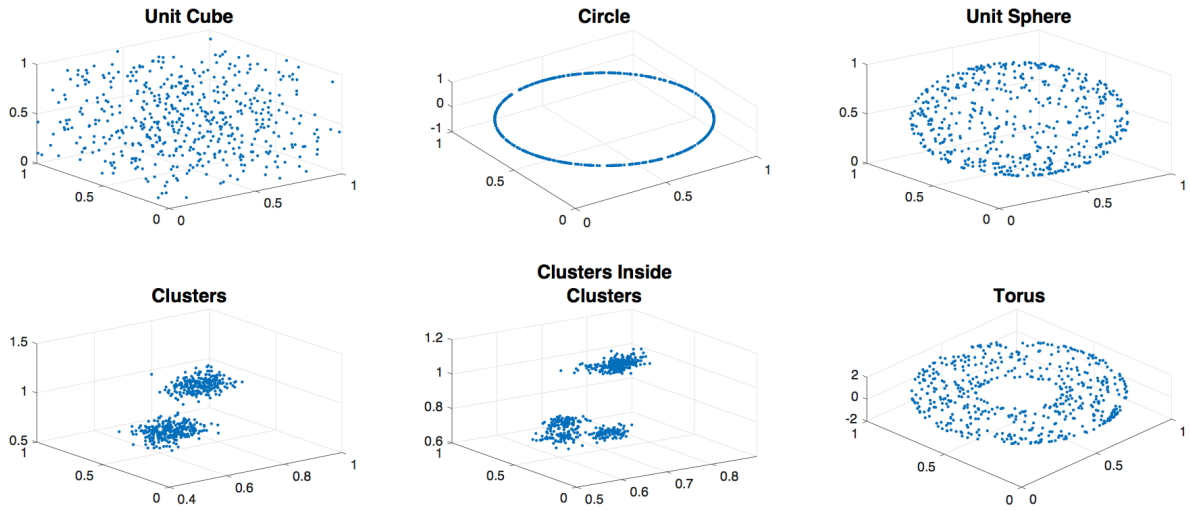


Same parameter:



Persistence images: A stable vector representation of persistent homology. Adams, Chepushtanova, Emerson, Hanson, Kirby, Motta, Neville, Peterson, Shipman, Ziegelmeier, 2017

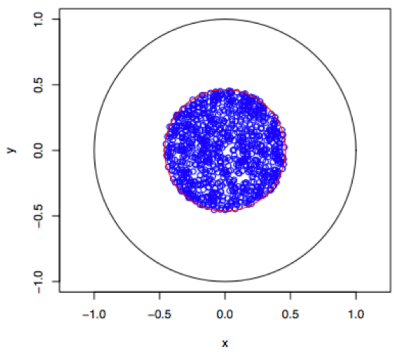
Local geometry



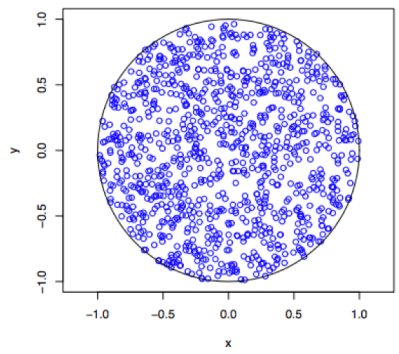
Persistence images: A stable vector representation of persistent homology. Adams, Chepushtanova, Emerson, Hanson, Kirby, Motta, Neville, Peterson, Shipman, Ziegelmeier, 2017

