Metric Thickenings of Euclidean Submanifolds

Advisor: Dr. Henry Adams
Committee: Dr. Chris Peterson, Dr. Daniel Cooley

Joshua Mirth
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Introduction
Can we recover the object on the right from the one on the left?
Motivation

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A reconstruction method should work given a perfect sample.
Background
Definition

Let $V$ be a set, called the set of vertices. An \textit{abstract simplicial complex} $K$ on vertex set $V$ is a subset of the power set of $V$ with the property that if $\sigma \in K$, then all subsets of $\sigma$ are in $K$. For example:

$V = \{a, b, c, d, e\}$

$K = \{\{abc\}, \{ac\}, \{bc\}, \{ac\}, \{ad\}, \{cd\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$
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Simplicial Complexes

Every simplicial complex has a geometric realization:

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The topology on a finite simplicial complex is the subspace topology of its geometric realization in $\mathbb{R}^n$. 
The Vietoris–Rips Complex

**Definition**

Let $X$ be a metric space and $r > 0$ a scale parameter. The **Vietoris–Rips complex**, $\text{VR}_\leq(X; r)$, of $X$, has vertex set $X$ and a simplex for every finite subset $\sigma \subseteq X$ such that $\text{diam}(\sigma) \leq r$. 
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![Diagram showing a Vietoris–Rips complex with vertices a, b, c, d, e connected by edges based on the scale parameter r.](attachment:diagram.png)
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![Diagram showing the Vietoris–Rips complex with vertices a, b, c, d, e and a triangle formed by b, c, and a, with e and d outside the triangle, indicating the complex's structure based on distance.]
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Definition

Let $X \subseteq Y$ be a submetric space and $r > 0$ a scale parameter. The Čech complex $\check{C}_\leq(X, Y; r)$, of $X$, has vertex set $X$ and a simplex for every finite subset $\sigma \subseteq X$ such that

$$\bigcap_{x_i \in \sigma} \overline{B}(x_i, r/2) \neq \emptyset.$$
The Čech Complex
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**Definition**

Let $f: X \to Y$ and $g: X \to Y$ be continuous maps. Then $f$ is homotopic to $g$, denoted $f \simeq g$, if there exists a continuous function $H: X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$. 

Let $X$ and $Y$ be topological spaces. Then $X$ is homotopy equivalent to $Y$, written $X \simeq Y$, if there exists a pair of continuous functions $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. 
**Homotopy Equivalence**

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Lemma (Nerve Lemma: Convex Version)

Let $U_\alpha$ for $\alpha \in A$ an index set be convex subsets of $\mathbb{R}^n$. Then $\mathcal{N}(\{U_\alpha\}) \simeq \bigcup_{\alpha \in A} U_\alpha$. 

The ˇCech complex is the nerve of balls of radius $r/2$, so it is homotopy equivalent to the underlying space for a good cover.
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Hausmann’s Theorem

**Theorem**

Let $M$ be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then $\text{VR}(M; r) \simeq M$ \[4\].
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- $\text{VR}(M; r)$ does not inherit the metric of $M$. Thus:
  - Hausmann’s proof only gives a map $T: \text{VR}(M; r) \to M$, and proves the equivalence using algebraic techniques.
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  - $T$ depends upon a total order of the points in $M$.
  - In particular, the inclusion $\iota: M \hookrightarrow \text{VR}(M; r)$ does not provide the inverse (in fact, $\iota$ is not even continuous.)
Metric Thickenings
The Metric thickening of a simplicial complex was first introduced by Adamaszek, Adams, and Frick [1].

It puts the 1-Wasserstein metric on the geometric realization of a simplicial complex.

This lets us use the theory of metric spaces to prove results analogous to Hausmann and the Nerve Lemma.
**Definition (Adamaszek, Adams, Frick)**

For a metric space $X$ and $r \geq 0$, the Vietoris–Rips thickening $\text{VR}^m(X; r)$ is the set

$$\text{VR}^m(X; r) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N}, x_i \in X, \text{ and diam} \{x_0, \ldots, x_k\} \leq r \right\},$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$, equipped with the 1-Wasserstein metric.[1]
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- By identifying $x \in X$ with $\delta_x \in \mathcal{P}(X)$, we can view $\text{VR}^m(X; r)$ as a subset of $\mathcal{P}(X)$, the set of all Radon probability measures on $X$. 
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- By identifying $x \in X$ with $\delta_x \in \mathcal{P}(X)$, we can view $\text{VR}^m(X; r)$ as a subset of $\mathcal{P}(X)$, the set of all Radon probability measures on $X$.
- This makes $\text{VR}^m(X; r)$ a (metric) thickening of $X$. 

