Research Statement, 2016

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I am interested in computational topology and geometry, combinatorial topology, and topology applied to data analysis and to sensor networks. My research:

§1. **Advances the theory of Vietoris–Rips simplicial complexes.** Given a set of points X sampled from a metric space M, what information can one recover about M? One approach is to build a Vietoris–Rips simplicial complex, which depends on the choice of a scale parameter r, on top of vertex set X. If M is a Riemannian manifold, X is sufficiently dense, and r is sufficiently small, then the Vietoris–Rips complex of X at scale r is homotopy equivalent to M. In practice (e.g. when trying to estimate the shape of a data set X) one does not know how to choose scale r, and so the philosophy of persistent homology is to vary the scale from small to large and to trust those topological features which persist. However, the theory of Vietoris–Rips complexes is very poorly understood as the scale r increases, even though such complexes arise naturally in applications of persistent homology. My research addresses the following question: how do Vietoris–Rips complexes of manifolds behave as we increase the scale parameter? Surprising answers arise: as a first example, the Vietoris–Rips complexes of the circle obtain the homotopy types of the circle, 3-sphere, 5-sphere, 7-sphere, $\ldots$, as scale r increases.

§2. **Applies topology to data analysis and sensor networks.** I work on provably-stable methods for combining persistent homology with machine learning techniques. I also study a Morse-theoretic approach, using the nudged elastic band method from computational chemistry, to build coarse cell complex models for data. These methods have been applied to data arising from social networks, image processing, computer vision, microarray analysis, discrete dynamical systems, and PDEs.

In minimal sensor network problems, one is given only local data measured by many weak sensors but tries to answer a global question. For example, if sensors are scattered in a domain and each cannot measure its location but instead only the identities of its neighboring sensors, can we determine if the entire domain is covered? Topological tools (including Vietoris–Rips complexes) are useful for this passage from local to global.

Together these topics form a unified program in applied topology, with persistent homology and Vietoris–Rips complexes as recurring tools.

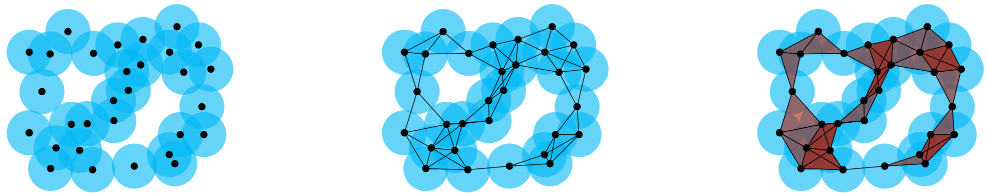
§1. The theory of Vietoris–Rips simplicial complexes

Large sets of high-dimensional data are common in most branches of science, and their shapes reflect important patterns within. The goal of topological data analysis is to measure and describe the shape of data, and one frequently used tool is persistent homology [14].

How can we recover the shape of a data set X sampled from a metric space M? Given a choice of scale r, the Vietoris–Rips complex connects nearby data points.

Definition. *Given metric space X and scale parameter $r \geq 0$, the* Vietoris–Rips simplicial complex $\mathrm{VR}(X, r)$ *contains a finite subset $\sigma \subseteq X$ as a simplex if its diameter is at most r.*

Since we do not know a priori how to choose the scale r, the idea of persistent homology is to compute the homology of the Vietoris–Rips complex of data set X over a large range of scale parameters r and to trust those topological features which persist. The motivation for



Data set or sensor network Connectivity graph Vietoris–Rips complex

using Vietoris–Rips complexes are Hausmann's and Latschev's reconstruction results [18, 19]: for M a Riemannian manifold, scale r sufficiently small, and data set $X \subset M$ sufficiently dense, we have a homotopy equivalence $\mathrm{VR}(X, r) \simeq M$. But as the main idea of persistence is to allow scale r to vary, the assumption that scale r is kept sufficiently small does not hold in practical applications.

Question. *How do the homotopy types of Vietoris–Rips complexes of manifolds change as the scale parameter r increases?*

The above question is fundamental for applications of persistent homology to data analysis, since as finite subset $X \subset M$ gets denser and denser, the persistent homology of X converges to the persistent homology of manifold M [16].

