

THE WALKER ABEL–JACOBI MAP DESCENDS

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ABSTRACT. For a complex projective manifold, Walker has defined a regular homomorphism lifting Griffiths’ Abel–Jacobi map on algebraically trivial cycle classes to a complex abelian variety, which admits a finite homomorphism to the Griffiths intermediate Jacobian. Recently Suzuki gave an alternate, Hodge-theoretic, construction of this Walker Abel–Jacobi map. We provide a third construction based on a general lifting property for surjective regular homomorphisms, and prove that the Walker Abel–Jacobi map descends canonically to any field of definition of the complex projective manifold. In addition, we determine the image of the l -adic Bloch map restricted to algebraically trivial cycle classes in terms of the coniveau filtration.

Let H be a pure integral Hodge structure of weight (-1) . The *Jacobian* attached to H is the complex torus

$$J(H) := F^0 H_{\mathbb{C}} \backslash H_{\mathbb{C}} / H_{\tau},$$

where $F^{\bullet} H_{\mathbb{C}}$ denotes the Hodge filtration on the complexification $H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C}$ and where, for an abelian group G , we denote G_{τ} its torsion-free quotient. If X is a complex projective manifold, then the cohomology groups $H^{2p-1}(X, \mathbb{Z}(p))$ are naturally endowed with the structure of a pure Hodge structure of weight (-1) . In the seminal paper [Gri69], Griffiths defined an Abel–Jacobi map for homologically trivial cycle classes $\mathrm{CH}^p(X)_{\mathrm{hom}} := \ker(\mathrm{CH}^p(X) \rightarrow H^{2p}(X, \mathbb{Z}(p)))$:

$$\mathrm{AJ} : \mathrm{CH}^p(X)_{\mathrm{hom}} \longrightarrow J^{2p-1}(X) := J(H^{2p-1}(X, \mathbb{Z}(p))),$$

which is in particular functorial with respect to the action of correspondences between complex projective manifolds. Since algebraically trivial cycles in $\mathrm{CH}^p(X)$ are parametrized by smooth projective complex curves, and since the Abel map $\mathrm{CH}^1(C)_0 \rightarrow J(C) := J(H^1(C, \mathbb{Z}(1)))$ on degree-0 zero-cycle classes on a curve C is an isomorphism, the image of the Abel–Jacobi map restricted to the subgroup $A^p(X) \subseteq \mathrm{CH}^p(X)$ of algebraically trivial cycle classes has image a subtorus

$$J_a^{2p-1}(X) \hookrightarrow J^{2p-1}(X)$$

which is algebraic, i.e., an abelian variety, and called the *algebraic intermediate Jacobian*. The resulting (surjective) Abel–Jacobi map

$$\psi^p : A^p(X) \longrightarrow J_a^{2p-1}(X)$$

defines a *regular homomorphism*, meaning that for all pointed smooth connected complex varieties (T, t_0) and all families of codimension- p cycles $Z \in \mathrm{CH}^p(T \times X)$ the map $T(C) \rightarrow J_a^{2p-1}(X), t \mapsto \psi^p(Z_t - Z_{t_0})$ is induced by a complex morphism $T \rightarrow J_a^{2p-1}(X)$ of complex varieties. The algebraic intermediate Jacobian $J_a^{2p-1}(X)$ can also be described Hodge-theoretically. For Λ a commutative ring, consider the *coniveau filtration* \mathbf{N}^{\bullet} :

$$\mathbf{N}^i H^j(X, \Lambda) := \sum \ker(H^j(X, \Lambda) \rightarrow H^j(X \setminus Z, \Lambda)),$$

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where the sum runs through all closed subvarieties Z of codimension $\geq i$ in X . Then the algebraic intermediate Jacobian $J_a^{2p-1}(X)$ is the subtorus of $J^{2p-1}(X)$ corresponding to the inclusion of rational Hodge structures $\mathbb{N}^{p-1} H^{2p-1}(X, \mathbb{Q}(p)) \subseteq H^{2p-1}(X, \mathbb{Q}(p))$; precisely,

$$J_a^{2p-1}(X) = J(H^{2p-1}(X, \mathbb{Z}(p)) \cap \mathbb{N}^{p-1} H^{2p-1}(X, \mathbb{C})).$$

On the other hand, the *Walker intermediate Jacobian* is the complex torus defined as

$$J_W^{2p-1}(X) := J(\mathbb{N}^{p-1} H^{2p-1}(X, \mathbb{Z}(p))).$$

The inclusion of lattices $\mathbb{N}^{p-1} H^{2p-1}(X, \mathbb{Z}(p)) \subseteq H^{2p-1}(X, \mathbb{Z}(p)) \cap \mathbb{N}^{p-1} H^{2p-1}(X, \mathbb{C})$ induces an isogeny of complex tori

$$\alpha : J_W^{2p-1}(X) \longrightarrow J_a^{2p-1}(X),$$

which in fact is an isogeny of complex abelian varieties, since the pull-back of an ample line bundle on $J_a^{2p-1}(X)$ along the finite map α is ample.