[1]
Let $x, x' \in \text{VR}^m(X; r)$ with $x = \sum_{i=0}^{k} \lambda_i x_i$ and $x' = \sum_{i=0}^{k'} \lambda'_i x'_i$. Define a matching $p$ between $x$ and $x'$ to be any collection of non-negative real numbers \{p_{i,j}\} such that $\sum_{j=0}^{k'} p_{i,j} = \lambda_i$ and $\sum_{i=0}^{k} p_{i,j} = \lambda'_j$. Define the cost of the matching $p$ to be $\text{cost}(p) = \sum_{i,j} p_{i,j} d(x_i, x'_j)$.

**Definition**

The 1-Wasserstein metric on $\text{VR}^m(X; r)$ is the distance $d_W$ defined by

$$d_W(x, x') = \inf \left\{ \text{cost}(p) \mid p \text{ is a matching between } x \text{ and } x' \right\}.$$
Euclidean Submanifolds
Sets of Positive Reach

We will prove an analogue of Hausmann’s theorem in the context of subsets of Euclidean space, rather than Riemannian manifolds.

This is a natural setting for data analysis.

Positive reach was first introduced by Federer [3].

In particular, any $C^k$ submanifold of $\mathbb{R}^n$ has positive reach, for $k \geq 2$, so sets of positive reach include many potentially interesting objects.
Sets of Positive Reach

The medial axis of $X \subseteq \mathbb{R}^n$ is the closure, $\overline{Y}$, of

$$Y = \{ y \in \mathbb{R}^n \mid \exists x_1 \neq x_2 \in M \text{ with } d(y, x_1) = d(y, x_2) = d(y, X) \}.$$

The reach, $\tau$, of $X$ is the minimal distance $\tau = d(X, \overline{Y})$ between $X$ and its medial axis.
Sets of Positive Reach

- Sets with “corners” have zero reach.

\[ \tau = 0 \]

\[ \tau = r \]
Sets of Positive Reach

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- Sets with “corners” have zero reach.
- Smooth manifolds embedded in $\mathbb{R}^n$ have positive reach.
- Reach is $\leq$ half the distance between non-connected components.
Define the $\alpha$-offset of $X \subseteq \mathbb{R}^n$:

$$\text{Tub}_\alpha = \{ x \in \mathbb{R}^n \mid d(x, X) < \alpha \} = \bigcup_{x \in X} B(x, \alpha).$$

If $X$ has reach $\tau$, then $\pi: \text{Tub}_\tau \to X$ where $x$ maps to its nearest point in $X$ is well-defined and continuous [3].
Proposition (Niyogi, Smale, Weinberger)

Let $X \subseteq \mathbb{R}^n$ have reach $\tau > 0$. Let $p \in X$ and suppose $x \in \text{Tub}_\tau \setminus X$ satisfies $\pi(x) = p$. If $c = p + \tau \frac{x - p}{\|x - p\|}$, then $B(c, \tau) \cap X = \emptyset$.

Proof.

For any $0 < t < \tau$, let $y_t = p + t \frac{x - p}{\|x - p\|}$. Since $y_t \in \text{Tub}_\tau$, we have $\overline{B}(y_t, t) \cap X = \{p\}$ and $d(y_t, p) = t$, so $B(c, t) \cap X = \emptyset$. Note that $B(c, \tau) = \bigcup_{0 < t < \tau} B(y_t, t)$. Indeed, to see the inclusion $\subseteq$, suppose that $z \in B(c, \tau)$, so that $d(z, c) = \tau - \epsilon$ for some $\epsilon > 0$. Let $t = \tau - \frac{\epsilon}{3}$. By the triangle inequality, $d(y_t, z) \leq d(y_t, c) + d(c, z) = \tau - \frac{2\epsilon}{3} < t$, giving $z \in B(y_t, t)$. The reverse inclusion $\supseteq$ is straightforward. It follows that $B(c, \tau) \cap X = \emptyset$. \qed
Results
Main Theorem

**Theorem (Metric Hausmann)**

Let $X \subseteq \mathbb{R}^n$ and suppose the reach $\tau$ of $X$ is positive. Then for all $r < \tau$, the metric Vietoris–Rips thickening $\text{VR}^m(X; r)$ is homotopy equivalent to $X$. 
Main Theorem

