The persistent topology of optimal transport based thickenings

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www.math.colostate.edu/~adams/talks/ATMCS10.pdf

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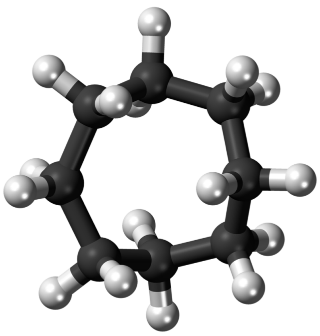
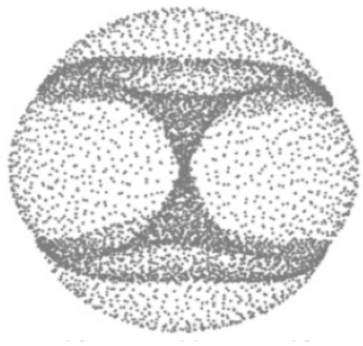
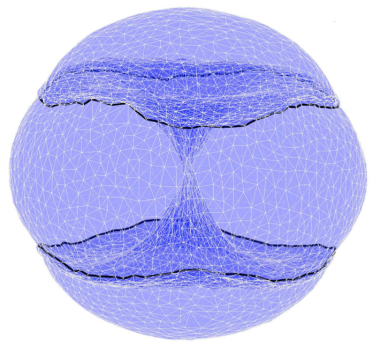
With Hana Dal Poz Kourimská, Teressa Heiss, Sara Kališnik, Bastian Rieck, Elchanan Solomon

Cyclo-octane molecule C_8H_{16}

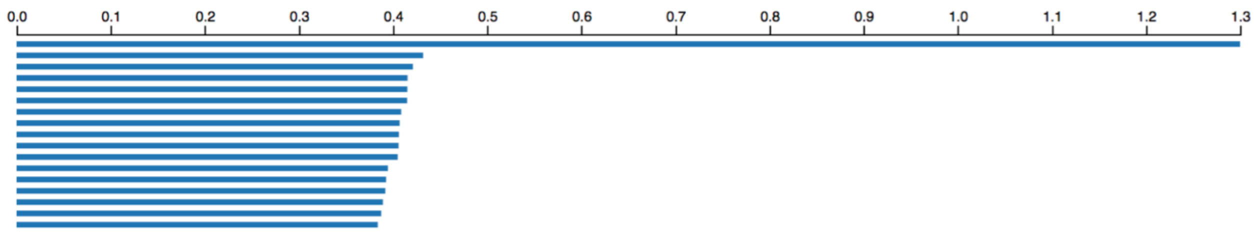
Martin, Thompson, Coutsias, Watson 2010



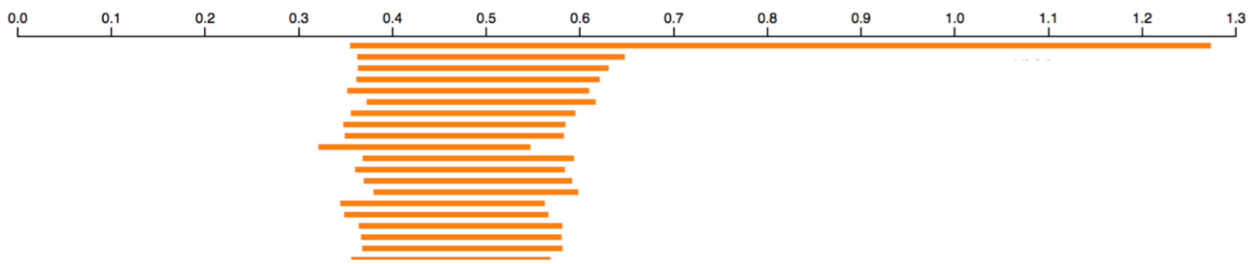




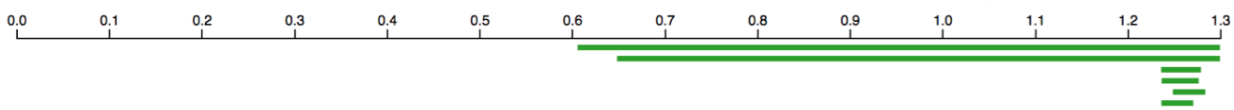
Persistence intervals in dimension 0:



Persistence intervals in dimension 1:



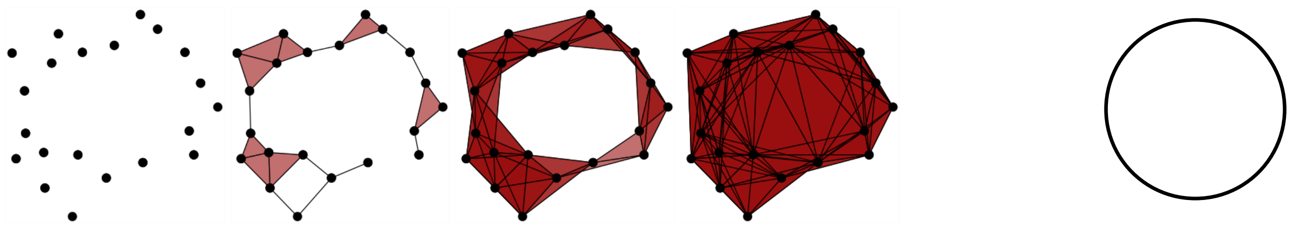
Persistence intervals in dimension 2:



X metric space, $r \geq 0$.

<u>Def</u> The <u>Vietoris-Rips simplicial complex $VR(X,r)$</u> has

- vertex set X
- finite simplex $\sigma \subseteq X$ when $\text{diameter}(\sigma) < r$.



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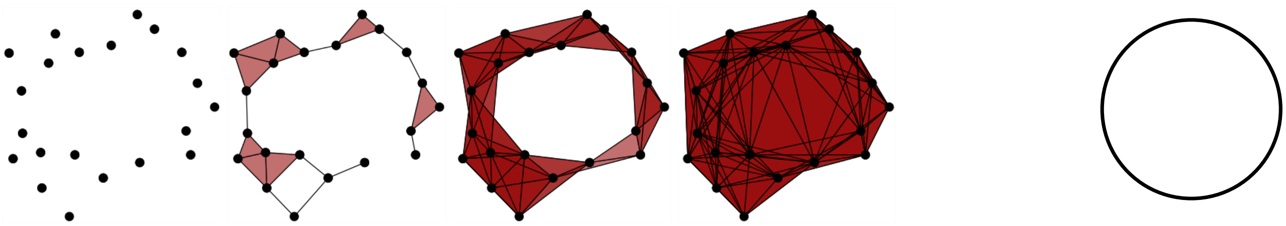
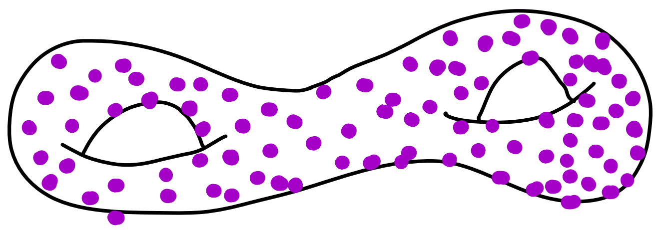
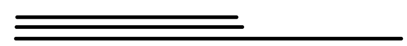


Stability



$PH_1(VR(M; r))$



$PH_1(VR(X; r))$

Chazal, de Silva, Oudot, 2014
Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot, 2009

Metric Reconstruction

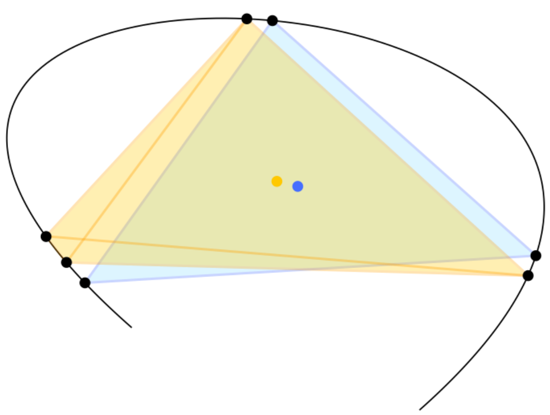
A simplicial complex whose vertex set is a metric space should often be equipped with an optimal transport metric (instead of the simplicial complex topology).

