Summary of earlier meetings & problem sets with old (pre 1984) & new numbering.

1975 Asilomar 75:01–75:23
1976 San Diego 1–65 i.e., 76:01–76:65
1977 Los Angeles 101–148 i.e., 77:01–77:48
1978 Santa Barbara 151–187 i.e., 78:01–78:37
1979 Asilomar 201–231 i.e., 79:01–79:31
1980 Tucson 251–268 i.e., 80:01–80:18
1981 Santa Barbara 301–328 i.e., 81:01–81:28
1982 San Diego 351–375 i.e., 82:01–82:25
1983 Asilomar 401–418 i.e., 83:01–83:18

[With comments on 76:60, 86:05, 88:06, 93:20, 95:18, and 97:22]

COMMENTS ON ANY PROBLEM WELCOME AT ANY TIME
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Comments on Earlier Problems

76:60 (Peter Weinberger) Let \(|f|\) denote the number of non-zero coefficients of a polynomial \(f\). Is there a function \(A\) such that \(|(f,g)| \leq A(|f|,|g|)\)? Can such an \(A\) be a polynomial? The example \(f = (x^{ab} + 1)(x^b + 1)/(x + 1)\), \(g = (x^{ab} + 1)(x^b + 1)/(x^a + 1)\) with \(a > b - 1\), \(a\) even, \(b\) odd shows that if such an \(A\) exists then \(A(n,n) \gg n^2\).

**Solution:** Andrzej Schinzel writes that the answer to this problem is negative, and a simple counterexample is \(f = x^{ab} - 1\), \(g = (x^a - 1)(x^b - 1)\), where \(|f| = 2\), \(|g| = 4\) and \(|(f,g)|\) can be arbitrarily large. The only difficult case in characteristic 0 is \(|f| = |g| = 3\).

86:05 (Michael Filaseta) Is \(f_n(x) = d/dx (x^n + x^{n-1} + \cdots + x + 1)\) irreducible for all positive integers \(n\)? For almost all \(n\)?

**Solution:** The “almost all” question is answered in the affirmative in

A. Borisov, M. Filaseta, T. Y. Lam, O. Trifonov, Classes of polynomials having only one non-cyclotomic irreducible factor, Acta Arith. 90 (1999) 121–153,

where Theorem 1 states that “if \(\epsilon > 0\) then for all but \(O(t^{1/3+\epsilon})\) positive integers \(n \leq t\) the derivative of the polynomial \(f(x) = 1 + x + x^2 + \cdots + x^n\) is irreducible.”

88:06 (Emil Grosswald) Mike Filaseta proved that almost all Bessel polynomials \([\text{polynomial solutions of } x^2 y'' + xy' - n(n+1)y = 0 \text{ with } y(0) = 1]\) are irreducible over \(\mathbb{Q}\). Get rid of “almost all”.

**Solution:** In work submitted for publication, Filaseta and Trifonov write the Bessel polynomials as

\[
y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{2^j(n-j)!j!} x^j
\]

and prove that if \(n\) is a positive integer and \(a_0, a_1, \ldots, a_n\) are arbitrary integers with \(|a_0| = |a_n| = 1\) then

\[
\sum_{j=0}^{n} a_j \frac{(n+j)!}{2^j(n-j)!j!} x^j
\]

is irreducible.

The techniques are similar to those used in


93:20 (Eugene Gutkin via Jeff Lagarias) [...] consider the polynomials

\[
p_n(x) = \frac{(n-1)(x^{n+1} - 1) - (n+1)(x^n - x)}{(x-1)^3}
\]

[which arise in the solution of \(\tan n\theta = n \tan \theta\)] for \(n \geq 1\).

**Conjecture.** \(p_n(x)\) is irreducible if \(n\) is even, and is \(x + 1\) times an irreducible if \(n\) is odd.

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Solution: This is true for almost all \( n \). Theorem 4 of the four-author paper cited above states that if \( \epsilon > 0 \) then for all but \( O(t^{1/5+\epsilon}) \) positive integers \( n \leq t \) the polynomial \( p(x) = (n-1)(x^{n+1} - 1) - (n+1)(x^n - x) \) is \((x-1)^3 \) times an irreducible polynomial if \( n \) is even and \((x-1)^3(x+1) \) times an irreducible polynomial if \( n \) is odd.