Walker has shown that the Abel–Jacobi map on algebraically trivial cycle classes lifts to the Walker intermediate Jacobian:

Theorem A (Walker, [Wal07]). *Let X be a complex projective manifold. There exists a regular homomorphism ψ_W^p lifting the Abel–Jacobi map ψ_p along the isogeny $\alpha : J_W^{2p-1}(X) \rightarrow J_a^{2p-1}(X)$, i.e., making the following diagram commute:*

$$\begin{array}{ccc} & & J_W^{2p-1}(X) \\ & \nearrow \psi_W^p & \downarrow \alpha \\ A^p(X) & \xrightarrow{\psi^p} & J_a^{2p-1}(X). \end{array}$$

The regular homomorphism $\psi_W^p : A^p(X) \rightarrow J_W^{2p-1}(X)$ will be called the *Walker Abel–Jacobi map*. It was first constructed by Walker [Wal07] using Lawson homology; recently, Suzuki [Suz20a] gave a Hodge-theoretic construction relying solely on Bloch–Ogus theory [BO74]. That ψ_W^p is regular is [Wal07, Lem. 7.3] or [Suz20a, Cor. 2.6]. In addition, it is shown in [Suz20a, Lem. 2.4] that ψ_W^p is compatible with the action of correspondences. In the case where $p = 1, 2, \dim X$, the usual Abel–Jacobi map ψ^p is universal among regular homomorphisms (see [Mur85, Thm. C]), and so the Walker Abel–Jacobi map coincides with the usual Abel–Jacobi map (i.e., the isogeny α is an isomorphism), while in general it differs (see Ottem–Suzuki [OS20, Cor. 4.2]) and hence provides a finer invariant for algebraically trivial cycles.

The first aim of this paper is to provide a new proof of Walker’s Theorem A; see §2.3. Our proof is based on the general lifting Theorem 1.5 for regular homomorphisms (see also Proposition 1.3), which we hope could prove useful in other situations, especially in positive characteristic.

As our main new result, we show that if X is defined over a field $K \subseteq \mathbb{C}$, then the Walker intermediate Jacobian descends to K in such a way that the diagram of Theorem A can be made $\text{Aut}(\mathbb{C}/K)$ -equivariant:

Theorem B (Distinguished model). *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. Then the isogeny $\alpha : J_W^{2p-1}(X_{\mathbb{C}}) \rightarrow J_a^{2p-1}(X_{\mathbb{C}})$ of complex abelian varieties descends uniquely to an isogeny $J_{W,X/K}^{2p-1} \rightarrow J_{a,X/K}^{2p-1}$ of abelian varieties over K in such a way that both ψ^p and ψ_W^p are $\text{Aut}(\mathbb{C}/K)$ -equivariant.*

The part of Theorem B stated for the algebraic intermediate Jacobian was proved in [ACMV20a, Thm. A] (see also [ACMV19a, Thm. 9.1]). We provide two proofs of Theorem B. The first one is presented in §2.5; it is based on [ACMV20a, Thm. A], on the universality of the Walker Abel–Jacobi map among lifts of the Abel–Jacobi map along isogenies (Theorem 2.3) and on the general descent statement of our lifting Theorem 1.5. The second one is presented in §3.3 and builds directly upon [ACMV20a]. We note also here that, as in [ACMV20a, Thm. A] and [ACMV19b, Prop. 3.1], which concern the case of the algebraic intermediate Jacobian, the K -structure in Theorem B for the Walker intermediate Jacobian and Walker Abel–Jacobi map is stable under field extensions $K \subseteq L \subseteq \mathbb{C}$ (Remark 3.5), and independent of the embedding of K into \mathbb{C} (Remark 3.6). As a consequence, the kernel of the Walker Abel–Jacobi map is independent of the choice of embedding of K into \mathbb{C} ; the analogous statement for the Abel–Jacobi map on algebraically trivial cycle classes is [ACMV19b, Rem. 3.4].

From our second approach to proving Theorem B we obtain two applications.

First, we obtain the following proposition, which provides further arithmetic significance to the Walker Abel–Jacobi map, by showing that the torsion-free quotient of $\mathbb{N}^{p-1} H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p))$ can be modeled by an abelian variety independently of ℓ :

Corollary C (Modeling coniveau integrally). *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. Then for all integers p , the model $J_{W,X/K}^{2p-1}$ over K of the Walker intermediate Jacobian $J_W^{2p-1}(X_{\mathbb{C}})$ (Theorem B) has the property that for all primes ℓ we have canonical isomorphisms of $\text{Aut}(\mathbb{C}/K)$ -representations*

$$T_{\ell} J_{W,X/K}^{2p-1} \simeq \mathbb{N}^{p-1} H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p))_{\tau}.$$

This result is established in §4.1. It was established with \mathbb{Q}_{ℓ} -coefficients in [ACMV20a, Thm. A] with the model of the algebraic intermediate Jacobian over K in place of that of the Walker intermediate Jacobian. We direct the reader to [ACMV20b] for more details, and in particular, the connection to a question of Mazur [Maz14].