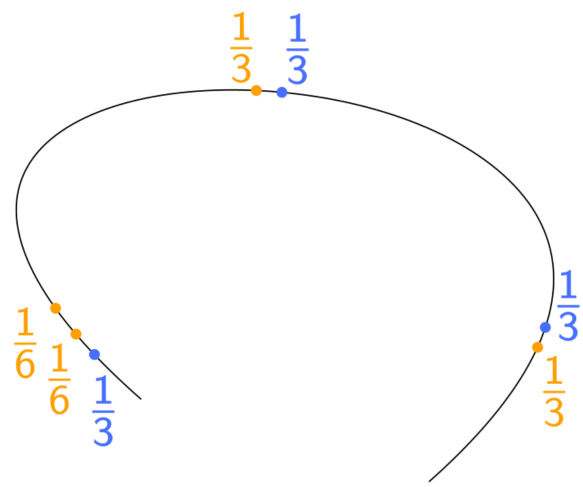
Adamaszek, A, Frick, 2018

<u>Def</u> The <u>Vietoris-Rips metric thickening</u> is

$$\begin{aligned}\mathrm{VR}_\infty(X;r) &= \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; x_i \in X,\ \lambda_i \geq 0,\ \Sigma_i \lambda_i = 1,\ \mathrm{diam}(\{x_0,\dots,x_k\}) < r \right\} \\ &= \{ \text{probability measures } \mu \mid \mathrm{diam}(\mathrm{supp}(\mu)) < r \},\end{aligned}$$

equipped with an optimal transport metric.





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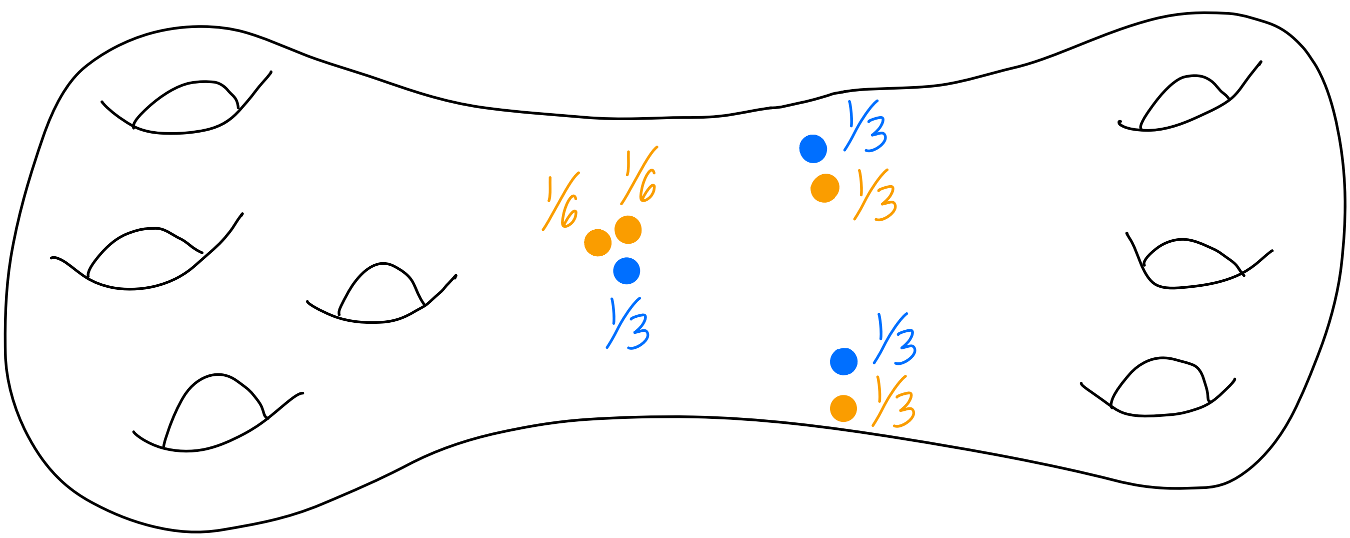
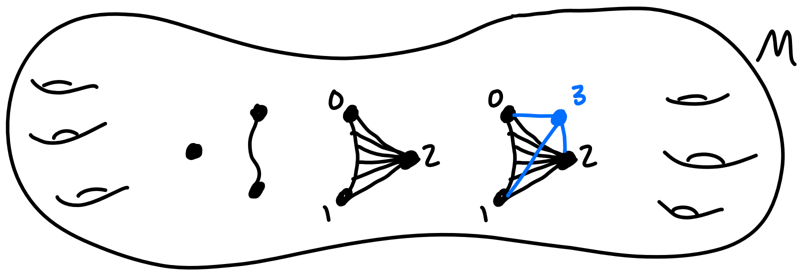


Thm (Hausmann 1995)

M compact Riemannian manifold.
Then $\exists\ r_0 > 0$ such that $VR(M;r) \simeq M$ $\forall r < r_0$.

