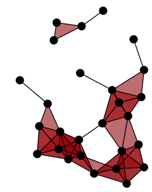
Statement of Recent Work: Henry Adams

Bridging applied and quantitative topology



Large sets of high-dimensional data are common in most branches of science, and their shapes reflect important patterns within. The goal of topological data analysis is to describe the shape of data [12]: How can we recover the shape of a data set X sampled from an underlying space M? One tool along these lines is the Vietoris–Rips complex, which connects nearby data points according to a choice of scale r.

Definition. *Given a metric space X and a scale parameter $r \geq 0$, the* Vietoris–Rips simplicial complex $\mathrm{VR}(X;r)$ *contains a finite simplex $\sigma \subseteq X$ if its diameter is at most r.*

Since we do not know a priori how to choose the scale parameter r, the idea of *persistent homology* [17] is to compute the homology of the Vietoris–Rips complex of data set X over a large range of scales r and to trust those topological features which persist. The motivation for using Vietoris–Rips complexes is a remarkable theorem due to Latschev [22]: for M a manifold, scale r sufficiently small depending on the curvature of M, and data set X close to M, we have a homotopy equivalence $\mathrm{VR}(X;r) \simeq M$. But as the main idea of persistence is to allow the scale r to vary, the assumption that r is kept sufficiently small fails in practice. Indeed, data scientists let the parameter r in the Vietoris–Rips complexes $\mathrm{VR}(X;r)$ vary from zero to relatively large scales (and there is efficient software designed to do this [10]), even though we do not understand how these simplicial complexes behave at large scales.

Motivating Question. *How do the homotopy types of Vietoris–Rips complexes of manifolds change as the scale parameter r increases?*

The above question is fundamental for applications of persistent homology to data analysis, since as a finite subset $X \subset M$ gets denser and denser, the Vietoris–Rips persistent homology of X converges to the Vietoris–Rips persistent homology of the manifold M [15, 14]. Nevertheless, the only connected non-contractible manifold such that the homotopy types of its Vietoris–Rips complexes are known at all scale parameters r is the circle. Indeed, in [1, 3] we prove that as the scale increases, the Vietoris–Rips complexes of the circle are homotopy equivalent to the circle, the 3-sphere, the 5-sphere, the 7-sphere, ..., until finally they are contractible. In [2] we find the first new homotopy types appearing in Vietoris–Rips thickenings of n-spheres, which we use in [5] to provide novel generalizations of the Borsuk–Ulam theorem for maps into higher-dimensional codomains, which we use in [4] we provide strong lower bounds on Gromov-Hausdorff distances between spheres of different dimensions. Within only 2–4 years, our work in [1, 2, 5, 6, 7, 8] has been expanded upon by many other mathematicians [13, 16, 18, 19, 23, 26, 27, 28, 30, 29, 31, 32].

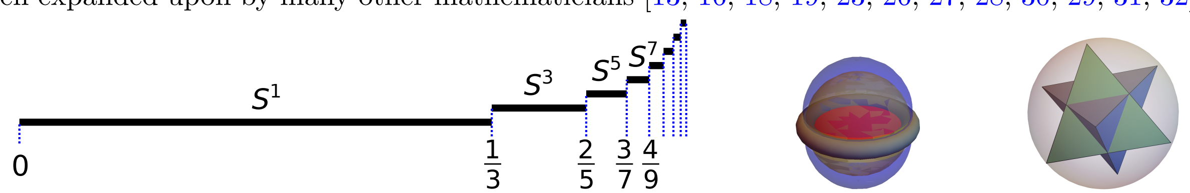
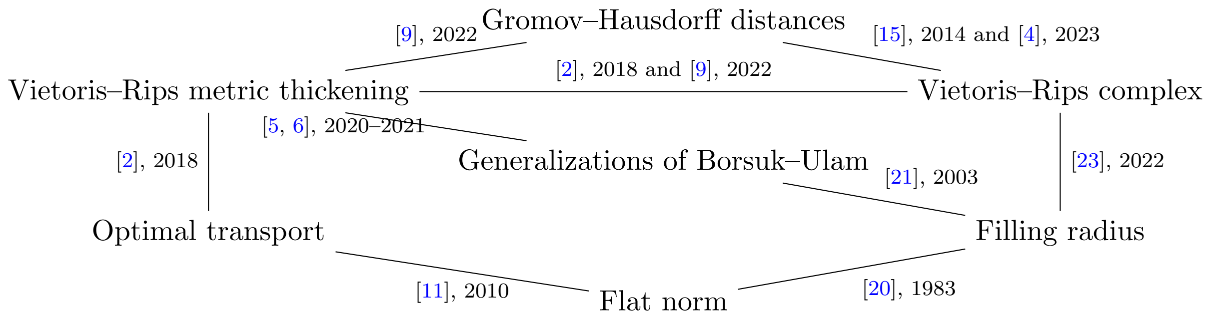


Figure 1. Homotopy types of $\mathrm{VR}(S^1;r)$ as the scale r increases [1]; the 3-sphere arises as a genus one Heegaard decomposition; critical tetrahedra first changing the homotopy type of $\mathrm{VR}(S^2;r)$.