Local geometry

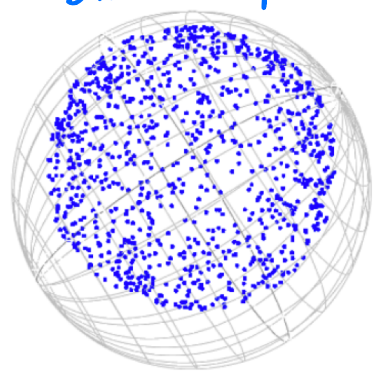
Hyperbolic disk



Flat disk

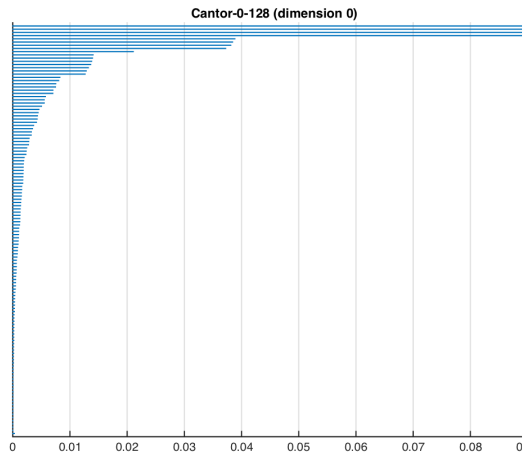


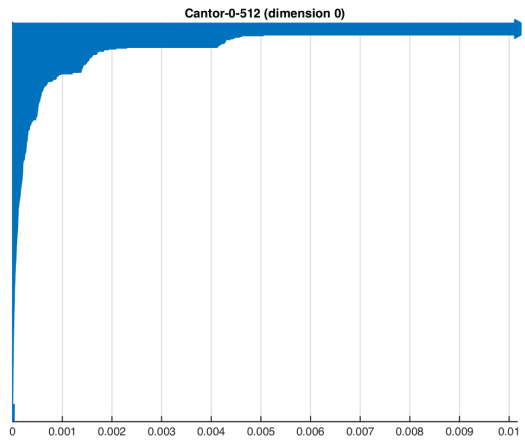
Disk on sphere



Persistent homology detects curvature
Bubenik, Hull, Patel, Whittle, 2019

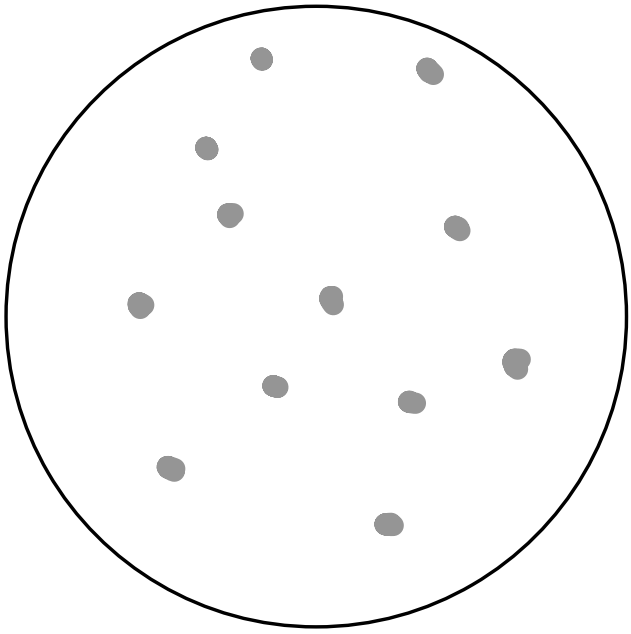
Local geometry





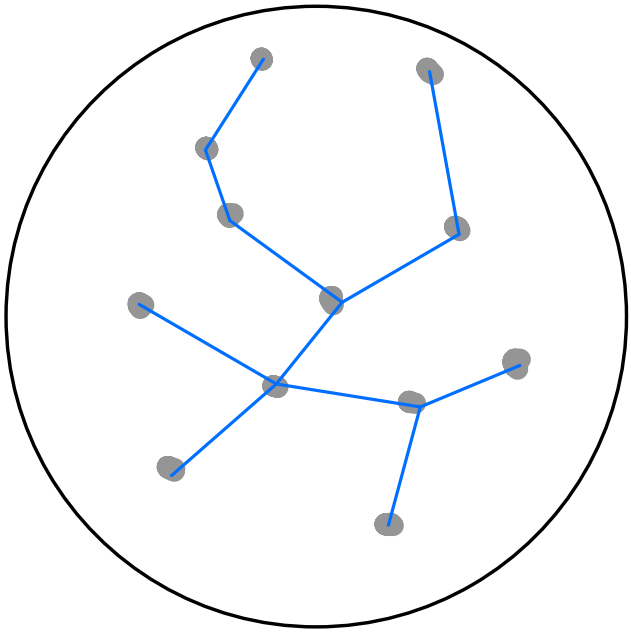
A fractal dimension for measures via persistent homology
Adams, Aminian, Farnell, Kirby, Peterson, Mirth, Neville, Shonkwiler, 2020