Proof

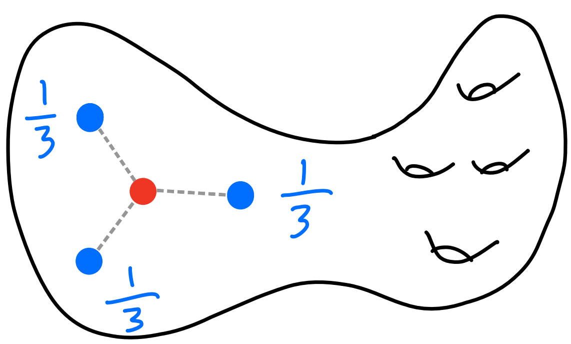
$$VR(M;r) \to M$$



- Not canonical
- $M \hookrightarrow VR(M;r)$ not continuous.

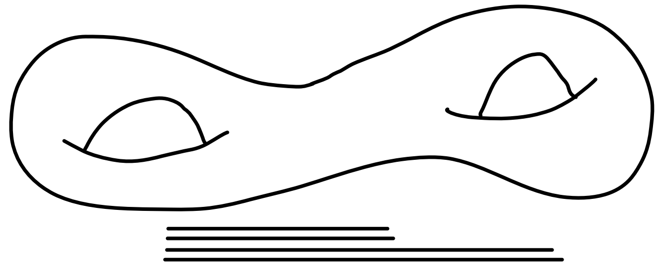
Our Proof for VR_∞

$$VR_\infty(M;r) \rightleftarrows M \qquad \sum \lambda_i \delta_{x_i} \mapsto \text{Frechét mean}$$



__Thm__ For X totally bounded, $VR_\infty(X;r)$ and $VR(X;r)$ have the same (undecorated) persistence diagrams.

A., Mémoli, Moy, Wang, 2021



__Corollary__ The persistence of $VR_\infty(X;r)$ is stable.

__Proof__

$$\begin{tikzcd}
VR(M;r) \arrow[r, hook] \arrow[dr] & VR(M;r+\varepsilon) \arrow[r, hook] \arrow[dr] & VR(M;r+2\varepsilon) \\
VR_\infty(M;r) \arrow[r, hook] \arrow[ur] & VR_\infty(M;r+\varepsilon) \arrow[r, hook] \arrow[ur] & VR_\infty(M;r+2\varepsilon)
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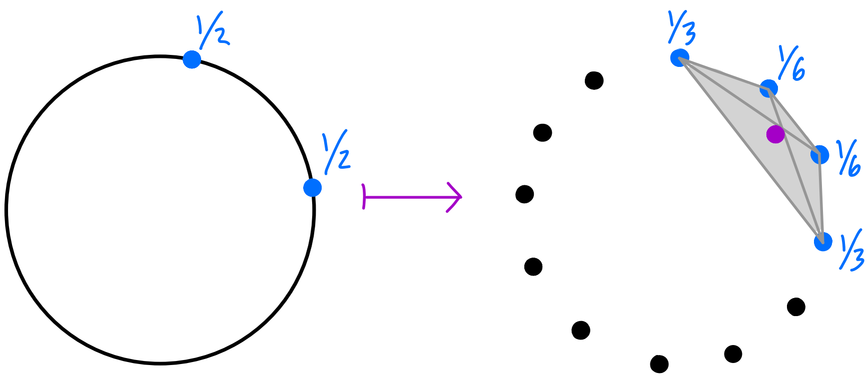

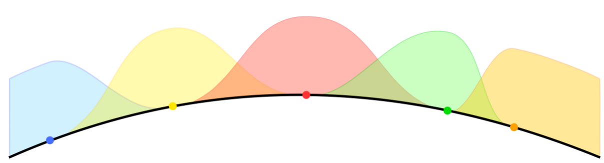

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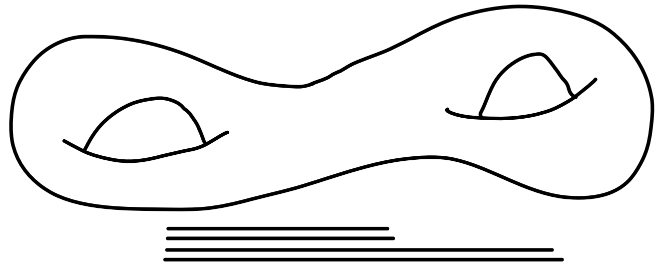

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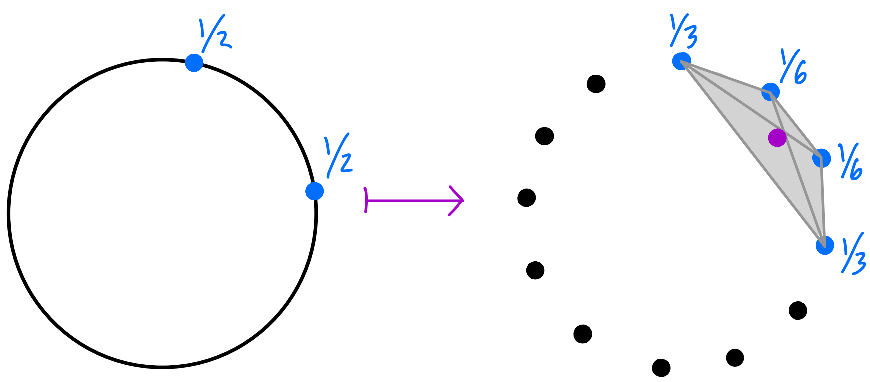