95:18 (Martin LaBar, via Richard Guy) Is there a 3 × 3 magic square with distinct square entries?

Remark: Comments on this problem have appeared in each problem set since it was first proposed.


97:22 (John Selfridge) Let \( n = rs^2 \), \( r \) square-free, \( r > 1 \). It is conjectured that for all such \( n \) except \( n = 8 \) and \( n = 392 \) there exist integers \( a, b \) with \( n < a < b < r(s+1)^2 \) such that \( nab \) is a square.


Selfridge reports that he and Aaron Meyerowitz have proved that if there is a counterexample \( n > 392 \) then \( n \) is at least on the order of \( 10^{30000} \).

Problems Proposed 16 & 19 Dec 99

99:01 (John Wolfskill) Let \( d \equiv 3 \pmod{4} \) be positive and squarefree. Let a fundamental unit in \( \mathbb{Z} [\sqrt{d}] \) be given by \( \epsilon = a + b \sqrt{d} > 1 \). Characterize those \( d \) for which \( \sqrt{2} \) is in \( \mathbb{Q}(\sqrt{\epsilon}) \).

Remarks: \( \sqrt{2} \) is in \( \mathbb{Q}(\sqrt{\epsilon}) \) for all prime \( d \) and for some but not all composite \( d \).

Gary Walsh shows that the following are equivalent:

a) \( \sqrt{2} \) is in \( \mathbb{Q}(\sqrt{\epsilon}) \);

b) at least one of the equations \( x^2 - dy^2 = \pm 2 \) is solvable in integers \( x \) and \( y \);

c) the prime over 2 in \( \mathbb{Q}(\sqrt{d}) \) is principal.

Characterizing \( d \) such that \( x^2 - dy^2 = -1 \) has a solution is a notorious open question, which suggests that there may be no simple solution to the current problem.

Walsh’s argument, as presented by Wolfskill, runs as follows. Let \( K = \mathbb{Q}(\sqrt{\epsilon}) \), let \( \alpha \) in \( K \) be such that \( \alpha^2 = \epsilon \). Note that the norm of \( \epsilon \) is 1, whence \( K/\mathbb{Q} \) is Galois and non-cyclic. Since \( \alpha \) is in \( K \) we have \( \alpha = r + s \sqrt{d} + t \sqrt{d'} + u \sqrt{dd'} \) for some rational \( r, s, t \) and \( u \) and some \( d' \) with \( \sqrt{d'} \) in \( K \). Let \( \sigma \) be the element of the Galois group of \( K/\mathbb{Q} \) fixing \( \sqrt{d} \) but not fixing \( \sqrt{d'} \). Then \( (\sigma(\alpha))^2 = \sigma(\alpha^2) = \sigma(\epsilon) = \epsilon = \alpha^2 \), so \( \sigma(\alpha) = \alpha \) or \( \sigma(\alpha) = -\alpha \). If \( \sigma(\alpha) = \alpha \) then \( \alpha \) is in \( \mathbb{Q}(\sqrt{d}) \) but then \( \alpha^2 = \epsilon \) contradicts the hypothesis that \( \epsilon \) is a fundamental unit in \( \mathbb{Q}(\sqrt{d}) \), so \( \sigma(\alpha) = -\alpha \), so \( \alpha = t \sqrt{d'} + u \sqrt{dd'} \).
Now assume \( \sqrt{2} \) is in \( K \), so \( \alpha = t\sqrt{2} + u\sqrt{2d} \), \( t \) and \( u \) rational. From \( \alpha^2 = \epsilon \) we get that \( 2(t^2 + du^2) = a \) and \( 4tu = b \) are both integers, from which it is easy to deduce that \( 2t = x \) (say) and \( 2u = y \) (say) are integers. Then \( (x^2 - dy^2)^2 = 4(a^2 - db^2) = 4 \), so \( x^2 - dy^2 = \pm 2 \).