Second, for any smooth projective variety X over an algebraically closed field and for any prime ℓ invertible in X , Bloch [Blo79] has defined a map $\lambda^p : \text{CH}^p(X)[\ell^{\infty}] \rightarrow H_{\text{ét}}^{2p-1}(X, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(p))$ on ℓ -primary torsion. The ℓ -adic Bloch map $T_{\ell}\lambda^p : T_{\ell}\text{CH}^p(X) \rightarrow H_{\text{ét}}^{2p-1}(X, \mathbb{Z}_{\ell}(p))_{\tau}$ is then obtained by taking Tate modules and making the identification $T_{\ell}H_{\text{ét}}^i(X, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(j)) = H_{\text{ét}}^i(X, \mathbb{Z}_{\ell}(j))_{\tau}$; we refer to [Suw88, (2.6.5)], and to [ACMV20b, §A.3.3], for more details. Here, the Tate module associated to an ℓ -primary torsion abelian group M is the group $T_{\ell}M := \varprojlim M[\ell^n]$. Thanks to our approach to lifting regular homomorphisms along isogenies, together with the existence of the Walker Abel–Jacobi map, we determine the image of $T_{\ell}\lambda^p$ restricted to algebraically trivial cycle classes:

Corollary D. *Let X be a smooth projective variety over a field K of zero characteristic. Then*

$$\text{im}(T_{\ell}\lambda^p : T_{\ell}A^p(X_{\overline{K}}) \longrightarrow H_{\text{ét}}^{2p-1}(X_{\overline{K}}, \mathbb{Z}_{\ell}(p))_{\tau}) = \mathbb{N}^{p-1} H_{\text{ét}}^{2p-1}(X_{\overline{K}}, \mathbb{Z}_{\ell}(p))_{\tau}$$

for all primes ℓ .

This extends [Suw88, Prop. 5.2] (see also [ACMV20b, Prop. 2.1]), where the images of the usual Bloch map λ^p and of $T_{\ell}\lambda^p \otimes \mathbb{Q}_{\ell}$, both restricted to algebraically trivial cycle classes, were determined.

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1. LIFTING REGULAR HOMOMORPHISMS ALONG ISOGENIES

1.1. An elementary fact. We start with the following elementary fact, which will be used recurrently throughout this note.

Fact 1.1. *Let $f : D \rightarrow G$ and $\alpha : G' \rightarrow G$ be homomorphisms of abelian groups. Assume D is divisible and that $\ker \alpha$ is finite. Then there exists at most one homomorphism $f' : D \rightarrow G'$ such that $\alpha \circ f' = f$, i.e., such that the following diagram commutes :*

$$\begin{array}{ccc} & & G' \\ & \nearrow f' & \downarrow \alpha \\ D & \xrightarrow{f} & G. \end{array}$$

Moreover, if there is a group H acting on D , G , and G' , and f and α are H -equivariant, then f' , if it exists, is H -equivariant, as well. □

As a first consequence, note that since for a smooth complex projective variety X one has that $A^p(X)$ is a divisible group (e.g., [BO74, Lem. 7.10]), there is at most one homomorphism $\psi_W^p : A^p(X) \rightarrow J_W^{2p-1}(X)$ such that $\alpha \circ \psi_W^p = \psi^p$; i.e., there is at most one lifting of the Abel–Jacobi map to the Walker intermediate Jacobian.

1.2. Lifting homomorphisms of abelian varieties along isogenies. We have the following elementary lemma on lifting morphisms of abelian varieties. (Recall that for an abelian variety A over a field K of positive characteristic, the N -torsion group scheme $A[N]$ may carry strictly more information than the abstract group of points $A[N](\bar{K})$.)

Lemma 1.2. *Let $f : B \rightarrow A$ be a morphism of abelian varieties over a field K , and let $\alpha : A' \rightarrow A$ be an isogeny of abelian varieties over K . The following are equivalent :*

(1) *There exists a lift $f' : B \rightarrow A'$ of f ; i.e., there is a commutative diagram*

$$\begin{array}{ccc} & & A' \\ & \nearrow f' & \downarrow \alpha \\ B & \xrightarrow{f} & A. \end{array}$$

(2) *There exists a lift of f restricted to torsion schemes; i.e., for each natural number N there is a commutative diagram of finite group schemes*

$$\begin{array}{ccc} & & A'[N] \\ & \nearrow (f[N])' & \downarrow \alpha[N] \\ B[N] & \xrightarrow{f[N]} & A[N] \end{array} \tag{1.1}$$

such that $(f[MN])'|_{B[N]} = (f[N])'$.