Theorem (Metric Hausmann)

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\[ \pi \quad \text{VR}^m(X; r) \quad \text{X with Tub}_\tau \]
Theorem (Metric Nerve Theorem)

Let $X$ be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach $\tau$ of $X$ is positive. Then for all $r < \tau$, the metric Čech thickening $\check{C}^m(X; 2r)$ is homotopy equivalent to $X$. 

\[
\xymatrix{ f \ar@{~}[r] & X \text{ with } \text{Tub}_\tau \ar@{~}[r]^-{\pi} & \check{C}^m(X; 2r) \ar@{~}[r]^-{i} & X }
\]
Lemma

For $X \subseteq \mathbb{R}^n$ and $r > 0$, the linear projection map $f : \text{VR}^m(X; r) \to \mathbb{R}^n$ has its image contained in $\overline{\text{Tub}_r}$. 
Lemma

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Proof.

Let $x = \sum_{i=0}^{k} \lambda_i x_i \in \text{VR}^m(X;r)$; we have

$$\text{diam}(\text{conv}\{x_0, \ldots, x_k\}) = \text{diam}([x_0, \ldots, x_k]) \leq r.$$ 

Since $f(x) \in \text{conv}\{x_0, \ldots, x_k\}$, it follows that $d(f(x), X) \leq d(f(x), x_0) \leq r$, and so $f(x) \in \overline{Tub}_r$. \qed
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Lemma

Let $x_0, \ldots, x_k \in \mathbb{R}^n$, let $y \in \text{conv}\{x_0, \ldots, x_k\}$, and let $C$ be a convex set with $y \notin C$. Then there is at least one $x_i$ with $x_i \notin C$.  

22
Lemma

Let $X \subseteq \mathbb{R}^n$ have positive reach $\tau$, let $[x_0, \ldots x_k]$ be a simplex in $\text{VR}(X; r)$ with $r < \tau$, let $x = \sum \lambda_i x_i \in \text{VR}^m(X; r)$, and let $p = \pi(f(x))$. Then the simplex $[x_0, \ldots, x_k, p]$ is in $\text{VR}(X; r)$.

Proof.
We are now prepared to prove our main result.

**Theorem**

Let $X$ be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach $\tau$ of $X$ is positive. Then for all $r < \tau$, the metric Vietoris–Rips thickening $\text{VR}^m(X; r)$ is homotopy equivalent to $X$. 
Proof.

By [1, Lemma 5.2], map \( f : \text{VR}^m(X; r) \to \mathbb{R}^n \) is 1-Lipschitz and hence continuous. It follows from Lemma 12 that the image of \( f \) is a subset of \( \text{Tub}_\tau \). Let \( i : X \to \text{VR}^m(X; r) \) be the inclusion map. Note that \( \pi \circ f \circ i = \text{id}_X \).
Proof.

Consider $H : \text{VR}^m(X; r) \times I \to \text{VR}^m(X; r)$ defined by $H(x, t) = t \cdot \text{id}_{\text{VR}^m(X; r)} + (1 - t)i \circ \pi \circ f$. $H$ is well-defined by Lemma 14, and continuous by [1, Lemma 3.8]. It follows that $H$ is a homotopy equivalence from $i \circ \pi \circ f$ to $\text{id}_{\text{VR}^m(X; r)}$. 
**Theorem**

Let $X$ be a subset of Euclidean space $\mathbb{R}^n$, equipped with the Euclidean metric, and suppose the reach $\tau$ of $X$ is positive. Then for all $r < \tau$, the metric Čech thickening $\check{C}^m(X; 2r)$ is homotopy equivalent to $X$.

**Proof.**

The proof uses similar techniques to that of Theorem 15.
Conclusion
Conclusions

• Metric analogue of Hausmann in Euclidean space.
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- For a Riemannian version see [1]. Or:

**Corollary**

*If $N$ is a smooth, compact, Riemannian manifold, there exists a $\tau > 0$ such that $\text{VR}^m(N; r) \simeq N$ for all $0 < r < \tau$.***

**Proof.**

This follows from the Nash Embedding theorem [7].
Conclusions

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*If* $N$ *is a smooth, compact, Riemannian manifold, there exists a* $\tau > 0$ *such that* $\VR^m(N; r) \simeq N$ *for all* $0 < r < \tau$.

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- The same techniques hold for metric Čech thickenings.
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**Proof.**

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- The same techniques hold for metric Čech thickenings.
- Worth considering version for dense-samplings [6, 2].