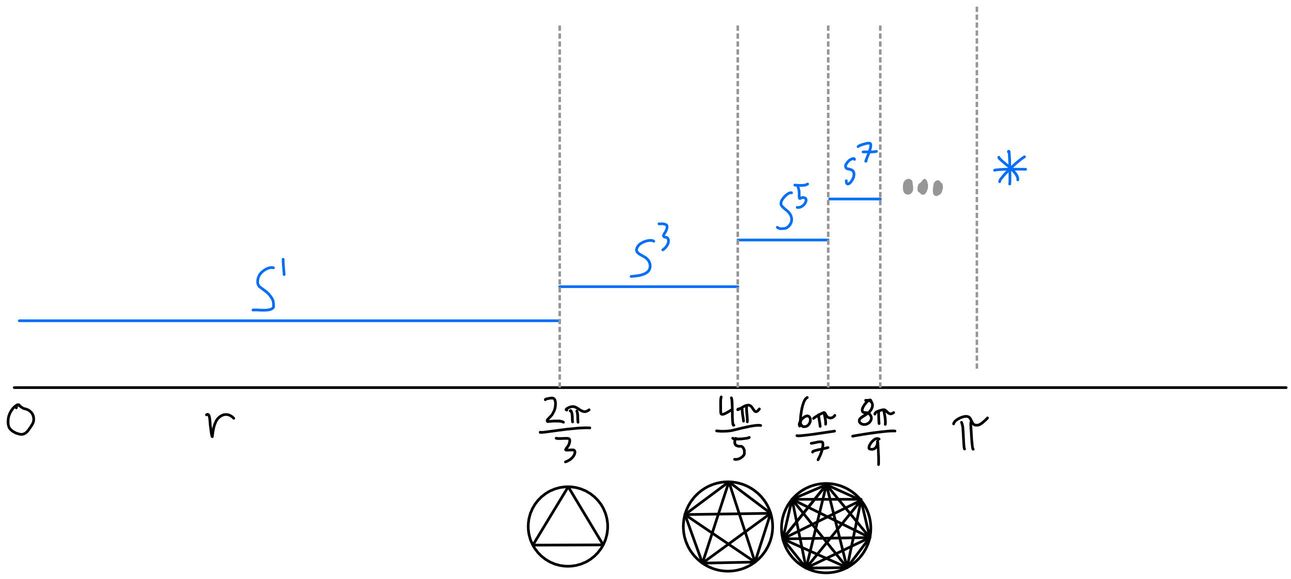
Question Is $VR_\infty(X;r) \simeq VR(X;r)$?

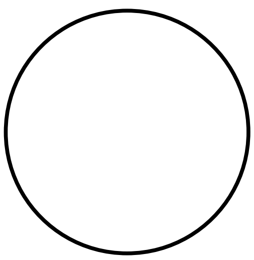
Remark $VR_\infty(X;r)$ and $VR(X;r)$ have the same homotopy groups.

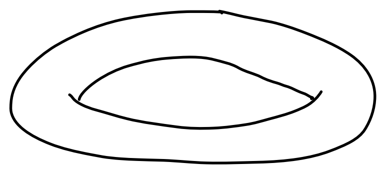
A., Frick, Virk 2022

Adamaszek, A. 2017 May 2022

<u>Thm</u> $VR(S^1;r) \simeq S^{2k+1} \simeq VR_\infty(S^1;r)$ if $\frac{2\pi k}{2k+1} < r < \frac{2\pi(k+1)}{2k+3}$