Theorem 1 ([1])**.** *As scale r increases, the Vietoris–Rips complex of the circle is homotopy equivalent to the circle, the 3-sphere, the 5-sphere, the 7-sphere, . . . , until finally it is contractible.*

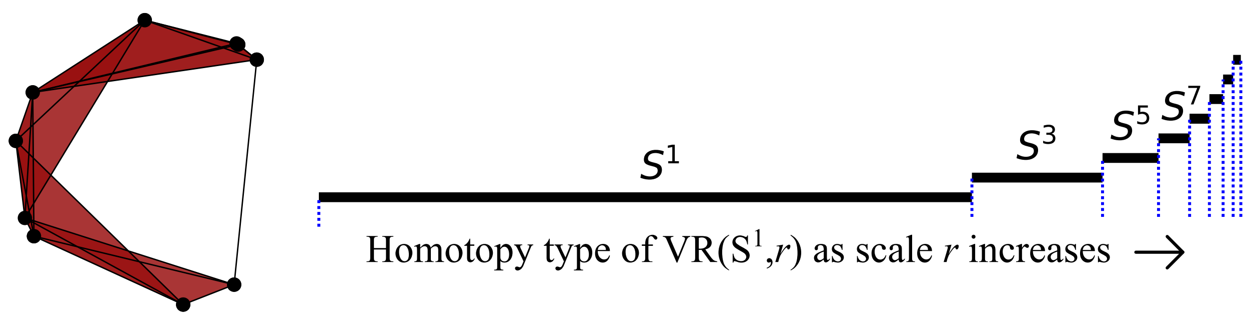


FIGURE 2. *(Left)* A Vietoris–Rips complex of 9 points on the circle. *(Right)* As the scale parameter increases, the Vietoris–Rips complex of the entire circle obtains the homotopy types of the circle, the 3-sphere, the 5-sphere, the 7-sphere, . . . , until finally it is contractible.

The circle is the first connected non-contractible Riemannian manifold for which the homotopy type of the Vietoris–Rips complex is known at all choices of scale. We also classify the homotopy types of Vietoris–Rips complexes of d-dimensional tori with the l_∞ metric.

Theorem 1 relies upon my joint work [3] describing the homotopy types of nerve complexes of circular acs, which also has applications to cyclic polytopes, to sizes of gaps between roots of trigonometric polynomials, and to the Lovász bound on the chromatic number of circular complete graphs. Theorem 1 is strengthened in my joint work [4] studying the Vietoris–Rips complexes of random samples from the circle, and generalized to ellipses and regular polygons in joint work with undergraduate researchers [5, 12]

My collaborators and I in [2] extend Hausmann and Latschev's results to metric reconstruction (strengthening homotopy type reconstruction); our proofs are also simpler. Our techniques also also allow us to identify the first new homotopy type that appears in the Vietoris–Rips thickening of a higher-dimensional sphere S^{n-1} as the scale increases (as the n-fold suspension of $\mathrm{SO}(n)$ modulo the alternating group on $n+1$ elements). This project, the theory of metric reconstruction via optimal transport and Vietoris–Rips thickening, will be the focus of my research for the next few years. Indeed, my student Joshua Mirth and I have extended Hausmann's theorem from Riemannian manifolds to Euclidean manifolds [13].

Future Work. *As the scale r increases, how does the homotopy type of the Vietoris–Rips complex of a broader class of spaces, such as d-spheres for $d \geq 2$, behave? For a Riemannian manifold M, is the connectivity (dimension up to which all homotopy groups vanish) of Vietoris–Rips complex $\mathrm{VR}(M,r)$ a non-decreasing function of r?*

§2. Persistent homology applied to data analysis and sensor networks

Data analysis. Persistent homology describes the shape of a data set, but in applications one often wants to perform machine learning tasks such as classification or feature identification. The most useful input to many machine learning algorithms is a vector. In [11] my collaborators and I stably convert persistent homology output into a vector, a *persistence image*, on which we apply a suite of machine learning tools (clustering, K-medoids, support vector machines).

Theorem 2 ([11]). *The map [persistent homology] $\to$ [persistence image] is Lipschitz.*

Though persistent homology may measure the homology groups of a data set, it remains to build a coarse geometric model realizing its shape. My collaborators and I [8] introduce a method for geometrically modelling data sets that may be genuinely nonlinear. We adapt Morse theory to the setting of point clouds, i.e. finite sets of points in Euclidean space, using a kernel density estimator as the analogue of the Morse function. To sample cells from the skeleton of the Morse complex we use the nudged elastic band method from computational chemistry, and in accordance with the idea of topological persistence, produce a multiscale sequence of cell complexes modeling the dense regions of the data. We test our approach on a variety of data sets arising in social networks, image processing, computer vision, and microarray analysis.