Conversely, suppose \( x \) and \( y \) are positive integers such that \( x^2 - dy^2 = \pm 2 \). Note that \( x \) and \( y \) are odd. Let \( a = (x^2 + dy^2)/2, b = xy \). Then \( a^2 - db^2 = 1 \), so \( a + b\sqrt{d} \) is a unit in \( \mathbb{Q}(\sqrt{d}) \). Also, \( \left( \frac{x}{2}\sqrt{2} + \frac{y}{2}\sqrt{2d} \right)^2 = a + b\sqrt{d} \), so \( a + b\sqrt{d} \) must be an odd power of the fundamental unit in \( \mathbb{Q}(\sqrt{d}) \)—otherwise, \( \frac{x}{2}\sqrt{2} + \frac{y}{2}\sqrt{2d} \) would be in \( \mathbb{Q}(\sqrt{d}) \). So, \( \sqrt{2} \) is in \( \mathbb{Q}(\sqrt{\epsilon}) \).

99:02 (Greg Martin) Consider the following “proof” that 4680 is perfect: \( 4680 = 2^3 \cdot 3^2 \cdot (-5) \cdot (-13) \), so \( \sigma(4680) = (1+2+2^2+2^3)(1+3+3^2)(1+(-5))(1+(-13)) = 9360 = 2 \times 4680 \). Allowing the use of \( \sigma(-p^n) = \sum_{j=0}^{n}(-p)^j \), is there a “spoof perfect number” with exactly 3 distinct prime factors?

Remark: If so, it must be negative.

Solution: Dennis Eichhorn found that \(-84 = 2^2(3)(-7)\) is spoof-perfect, and Eichhorn and Peter Montgomery independently found that \(-120 = 2^3(3)(-5)\) is spoof-perfect. Montgomery also found that \(-672 = (-2)^5(3)(7)\) leads to

\[
\sigma(-672) = (1 - 2 + 4 - 8 + 16 - 32)(1 + 3)(1 + 7) = -672.
\]

Alf van der Poorten asked whether there are any odd spoof-perfects.

John Selfridge asked whether 4680 is the smallest positive spoof-perfect.

See also 99:08, below.

99:03 (Mike Filaseta) Find \( m_0 \) such that if \( m \geq m_0 \) and \( m(m - 1) = 2^a 3^b m' \) and \( (m', 6) = 1 \) then \( m' > \sqrt{m} \).

Remark: See


for a similar but ineffective result derived from work of Mahler.

99:04 (Mike Filaseta) Show that every \( n \times n \) integer matrix, \( n \geq 2 \), is a sum of 3 squares of \( n \times n \) integer matrices.

Remark: What is wanted is an argument more transparent than that in


99:05 (Zachary Franco) Call \( n \) equidigital if each digit occurs equally often in the repeating block in the decimal expansion of \( 1/n \). It is easy to see that if \( p \) is prime and 10 is a primitive root \( \pmod{p} \) then \( p \) is equidigital. Are there any equidigital primes \( p \) for which 10 is not a primitive root?

Remarks: The answer to the corresponding question in base 2 is yes; 2 is not a primitive root \( \pmod{17} \) but the binary expansion of 1/17 is .00001111.

There are equidigital composites, e.g., \( n = 1349 = 19 \times 71 \).
Mike Filaseta notes that if \( p \equiv 11 \pmod{20} \) is prime and 10 is of order \((p - 1)/2 \pmod{p}\) then \(10^k\) runs through the quadratic residues \(\pmod{p}\), and since there are more quadratic residues in \([1, (p - 1)/2]\) than in \([(p + 1)/2, p - 1]\) for such \( p \) \((p \equiv 3 \pmod{4})\) \( p \) can’t be equidigital. For example, \(1/31 = .03225806451612\) has 9 small digits and 6 large ones. Perhaps there are similar results for 10 of order \((p - 1)/k\) for \( k = 3, 4, \ldots \).

99:06 (Kevin O’Bryant) Write \(\sqrt{a_1,a_2,\ldots}\) for the continued square root

\[
\frac{1}{\sqrt{a_1 + \frac{1}{\sqrt{a_2 + \cdots}}}}
\]

where \(a_1, a_2, \ldots\) are positive integers. Every real number \( r, 0 < r < 1, \) has such an expression, and the expression is unique in the same sense as for simple continued fractions. Does \(3/4\) have a finite continued root?