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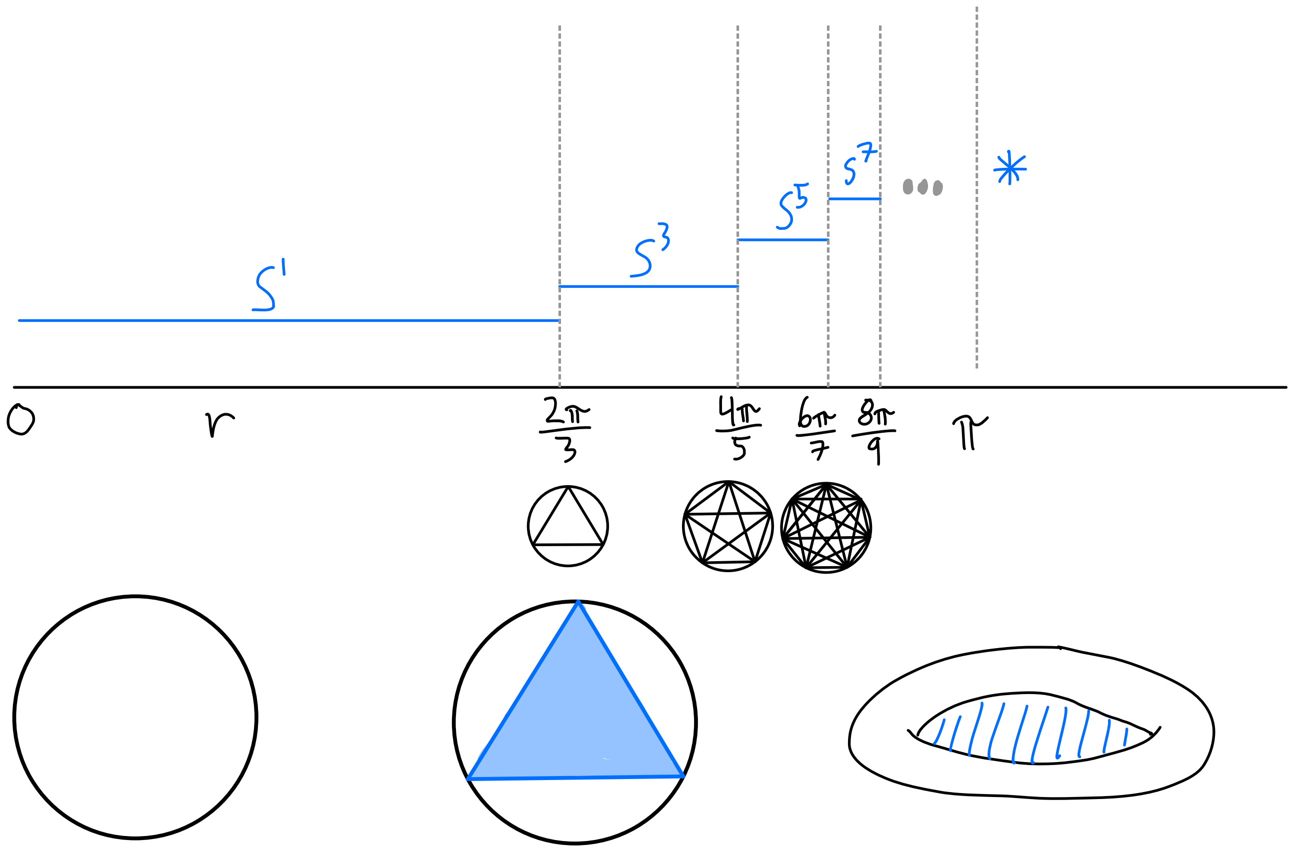