If α is separable (equivalently, étale), and Ω/K is any field extension with Ω algebraically closed, then (1) and (2) are also equivalent to either of the following conditions :

(3) There exists a group-theoretic lift $(f(\Omega)_{\text{tors}})' : B(\Omega)_{\text{tors}} \rightarrow A'(\Omega)_{\text{tors}}$ of $f(\Omega)$ restricted to torsion points; i.e., there is a commutative diagram of torsion abelian groups

$$\begin{array}{ccc} & & A'(\Omega)_{\text{tors}} \\ & \nearrow (f(\Omega)_{\text{tors}})' & \downarrow \alpha(\Omega)_{\text{tors}} \\ B(\Omega)_{\text{tors}} & \xrightarrow{f(\Omega)_{\text{tors}}} & A(\Omega)_{\text{tors}} \end{array}$$

(4) For all prime numbers l there exists a group-theoretic lift $(T_l f)' : T_l B \rightarrow T_l A'$ of $T_l f$, the map on Tate modules; i.e., there is a commutative diagram

$$\begin{array}{ccc} & & T_l A' \\ & \nearrow (T_l f)' & \downarrow T_l \alpha \\ T_l B & \xrightarrow{T_l f} & T_l A \end{array}$$

Finally, if any of the lifts in (1)–(4) exist, they are unique. In particular, $(f')_{\text{tors}} = (f_{\text{tors}})'$, $T_l(f') = (T_l f)'$ and, for any extension field L/K , $(f(L)_{\text{tors}})' = f'(L)_{\text{tors}}$.

Proof. The uniqueness of the lift f' follows from Fact 1.1; and (1) clearly implies (2). Moreover, (2) implies (3), (3) implies (2) over an algebraically closed field of characteristic zero, and (3) and (4) are equivalent.

To show (2) implies (1), suppose there exists a suitable lift of f on torsion schemes. By rigidity of homomorphisms of abelian varieties, we may assume that K is perfect. Using the uniqueness of f' and Galois descent, we may and do assume K is algebraically closed.

We start by reducing to the case where f is an isogeny. To this end, consider the diagram

$$\begin{array}{ccccc} & & A'' \hookrightarrow & \xrightarrow{\iota'} & A' \\ & & \downarrow \alpha' & & \downarrow \alpha \\ B & \xrightarrow{f_{\text{conn}}} & B' & \xrightarrow{f_{\text{fin}}} & B'' \hookrightarrow & \xrightarrow{\iota} & A \end{array}$$

where $B'' := \text{Im}(f) \subseteq A$, the morphisms $f : B \xrightarrow{f_{\text{conn}}} B' \xrightarrow{f_{\text{fin}}} B''$ give the Stein factorization, ι is the natural inclusion, and $A'' = B'' \times_A A'$. Explicitly, $B' = B / ((\ker f)_{\text{red}}^0)$ is the quotient of B by the largest sub-abelian variety contained in $\ker(f)$.

Fix a prime l and consider l -primary torsion. Using the lift $(f[l^\infty])'$, we have the diagram:

$$\begin{array}{ccccc} & & A''[l^\infty] \hookrightarrow & \xrightarrow{\iota'[l^\infty]} & A'[l^\infty] \\ & & \downarrow \alpha'[l^\infty] & & \downarrow \alpha[l^\infty] \\ B[l^\infty] & \xrightarrow{f_{\text{conn}}[l^\infty]} & B'[l^\infty] & \xrightarrow{f_{\text{fin}}[l^\infty]} & B''[l^\infty] \hookrightarrow & \xrightarrow{\iota[l^\infty]} & A[l^\infty] \end{array}$$

The splitting of the map $f_{\text{conn}}[l^\infty]$ is elementary, since whenever one has a short exact sequence of abelian varieties the induced maps on l -primary torsion give a split exact sequence (taking l -primary torsion is exact since the kernel is divisible, and then free modules are projective). (If $l = \text{char}(K)$, an appeal to Dieudonné modules gives the same conclusion.)

Thus, we now assume f and α are isogenies. Suppose briefly that $\text{char}(K) = 0$; then f and α are étale. The cover f factors through α if and only if the induced map on étale fundamental groups $f_* : \pi_1^{\text{ét}}(B, 0_B) \rightarrow \pi_1^{\text{ét}}(A, 0_A)$ factors through $\alpha_* : \pi_1^{\text{ét}}(A', 0_{A'}) \rightarrow \pi_1^{\text{ét}}(A, 0_A)$. For an abelian variety D/K , there is a canonical isomorphism $\pi_1^{\text{ét}}(D, 0_D) \simeq \varprojlim_N D[N](K)$. By taking the inverse

limit of the maps of finite groups $(f[N])'(K)$, we see that the condition on fundamental groups is equivalent to (2).

Now suppose instead that K is algebraically closed of positive characteristic. Then f , while possibly not étale, is at least a torsor over X under the finite commutative group scheme $\ker(f)$. Consequently, it is classified by a quotient of Nori's fundamental group scheme $\pi_1^{\text{Nori}}(A)$ [Nor76]. Moreover, for an abelian variety D/K , we have $\pi_1^{\text{Nori}}(D) = \lim_{\leftarrow N} D[N]$ [Nor83]. Consequently, condition (2) is again equivalent to the hypothesis that the cover f factors through α .