**Remark:** \(2/3 = \sqrt{2,16}],\ 22/47 = \sqrt{3,1098,2892,410,256}].\)

99:07 (Bart Goddard) Let \( f : (0, \infty) \rightarrow (0, \infty) \) be strictly decreasing and onto with \( f(1) = 1 \). Let \( g \) be the functional inverse \( f^{-1} \) of \( f \). For \( \alpha_0 \) real and positive, define integers \( a_0, a_1, \ldots \) and reals \( \alpha_1, \alpha_2, \ldots \) by \( a_j = \lfloor \alpha_j \rfloor, \alpha_j = g(\alpha_{j-1} - a_{j-1}) \). Write \((\alpha_0)_f\) for the sequence \(a_0, a_1, \ldots\). Let \( c_0 = a_0, c_1 = a_0 + f(a_1), c_2 = a_0 + f(a_1 + f(a_2)) \), etc. Note that \( f(x) = 1/x \) gives the usual continued fraction expansion of \( \alpha_0 \), and \( f(x) = 1/\sqrt{x} \) gives the expansion of 99:06.

Some interesting examples are
\[
\begin{align*}
f(x) &= x^{-5}, \quad (\sqrt[7]{2})_f = (1, 1, 1, \ldots) \\
f(x) &= 1/\Omega(ex), \quad \text{where } \Omega \text{ is the Lambert } \Omega \text{-function}, \\
(\pi)_f &= (3, 3033, 23766810023426903113005, 2279, 2, 864, \ldots)
\end{align*}
\]

1. Given \( f \), which numbers have finite expansions? periodic expansions? Is it true that if \( f(x) = x^{-2/3} \) then \((\sqrt[3]{3})_f = (1, 1, 1, 2)\)?
2. Is there an \( f \) such that \((\alpha)_f \) is periodic for all algebraic \( \alpha \) of degree 3?
3. Find \( f \) such that \((\pi)_f \) has a recognizable pattern.
4. Find \( f \) such that \((e)_f \) is periodic.
5. Find conditions on \( f \) and \( \alpha \) for \( \lim_{n \to \infty} c_n = \alpha \).

**Solution:** (to question 4) Greg Martin notes that if \( f(x) = x^{\log(e-2)/\log(e-1)} \) then \((e)_f = (2, 1, 1, 1, \ldots)\).

**Remark:** Jeff Lagarias refers to


Many later papers refer to this one, as may be seen from the listing on MathSciNet.
99:08 (Greg Martin) Define a multiplicative function $\tilde{\sigma}$ (or $\tilde{\sigma}$ if you are left-handed) by $\tilde{\sigma}(p^r) = p^r - p^{r-1} + p^{r-2} - \cdots + (-1)^r$. Note that $\tilde{\sigma}(n) \leq n$ with equality only for $n = 1$. Call $n$ $\tilde{\sigma}$-perfect if $2\tilde{\sigma}(n) = n$; examples are $n = 2, 12, 40, 252, 880, 10880, 75852$. Call $n$ $\tilde{\sigma}$-$k$-perfect (or, more generally, $\tilde{\sigma}$-multiply perfect) if $k\tilde{\sigma}(n) = n$ for a positive integer $k$.

Two examples of $\tilde{\sigma}$-3-perfects are $n = 30240$ and $n = 2^{10}3^45^411\cdot 13^2\cdot 31\cdot 61\cdot 157\cdot 521\cdot 683$—there are at least 40 $\tilde{\sigma}$-3-perfects.

1. Are there any $\tilde{\sigma}$-$k$-perfect numbers with $k \geq 4$?
2. Are there infinitely many $\tilde{\sigma}$-$k$-perfect numbers?
3. Are there any odd $\tilde{\sigma}$-3-perfect numbers? Any such number must be a square.

Remark: Paraphrasing email from Greg: let $\tau(n) = n/\tilde{\sigma}(n)$, so $\tau(n) = k$ means $n$ is a $\tilde{\sigma}$-$k$-perfect number. Suppose $n = p^{2k-1}m$, $p$ prime, and $\tilde{\sigma}(p^{2k}) = q$ is prime, and $(m, pq) = 1$. Then it’s not hard to prove that $\tau(n) = \tau(npq)$. In particular, if $n$ is $\tilde{\sigma}$-$k$-perfect, so is $npq$.