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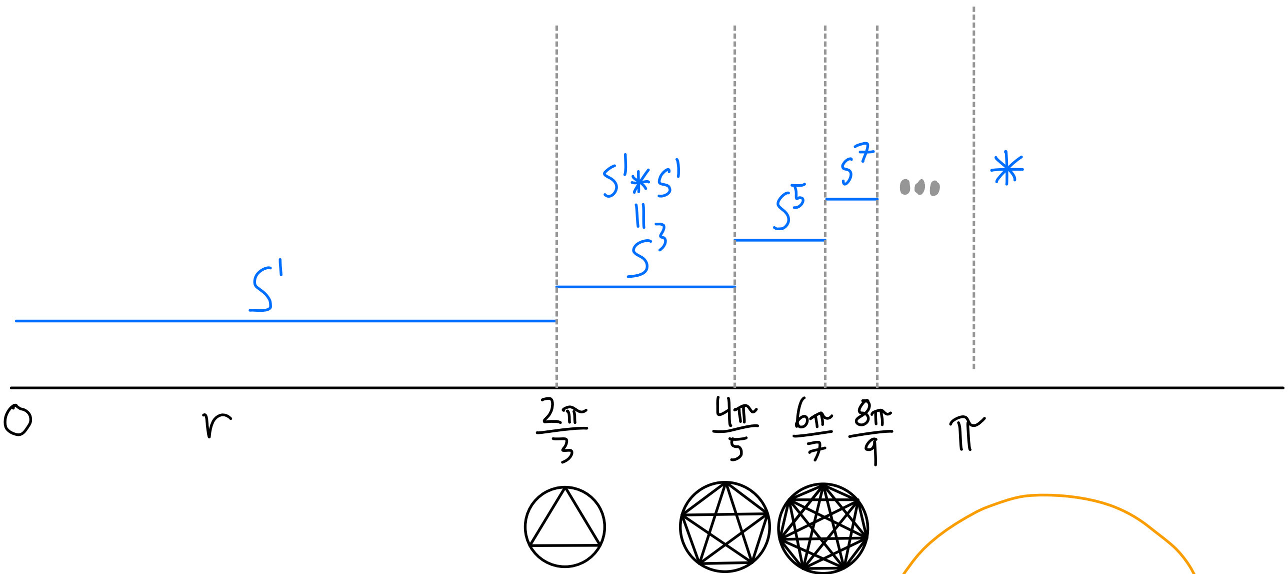
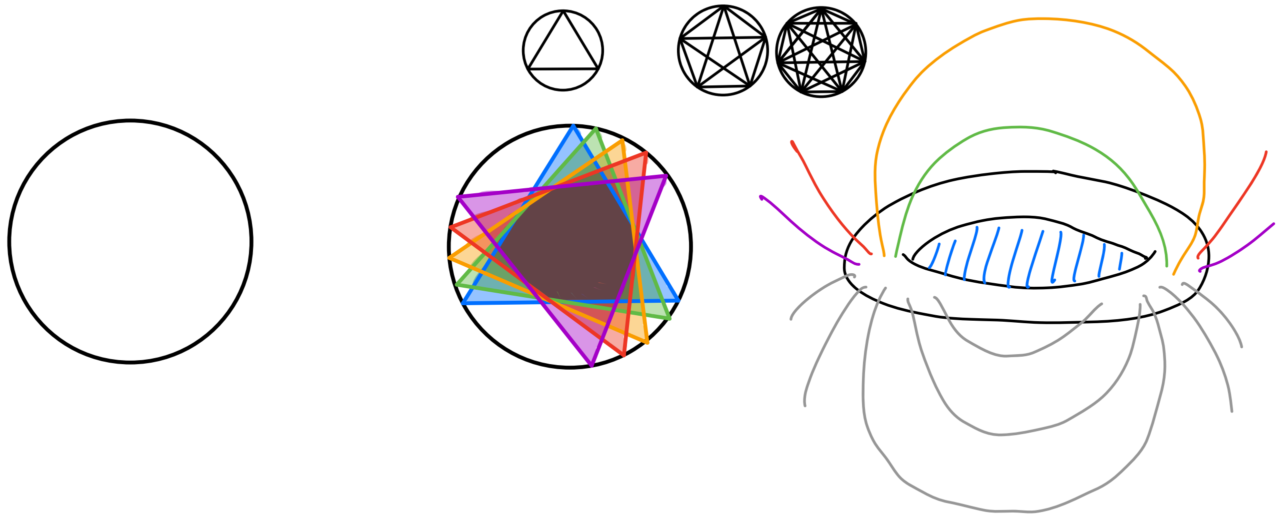
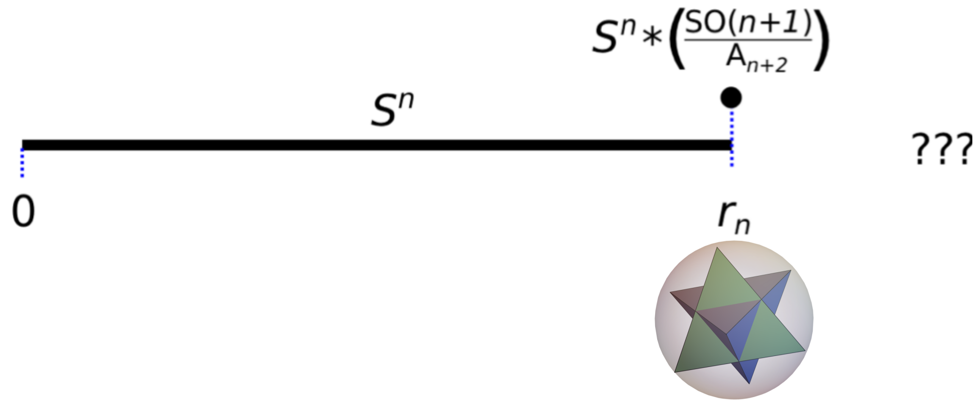