Finally, suppose α is étale by hypothesis and that (3) holds. As noted above, it suffices to consider the case where K is algebraically closed of positive characteristic and $f : B \rightarrow A$ is an isogeny. Now, any isogeny $g : D \rightarrow C$ of abelian varieties over K admits a canonical factorization $g = g_{\text{ét}} \circ g_i$, where $g_i : D \rightarrow D_i := D/(\ker g)^0$ is purely inseparable and $g_{\text{ét}}$ is étale. Since α is étale, f factors through α if and only if $f_{\text{ét}} : B_i \rightarrow A$ factors through α . Because f_i is a universal homeomorphism, the map of groups $\tilde{f}_{\text{tors}}(K)$ descends to a map of groups $\tilde{f}_{\text{ét,tors}}(K) : B_{i,\text{tors}}(K) \rightarrow A'_{\text{tors}}(K)$. Now $f_{\text{ét}}$ and α are étale isogenies and we may argue using fundamental groups as before, while recalling that (in all characteristics) $\pi_1^{\text{ét}}(D, 0_D) \simeq \lim_{\leftarrow N} D[N](K)$. The same argument, combined with the canonical isomorphism $\pi_1^{\text{ét}}(D, 0_D) \simeq \bigoplus_l T_l D$, shows that (4) implies (1), as well. \square

1.3. Lifting regular homomorphisms along isogenies. From Lemma 1.2 we get the following lifting criterion for regular homomorphisms :

Proposition 1.3. *Let K be a field, and Ω/K an algebraically closed extension. Let X/K be a smooth projective variety, let A/K be an abelian variety over K , let $\phi : A^p(X_\Omega) \rightarrow A(\Omega)$ be an $\text{Aut}(\Omega/K)$ -equivariant regular homomorphism, and let $\alpha : A' \rightarrow A$ be an étale isogeny of abelian varieties over K . Then the following are equivalent :*

- (1) *The $\text{Aut}(\Omega/K)$ -equivariant regular homomorphism ϕ lifts to A' , in the sense that there is a commutative diagram of $\text{Aut}(\Omega/K)$ -equivariant regular homomorphisms*

$$\begin{array}{ccc} & & A'(\Omega) \\ & \nearrow \phi' & \downarrow \alpha(\Omega) \\ A^p(X_\Omega) & \xrightarrow{\phi} & A(\Omega). \end{array}$$

- (2) *The homomorphism ϕ lifts on torsion, in the sense that there is a commutative diagram of torsion abelian groups*

$$\begin{array}{ccc} & & A'(\Omega)_{\text{tors}} \\ & \nearrow (\phi_{\text{tors}})' & \downarrow \alpha(\Omega)_{\text{tors}} \\ A^p(X_\Omega)_{\text{tors}} & \xrightarrow{\phi_{\text{tors}}} & A(\Omega)_{\text{tors}}. \end{array}$$

- (3) *For all prime numbers l there exists a group-theoretic lift $(T_l \phi)' : T_l A^p(X_\Omega) \rightarrow T_l A'$ of $T_l \phi$, the map on Tate modules ; i.e., there is a commutative diagram*

$$\begin{array}{ccc} & & T_l A' \\ & \nearrow (T_l \phi)' & \downarrow T_l \alpha \\ T_l A^p(X_\Omega) & \xrightarrow{T_l \phi} & T_l A. \end{array}$$

- (4) *For all primes numbers l , we have $\text{Im}(T_l \phi) \subseteq \text{Im}(T_l \alpha)$.*

Finally, if any of the lifts in (1)–(3) exist, then they are unique and $\text{Aut}(\Omega/K)$ -equivariant. In particular, $(\phi')_{\text{tors}} = (\phi_{\text{tors}})'$ and $(T_l\phi)' = T_l(\phi')$.

Proof. The uniqueness and $\text{Aut}(\Omega/K)$ -equivariance of the lifts follows from Fact 1.1. Clearly (1) implies (2), and (2) implies (3) by taking Tate modules. The equivalence of (3) and (4) is obvious since the l -th Tate module of an abelian variety is a free module of finite rank over the PID \mathbb{Z}_l , and so any submodule (in particular, the image of $T_l\phi$) is free of finite rank, and therefore projective. Thus we will show (3) implies (1).

Let (T, t_0) be a smooth pointed variety over Ω , and let $\Gamma \in \text{CH}^p(T \times_{\Omega} X_{\Omega})$. Then we have a commutative diagram

$$\begin{array}{ccccc} T & \xrightarrow{t \mapsto t - t_0} & A_0(T) & \xrightarrow{\text{alb}} & \text{Alb}(T)(\Omega) & & A'(\Omega) \\ & \searrow^{w_{\Gamma}; t \mapsto \Gamma_t - \Gamma_{t_0}} & \downarrow \Gamma_* & & \downarrow f & \swarrow \alpha & \\ & & A^p(X_{\Omega}) & \xrightarrow{\phi} & A(\Omega) & & \end{array}$$

where the top row is the pointed Albanese, and the right vertical arrow f comes from the universal property of algebraic representatives, together with the facts that Albaneses are algebraic representatives, and that $\phi \circ \Gamma_*$ can easily be confirmed to be a regular homomorphism.