Bridging applied and quantitative topology. Our recent work has laid a rich foundation for connecting applied topology to quantitative topology, allowing for rapid development in both areas. The below diagram summarizes important connections between Vietoris–Rips complexes and thickenings (2nd row) and quantitative topology (all other nodes). The connections between the quantitative topology nodes were all formed over a decade ago [11, 20, 21]. By contrast, the connections to Vietoris–Rips complexes and thickenings have been discovered in the past three years [2, 4, 5, 6, 9, 23]. We emphasize in particular the recent work by Lim, Mémoli, and Okutan [23]



connecting Vietoris–Rips complexes to the filling radius, as used by Gromov to prove the systolic inequality and the "waist of the sphere" theorem. The "waist of the sphere" theorem generalizes the Borsuk–Ulam theorem for maps from the n-sphere into lower-dimensional codomains.

Borsuk–Ulam theorems into higher-dimensional codomains. The Borsuk–Ulam states that any continuous map $f\colon S^n \to \mathbb{R}^n$ assigns two antipodal points in S^n to the same point in $\mathbb{R}^n$. In [5, 6] we give generalizations of Borsuk–Ulam for maps $S^n \to \mathbb{R}^k$ for any $k \geq n$, providing diameter bounds so that the convex hull of the image of some finite subset $X \subseteq S^n$ of bounded diameter collides with that of the antipodal set $-X$. These proofs rely on the homotopy connectivity of Vietoris–Rips thickenings of spheres. For $n = 1$ or for $k \leq n + 2$ our diameter bounds are optimal. *What are optimal diameter bounds for any $k \geq n$?* The Borsuk–Ulam theorem has many important corollaries, including the ham-sandwich theorem, the necklace splitting thoerem, and the Lyusternik–Shnirelman–Borsuk covering theorem [25]. *What generalizations of these corollaries follow from our Borsuk–Ulam theorems for $S^n \to \mathbb{R}^k$ with $k \geq n$?*

Gromov–Hausdorff distances. In [4], we use Vietoris–Rips complexes and generalized Borsuk–Ulam theorems to provide new lower bounds on Gromov–Hausdorff distances between spheres. The idea is as follows. The Borsuk–Ulam states there is no odd continuous map $S^k \to S^n$ for $k > n$. Given such an odd map, we use Vietoris–Rips complexes and tools from [6] to quantify how discontinuous such a map must be. We then deduce how these values provide conjecturally tight lower bound the Gromov–Hausdorff distance between S^k and S^n, generalizing and improving upon the bounds in [24]. Our machinery, involving equivariant topology and Vietoris–Rips complexes, can bound the Gromov–Hausdorff distances between many families of metric spaces.

The Applied Algebraic Topology Research Network (AATRN). As the Executive Director of the Applied Algebraic Topology Research Network, I provide opportunities for members of our community to advertise their work. Our network hosts an online seminar, with over 488 recorded videos posted to our YouTube Channel, which has 4,760 subscribers and around 24 hours watched per day. AATRN is inviting the community to help deepen the connections between applied and quantitative topology: I co-organize the AATRN Vietoris–Rips seminar (36 talks in 2020–2022), and I co-organized a 2022 AATRN workshop on *Bridging Applied and Quantitative Topology.*

Conclusion. Demands for ways to process, analyze, and understand complex datasets have grown over the past decade, motivating the need for techniques that summarize the shape of data. The main algorithm in applied topology, persistent homology, uses Vietoris–Rips complexes to provide insight into the shape of nonlinear, high-dimensional, time-varying, and noisy datasets at multiple resolutions or scales. A proper treatment of the mathematical foundations behind this novel data analysis technique is necessary in order to develop algorithms which are interpretable, so that scientists can trust the conclusions they draw from the shape of data. Our work provides these mathematical foundations by connecting applied topology to quantitative topology.

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