Adamaszek, A., Frick, 2018

More generally,

<u>Thm</u> $VR_\infty(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$

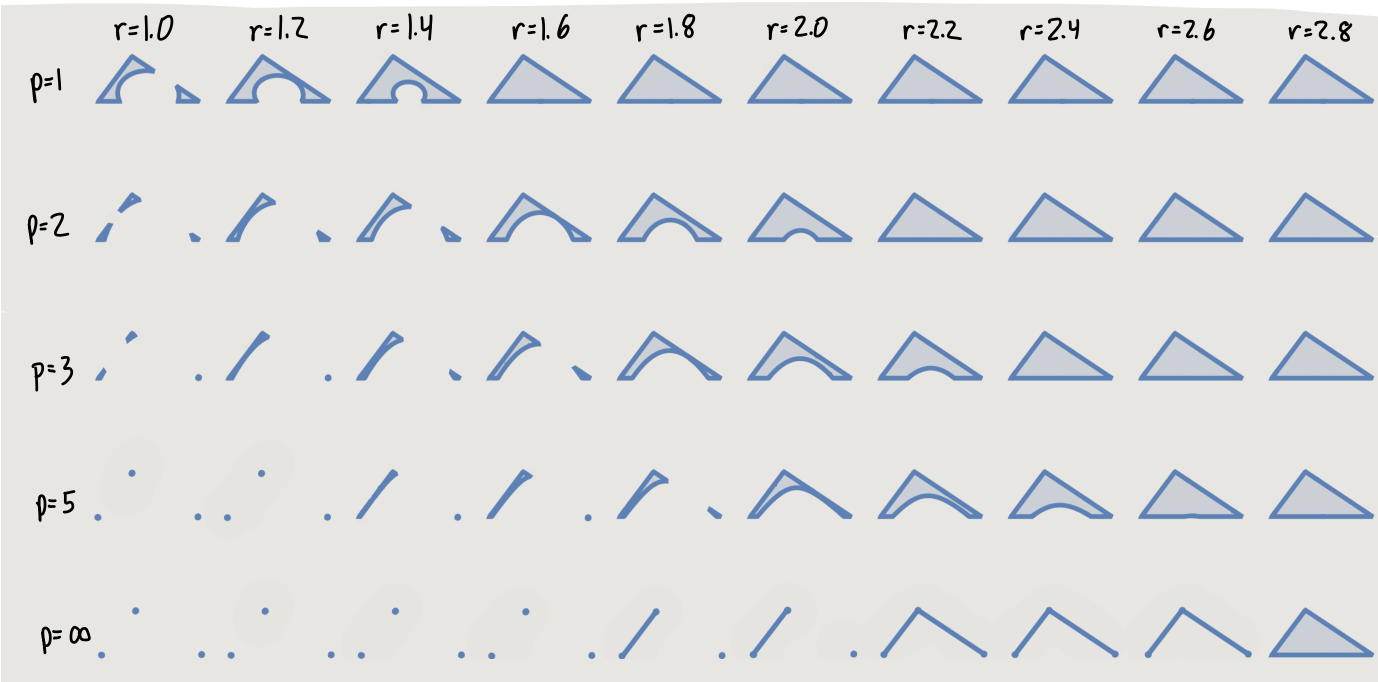


Katz 1991 | Zaremsky 2018 | Lim, Mémoli, Okutan 2021 | A., Bush, Frick 2021

Relaxation $\mathrm{VR}_p(X;r) := \{\text{probability measures } \mu \mid \mathrm{diam}_p(\mu) < r\}$.

A., Mémoli, Moy, Wang, 2021

$$\mathrm{diam}_p(\mu) = \left(\int_{X\times X} d(x,x')^p \, d(\mu\times\mu) \right)^{1/p}$$



Bifiltration $\mathrm{VR}_p(X;r) \hookrightarrow \mathrm{VR}_{p'}(X;r')$ for $r \le r'$, $p \ge p'$.

Theorem $\mathrm{VR}_p(X;r)$ is stable for all $p \in [1,\infty]$.

Theorem $\mathrm{VR}_2(S^n;r) \simeq S^n$ or $*$ for all r.

Connections to Morse theory

Katz 1991 Adamaszek, A., Frick, 2018 Mirth 2020 Moy 2022 Katz, Mémoli, Wang 2022

<u>Questions</u>

(1) $VR_p(S^n; r)$ for larger r?

(2) $\check{C}ech_p(S^n; r)$?

(3) Other manifolds? Tori, ellipsoids, $\mathbb{R}P^n$, $\mathbb{C}P^n$

(4) Morse and Morse-Bott theories

(5) Deeper connections to Gromov-Hausdorff distances