On Tate modules we obtain a diagram

$$\begin{array}{ccc} T_l A_0(T) & \xrightarrow[\sim]{T_l \text{alb}} & T_l \text{Alb}(T) & & T_l A' \\ \downarrow \Gamma_* & & \downarrow T_l f & \swarrow T_l \alpha & \\ T_l A^p(X_{\Omega}) & \xrightarrow{T_l \phi} & T_l A & & \end{array} \quad (1.2)$$

where the lift $(T_l\phi)'$ is provided by assumption (3). The isomorphism on Tate modules for the Albanese map comes from Roitman's theorem (see, e.g., [ACMV20b, Prop. A.29]). Since we assume α is étale, by Lemma 1.2 we obtain a lift $f' : \text{Alb}(T) \rightarrow A'$ of f giving a commutative diagram

$$\begin{array}{ccccc} T(\Omega) & \xrightarrow{t \mapsto [t] - [t_0]} & A_0(T) & \xrightarrow{\text{alb}} & \text{Alb}(T)(\Omega) & \xrightarrow{f'(\Omega)} & A'(\Omega) \\ & \searrow^{w_{\Gamma}; t \mapsto \Gamma_t - \Gamma_{t_0}} & \downarrow \Gamma_* & & \downarrow f(\Omega) & \swarrow \alpha(\Omega) & \\ & & A^p(X_{\Omega}) & \xrightarrow{\phi} & A(\Omega) & & \end{array} \quad (1.3)$$

It follows immediately that if ϕ lifts to an abstract homomorphism $\phi' : A^p(X_{\Omega}) \rightarrow A'(\Omega)$, then ϕ' is a regular homomorphism. Thus we have reduced the problem to showing that ϕ lifts as an abstract homomorphism to a homomorphism $\phi' : A^p(X_{\Omega}) \rightarrow A'(\Omega)$.

Over an algebraically closed field, algebraically trivial cycles are parameterized by smooth projective curves [Ful98, Ex. 10.3.2]. In other words, $A^p(X_{\Omega})$ is covered by the images of $\Gamma_* : A_0(T) \rightarrow A^p(X_{\Omega})$, where T runs through pointed smooth projective curves over Ω and Γ over correspondences in $\text{CH}^p(T \times_{\Omega} X_{\Omega})$. Now since $A_0(T)$ is divisible, it follows that $\Gamma_*(A_0(T))$ is divisible; therefore, by the uniqueness of lifts (Fact 1.1) it is enough to show that $f' \circ \text{alb}$ in (1.3) factors through $\Gamma_*(A_0(T))$ in the case where T is a smooth projective curve. In other words, taking T to be a smooth projective curve over Ω , and given any $\gamma \in A_0(T)$ such that $\Gamma_*(\gamma) = 0$, we must show that $(f' \circ \text{alb})(\gamma) = 0$.

The first observation is that this is clear if Ω is the algebraic closure of a finite field. Indeed, in that case $A_0(T)$ is a torsion group, since the Albanese map $A_0(T) \rightarrow \text{Alb}_T(\Omega)$ is an isomorphism

and closed points of an abelian variety over a finite field are torsion. Thus γ is torsion. Decomposing torsion in $A_0(T)$ into a direct sum of l -power torsion, we can work one prime at a time. Now we make the following elementary observation: given any homomorphism of groups $h : D \rightarrow G$ where D is divisible, and any $x \in D[l^\infty]$, we have that $h(x) = 0$ if for some lift x_l of x to $T_l D$ (which exists since D is divisible), we have that $(T_l h)(x_l) = 0$. Consequently, taking Tate modules in (1.3) and using the lift $(T_l \phi)'$ (1.2), we see that $\text{alb}(\gamma) = 0$.

We now deduce the general case from the case of finite fields, *via* a specialization argument. For this we use the terminology of regular homomorphisms from [ACMV19a], which is much better suited to the relative setting. Since all objects considered here are of finite type, the data $X, T, \Gamma, A, A', \alpha$ and γ descend to a field L which is finitely generated over the prime field. A standard spreading argument produces a smooth ring R , finitely generated as a \mathbb{Z} -algebra and with fraction field L , and smooth X, T, A, A' over $S = \text{Spec}(R)$, as well as $\gamma \in \mathcal{A}_{T/S}^1(S)$, whose generic fibers are the corresponding original data. Let $|S|^{\text{cl}}$ be the set of points of S with finite residue fields; then $|S|^{\text{cl}}$ is topologically dense in S .

From [ACMV19a], there exists a diagram

$$\begin{array}{ccccc}
 \mathcal{A}_{T/S}^1 & \xrightarrow{\text{alb}} & \text{Alb}_{T/S} & \xrightarrow{f'_S} & A' \\
 \Gamma_{S*} \downarrow & & \downarrow f_S & \swarrow \alpha_S & \\
 \mathcal{A}_{X/S}^p & \xrightarrow{\Phi} & A & &
 \end{array} \tag{1.4}$$

where $\Phi : \mathcal{A}_{X/S}^p \rightarrow A$ is a regular homomorphism, the Albanese homomorphism is the universal regular homomorphism for 0-cycles [ACMV19a, Lem. 7.5] and the remaining morphisms are extensions of those in (1.3). Set $a' = (f'_S \circ \text{alb})(\gamma) \in A'(S)$.