Some examples of prime powers $p^{2k-1}$ such that $\tilde{\sigma}(p^{2k})$ is prime are

$$2^1, 2^3, 2^5, 2^9, 3^1, 3^3, 5^3, \tau^1, 13^1.$$  

This makes it possible to find 40 $\tilde{\sigma}$-3-perfects from the four examples $2^33^35^27$, $2^53^5\cdot 7$, $2^53^52^73^13$, and $2^93^53^111\cdot 13\cdot 31$.

Jeff Lagarias suggested looking at the Dirichlet series generating function for $\tilde{\sigma}$, in analogy with

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} n^{-s} = \zeta(s+1)\zeta(s).$$

Greg finds that

$$\sum_{n=1}^{\infty} \frac{1}{\tau(n)} n^{-s} = \zeta(2s+2)\zeta(s)/\zeta(s+1),$$

but no such tidy form for $\sum_{n=1}^{\infty} \tau(n)n^{-s}$.

99:09 (Jean-Marie De Koninck) Given an integer $k$, $k \geq 2$, not a multiple of 3,

1. prove that there is a prime whose digits sum to $k$,
2. prove that if $k \geq 4$ then there are infinitely many primes whose digits sum to $k$.

Remarks: Jean-Marie provided a table of values of $\rho(k)$, the smallest prime whose digits add up to $k$, for $2 \leq k \leq 83$, $k$ not a multiple of 3. Your editor notes that $\rho(56) - \rho(55) = 2999999 - 2998999 = 1000$ and asks whether there are infinitely many $k$ with $\rho(k+1) - \rho(k) = 1000$, or with $\rho(k+1) - \rho(k) = 10^m$ for some $m$, or whether there is an integer $r$ with $\rho(k+1) - \rho(k) = r$ for infinitely many $r$.

Your editor further notes that $\rho(34)/\rho(32) = 17989/6899 = 2.61$ (to two decimals), $\rho(37)/\rho(35) = 29898/8999 = 3.33$, $\rho(70)/\rho(68) = 189997999/59999999 = 3.17$, and $\rho(73)/\rho(71) = 289999999/89999999 = 3.22$, and asks whether $\rho(3k+1)/\rho(3k-1)$ is unbounded. Moreover, your editor also notes that $\rho(34)/\rho(35) = 17989/6899 = 2.61$ and asks whether $\rho(k) > \rho(k+1)$ infinitely often.
Further questions: is it true that \( k > 25 \) implies \( \rho(k) \equiv 9 \pmod{10} \)? that \( k > 38 \) implies \( \rho(k) \equiv 99 \pmod{100} \)? that \( k > 59 \) implies \( \rho(k) \equiv 999 \pmod{1000} \)?

Jean-Marie also notes that it is trivial that \( \rho(k) \geq (a + 1)10^b - 1 \), where \( b = [k/9] \) and \( a = k - 9b \); and asks whether equality holds infinitely often. For instance, it is the case when \( k = 5, 7, 10, 11, 14, 16, 17, 19, 22, 23, 28, 29, 31, 35, 40 \).

99:10 (Jeff Lagarias) Is there a field with Galois group \( S_n \), \( n \geq 5 \), whose ring of integers has a power basis?

99:11 (Sinai Robins) Let \( q \) be real, \( |q| < 1 \). Is the function given by \( f(x) = \sum_{n=1}^{\infty} [nx]q^n \) real analytic in \( x \)?

Remark: A starting place for the analytic properties of this and related series is


See also


99:12 (Jeff Lagarias) Given \( n > 3 \), find upper and lower bounds for the number of solutions \( 1 < q_1 < \cdots < q_n \) of the system \( q_j^{-1} \prod_{1 \leq i \leq n} q_i \equiv 1 \pmod{q_j} \), \( j = 1, \ldots, n \).

Remark: It is known that there are only finitely many solutions for each \( n \), in fact there is an upper bound for \( q_n \), but it does not give a good estimate for the number of solutions. \( (2, 3, 5) \) is the only solution for \( n = 3 \). The problem is discussed in

Lawrence Brenton, Mi-Kyung Joo, On the system of congruences \( \prod_{j \neq i} n_j \equiv 1 \pmod{n_i} \), Fib. Q. 33 (1995) 258–267.

The review, MR 96k:11039, is also worth reading.