Sensor networks. The idea behind minimal sensor network applications is to deploy a large collection of cheap devices, perhaps not even equipped with GPS receivers, and then to combine local measurements to answer a global question. Topological tools are useful for this passage from local to global [17]. For example, suppose that ball-shaped sensors wander in a bounded domain. A sensor can't measure its location (no GPS) but does know when it overlaps a nearby sensor. Using only this local information, we would like to know if an evasion path exists, that is, if a moving evader can avoid being detected by the sensors.

Theorem 3 ([7, 10]). *If there is an evasion path in a mobile sensor network, then there is a full-length interval in the (top–1)-dimensional "zigzag" persistent homology of the time-varying covered region (approximated by a Vietoris–Rips complex).*

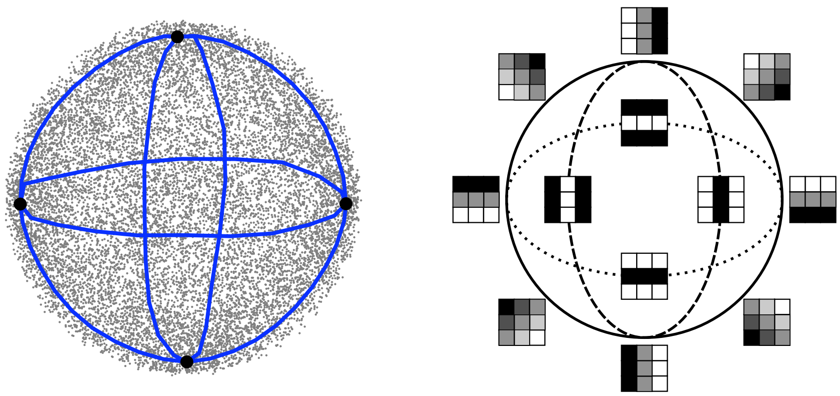


FIGURE 3. *(Left)* An image processing data set of 3×3 pixel patches, and our cellular model containing four 0-cells and eight 1-cells [8]. *(Right)* The model's interpretation. The most common non-constant 3×3 patches are linear gradients at all angles and quadratic gradients in the preferred horizontal and vertical directions. We obtain a geometric representation of the three-circle model from [15], as opposed to only computing its homology groups.

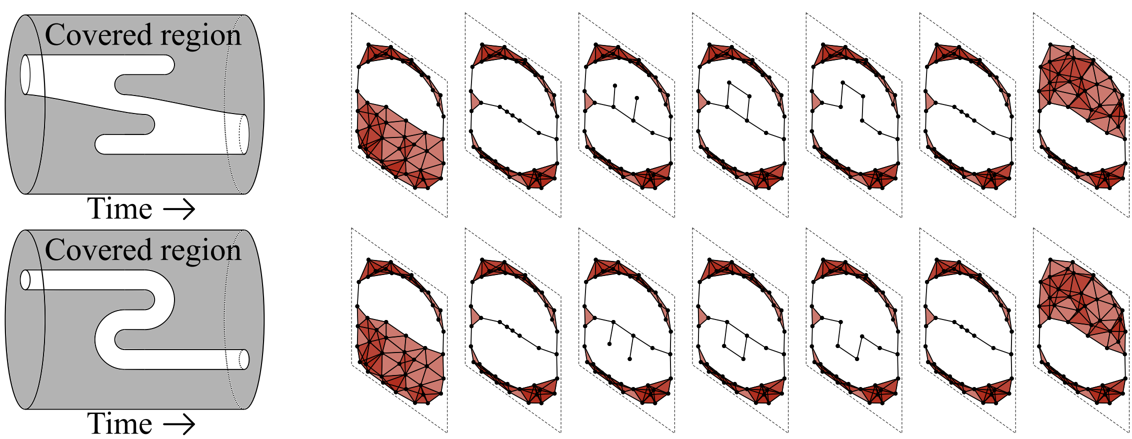


FIGURE 4. Each row is a sensor network, represented both as a covered region in spacetime and by seven sequential Vietoris–Rips simplicial complexes. Surprisingly, the two covered regions are homotopy equivalent through time-preserving maps. The top network contains an evasion path, but the bottom network does not since the evader cannot travel backwards in time. Hence the existence of evasion path depends not only on the time-varying homotopy type, but also on the embedding.