Now suppose $s \in |S|^{\text{cl}}$. Then pullback of (1.4) yields a diagram of objects over $s = \text{Spec}(\kappa(s))$, where specialization of cycles is provided by [Ful98, 20.3.5]. We have seen that for each such s , $a'_s = 0 \in A'(s)$. Using the density of $|S|^{\text{cl}}$, we see that $a' = 0$, and in particular its generic fiber $(f' \circ \text{alb})(\gamma)$ is zero. \square

Remark 1.4 (Regular homomorphisms and mini-versal cycle classes). Given a surjective $\text{Aut}(\Omega/K)$ -equivariant regular homomorphism $\phi : A^p(X_\Omega) \rightarrow A(\Omega)$, there is a cycle class $\Gamma \in \text{CH}^p(A \times_K X)$ (which we call a *mini-versal cycle class*) such that the associated map $\psi_\Gamma : A \rightarrow A$, induced on Ω -points by $a \mapsto \Gamma_a - \Gamma_0 \mapsto \phi(\Gamma_a - \Gamma_0)$, is given by multiplication by some non-zero integer r [ACMV19a, Lem. 4.7]. One can immediately see from the definition that given any étale isogeny $\alpha : A' \rightarrow A$ through which ϕ factors, one has $(\deg \alpha) \mid (\deg r \cdot \text{Id}_A) = r^{2 \dim A}$. In particular, if there is a universal cycle class (i.e., $r = 1$), then ϕ does not factor through any non-trivial isogeny $A' \rightarrow A$.

We obtain the following consequence of Proposition 1.3, establishing the existence of a universal lifting of a surjective regular homomorphism along isogenies. Together with Corollary 1.6, this extends [BF84, Thm. 0.1] to the case of arbitrary fields. Note also that the proof of [BF84, Thm. 0.1] is incorrect. (On the bottom of [BF84, p.362], it is assumed that the map $u : B(k) \rightarrow A^q(X)$ is a homomorphism, so that the image of u is a subgroup of $A^q(X)$. There, X is a smooth projective variety over an algebraically closed field k , B is an abelian variety over k , and $u : b \mapsto Z_*([u] - [0])$ is the map induced by a cycle $Z \in \text{CH}^p(B \times_k X)$. However, this is not the case in general. Indeed, consider the special instance where $X = B$ is an abelian variety of dimension > 1 over an uncountable algebraically closed field k and where $Z = \Delta_B$ is the diagonal cycle class. Then the

map $u : B(k) \rightarrow A_0(B), b \mapsto [b] - [0]$ is not a homomorphism since by [Blo76, Thm. 3.1(a)] there exist b_1 and b_2 in $B(k)$ such that $[b_1 + b_2] \neq [b_1] + [b_2] - [0]$ in $A_0(B)$; see also [Mur00, p.309].)

Theorem 1.5. *Let K be a field, and Ω/K an algebraically closed extension. Let X/K be a smooth projective variety, let A/K be an abelian variety over K , and let $\phi : A^p(X_\Omega) \rightarrow A(\Omega)$ be a surjective regular homomorphism. Then there exists an isogeny $\alpha : \tilde{A} \rightarrow A_\Omega$ and a surjective regular homomorphism $\tilde{\phi} : A^p(X_\Omega) \rightarrow \tilde{A}(\Omega)$ which is initial among all regular lifts of ϕ along étale isogenies $A' \rightarrow A_\Omega$.*

Moreover, if ϕ is $\text{Aut}(\Omega/K)$ -equivariant, then \tilde{A} admits a unique model over K making $\tilde{\phi}$ $\text{Aut}(\Omega/K)$ -equivariant, and the isogeny α descends to K .

Proof. Using a mini-versal cycle class as in Remark 1.4, one sees that $(\oplus T_l \phi)(A^p(X_\Omega))$ has finite index in $\oplus T_l A \simeq \pi_1^{\text{ét}}(A_\Omega, 0)$. Consequently, it determines an étale isogeny $\tilde{A} \rightarrow A_\Omega$ over Ω ; by Proposition 1.3(4), there is a surjective regular homomorphism $\tilde{\phi} : A^p(X) \rightarrow \tilde{A}(\Omega)$ which lifts ϕ and which is initial among all regular lifts of ϕ along étale isogenies $A'' \rightarrow A_\Omega$ over Ω .

Suppose now that ϕ is $\text{Aut}(\Omega/K)$ -equivariant, and briefly assume K perfect. The unicity of the model over K follows from the elementary Fact 1.1. Its existence follows from the universality of $\tilde{\phi}$: for all $\sigma \in \text{Aut}(\Omega/K)$, one obtains an isomorphism $g_\sigma : \tilde{A} \rightarrow \tilde{A}^\sigma$ over Ω , where \tilde{A}^σ is the pull-back of \tilde{A} along $\sigma : \Omega \rightarrow \Omega$, making the following diagram commute

$$\begin{array}{ccccc}
 & & & & \tilde{A}(\Omega) \\
 & & & \nearrow \alpha & \downarrow g_\sigma(\Omega) \\
 A^p(X_\Omega) & \xrightarrow{\tilde{\phi}} & & & \tilde{A}^\sigma(\Omega) \\
 & \searrow \phi & \rightarrow & A_\Omega(\Omega) & \\
 & & \searrow \alpha^\sigma & & \\
 & & & & \tilde{A}^\sigma(\Omega)
 \end{array}$$