Local geometry



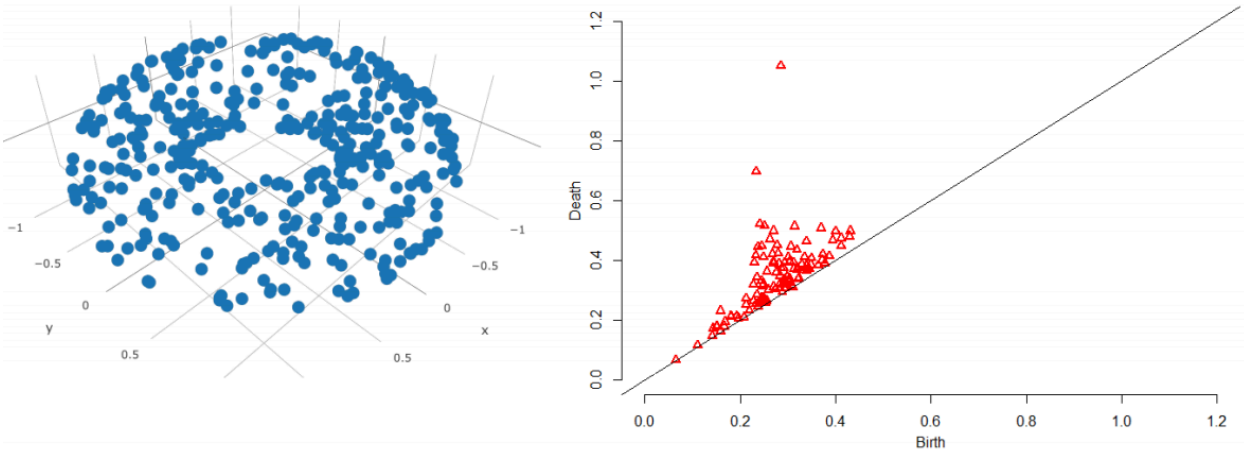
A fractal dimension for measures via persistent homology
Adams, Aminian, Farnell, Kirby, Peterson, Mirth, Neville, Shonkwiler, 2020

Local geometry



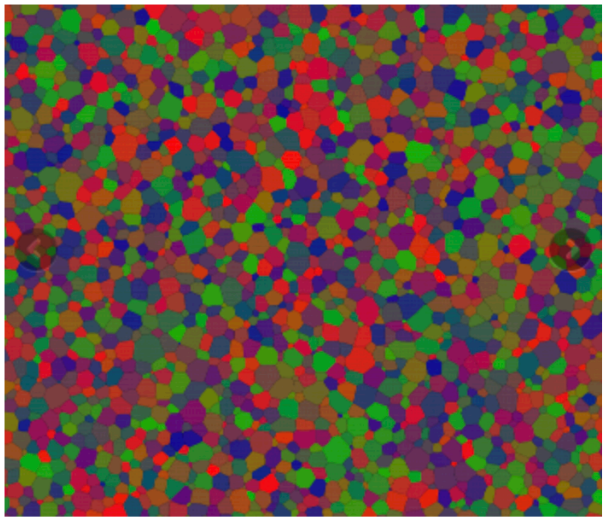
A fractal dimension for measures via persistent homology
Adams, Aminian, Farnell, Kirby, Peterson, Mirth, Neville, Shonkwiler, 2020