We compute this criterion in a streaming fashion. My work [6] with two undergraduates studies the shortest length of sensor curve necessary to provide a sweep of the domain.

I am a coauthor of the tutorial for Javaplex [20], a software package for persistent homology and data analysis. The tutorial, which contains exercises, solutions, and real life data sets including laser scanner images [9], is a popular tool for learning to apply topology to data.

References

[1] Michał Adamaszek and Henry Adams. The Vietoris–Rips complexes of a circle. *Pacific Journal of Mathematics*, 290, 2017.

[2] Michał Adamaszek, Henry Adams, and Florian Frick. Metric reconstruction via optimal transport. Preprint, arXiv:1612.03540, 2017.

[3] Michał Adamaszek, Henry Adams, Florian Frick, Chris Peterson, and Corrine Previte-Johnson. Nerve complexes of circular arcs. *Discrete & Computational Geometry*, 56:251–273, 2016.

[4] Michał Adamaszek, Henry Adams, and Francis Motta. Random cyclic dynamical systems. *Advances in Applied Mathematics*, 83:1–23, 2017.

[5] Michał Adamaszek, Henry Adams, and Samadwara Reddy. On Vietoris–Rips complexes of ellipses. Preprint, `arxiv/1704.04956`, 2017.

[6] Brooks Adams, Henry Adams, and Colin Roberts. Sweeping costs of planar domains. To appear in the AWM-IMA Springer series. Available at `arXiv/1612.03540`, 2017.

[7] Henry Adams. *Evasion paths in mobile sensor networks*. PhD thesis, Stanford University, 2013.

[8] Henry Adams, Atanas Atanasov, and Gunnar Carlsson. Nudged elastic band in topological data analysis. *Topological Methods in Nonlinear Analysis*, 45(1):247–272, 2015.

[9] Henry Adams and Gunnar Carlsson. On the nonlinear statistics of range image patches. *SIAM Journal on Imaging Sciences*, 2(1):110–117, 2009.

[10] Henry Adams and Gunnar Carlsson. Evasion paths in mobile sensor networks. *International Journal of Robotics Research*, 34(1):90–104, 2015.

[11] Henry Adams, Sofya Chepushtanova, Tegan Emerson, Eric Hanson, Michael Kirby, Francis Motta, Rachel Neville, Chris Peterson, Patrick Shipman, and Lori Ziegelmeier. Persistence images: A stable vector representation of persistent homology. *Journal of Machine Learning Research*, 18(8):1–35, 2017.

[12] Henry Adams, Samir Chowdhury, Adam Jaffe, and Bonginkosi Sibanda. Vietoris–Rips complexes of regular polygons. Preprint, 2017.

[13] Henry Adams and Joshua Mirth. Vietoris–Rips thickenings of Euclidean submanifolds. Preprint, 2017.

[14] Gunnar Carlsson. Topology and data. *Bulletin of the American Mathematical Society*, 46(2):255–308, 2009.

[15] Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian. On the local behavior of spaces of natural images. *International Journal of Computer Vision*, 76(1):1–12, 2008.

[16] Frédéric Chazal, Vin de Silva, and Steve Oudot. Persistence stability for geometric complexes. *Geometriae Dedicata*, pages 1–22, 2013.

[17] Vin de Silva and Robert Ghrist. Coordinate-free coverage in sensor networks with controlled boundaries via homology. *The International Journal of Robotics Research*, 25(12):1205–1222, 2006.

[18] Jean-Claude Hausmann. On the Vietoris–Rips complexes and a cohomology theory for metric spaces. *Annals of Mathematics Studies*, 138:175–188, 1995.

[19] Janko Latschev. Vietoris–Rips complexes of metric spaces near a closed Riemannian manifold. *Archiv der Mathematik*, 77(6):522–528, 2001.

[20] Andrew Tausz, Mikael Vejdemo-Johansson, and Henry Adams. Javaplex: A research software package for persistent (co)homology. In *International Congress on Mathematical Software*, pages 129–136, 2014. Software available at `http://appliedtopology.github.io/javaplex/`.