Local geometry

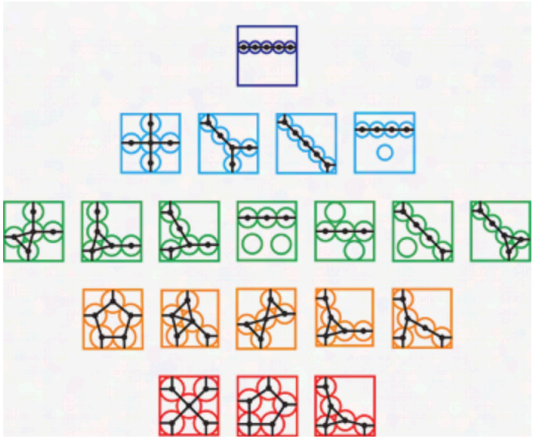


On the choice of weight functions for linear representations of persistence diagrams
Divol and Polonik, 2019

Local geometry



Cellular automata simulation of grain growth.

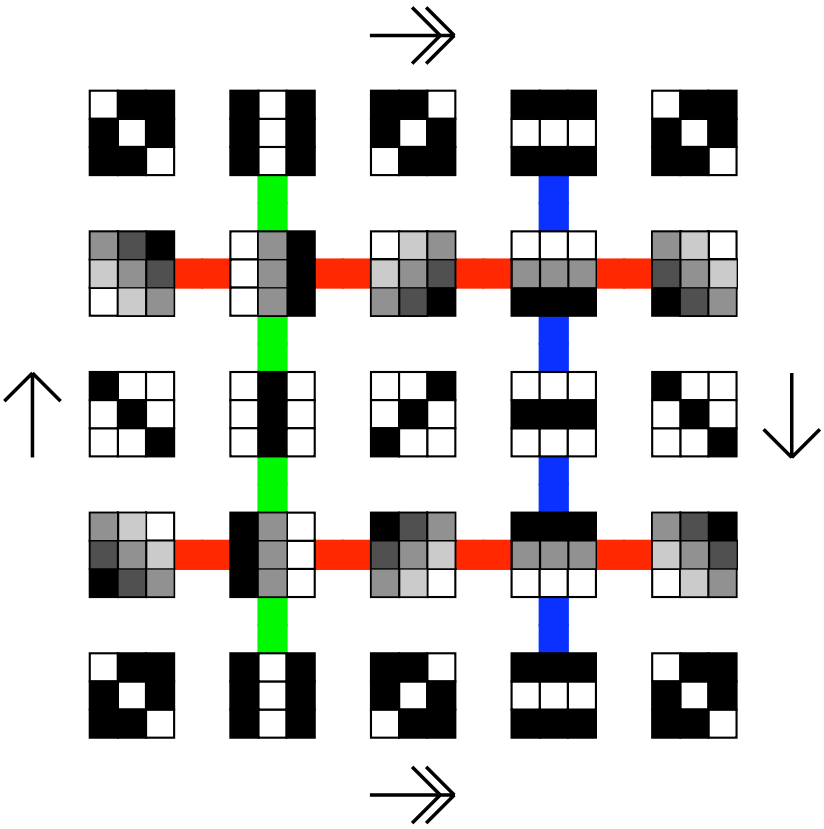


Jeremy Mason research group @UC Davis

Where can I find resources if I am interested in applied topology?

- You may be interested in the Applied Algebraic Topology Research Network. Become a member to receive email invites to the online research seminars. Recorded talks are available at the YouTube Channel. There is also a forum.
- Another source of applied topology news is appliedtopology.org.
- A second online research seminar is GEOTOP-A: Applications of Geometry and Topology.
- Mailing lists with announcements in applied topology include WinCompTop and ALGTOP-L.

https://www.math.colostate.edu/~adams/advising



Thank you!