Here $\tilde{\phi}^\sigma$ and α^σ are obtained from the action of σ on $A^p(X_\Omega)$ and on A_Ω , and from the canonical σ -morphism $\tilde{A}^\sigma \rightarrow \tilde{A}$. To conclude, one checks as in the proof of [ACMV17, Thm. 4.4] that the isomorphisms g_σ^{-1} for $\sigma \in \text{Aut}(\Omega/K)$ define a Galois-descent datum on the isogeny $\alpha : \tilde{A} \rightarrow A_\Omega$.

If K is a non-perfect field, let K^{perf} be the perfect closure of K inside Ω . From what we have seen, since $\text{Aut}(\Omega/K^{\text{perf}}) \subseteq \text{Aut}(\Omega/K)$, \tilde{A} descends to K^{perf} . Because in fact $\text{Aut}(\Omega/K^{\text{perf}}) = \text{Aut}(\Omega/K)$, it suffices to show that $\alpha : \tilde{A} \rightarrow A_{K^{\text{perf}}}$ descends to K . Now, by definition, the homomorphism α factors through the K^{perf}/K -image $\tilde{A} \rightarrow \text{im}_{K^{\text{perf}}/K}(\tilde{A})_{K^{\text{perf}}}$, which exists due to [Con06, Thm. 4.3]. Since $\alpha : \tilde{A} \rightarrow A_{K^{\text{perf}}}$ is étale and K^{perf}/K is primary, the canonical map $\tilde{A} \rightarrow \text{im}_{K^{\text{perf}}/K}(\tilde{A})_{K^{\text{perf}}}$, which always has connected kernel [Con06, Thm. 4.5(3)], is an isomorphism, and \tilde{A} and α descend canonically to K . \square

We derive the following characterization of surjective regular homomorphisms that do not lift along non-trivial isogenies in terms of their kernels:

Corollary 1.6. *Let X be a smooth projective variety over an algebraically closed field Ω and let $\phi : A^p(X) \rightarrow A(\Omega)$ be a surjective regular homomorphism. Then the following statements are equivalent:*

- (1) $\ker \phi$ is divisible.
- (2) $\ker \phi_{\text{tors}}$ is divisible.
- (3) $T_l \phi$ is surjective for all primes l .
- (4) ϕ does not factor through any non-trivial étale isogeny $\alpha : A' \rightarrow A$.

Proof. The argument in the proof of Theorem 1.5 says that (3) and (4) are equivalent (recall from Proposition 1.3 that a group-theoretic lift of a regular homomorphism along an isogeny is a regular homomorphism). The elementary commutative algebra Lemma 1.7 below gives the equivalence of (1) and (3). Finally, since surjective regular homomorphisms are surjective on torsion (see [ACMV20a, Rem. 3.3]), Lemma 1.7 below also gives that $T_l\phi$ being surjective for all l is equivalent to $\ker(\phi_{\text{tors}})$ being l -divisible for all primes l , i.e., that (2) is equivalent to (3). \square

Lemma 1.7. *Suppose that we have a short exact sequence of abelian groups*

$$0 \rightarrow H \rightarrow D \rightarrow G \rightarrow 0$$

with D an l -divisible group. Then the left exact sequence

$$0 \rightarrow T_l H \rightarrow T_l D \rightarrow T_l G$$

is right exact if and only if H is l -divisible.

If in addition $D_{\text{tors}} \rightarrow G_{\text{tors}}$ is surjective, then this is also equivalent to H_{tors} being l -divisible.

Proof. Since D is l -divisible, we have for all $n > 0$ exact sequences

$$0 \rightarrow H[l^n] \rightarrow D[l^n] \rightarrow G[l^n] \rightarrow H/l^n H \rightarrow 0.$$

Using that $A/l^n A = 0$ and $\varprojlim_n A[l^n] = 0$ for any l -divisible abelian group A , we obtain by passing to the inverse limit a short exact sequence

$$0 \rightarrow T_l H \rightarrow T_l D \rightarrow T_l G \rightarrow 0.$$

Conversely, if H is not l -divisible, let us assume that $H/l^n H \neq 0$ for all $n \geq n_0$. In particular $D[l^n] \rightarrow G[l^n]$ is not surjective for every $n \geq n_0$. Now let $g_{n_0} \in G[l^{n_0}]$ be an element that is not in the image of the map $D[l^{n_0}] \rightarrow G[l^{n_0}]$. Since G is l -divisible (being the image of the l -divisible group D), we can lift g_n to an element $(g_n) \in T_l G$. Clearly (g_n) is not the image of any element $(d_n) \in T_l D$, since then $d_{n_0} \mapsto g_{n_0}$. Thus $T_l D \rightarrow T_l G$ is not surjective. This completes the proof of the converse.

Finally assume that $D_{\text{tors}} \rightarrow G_{\text{tors}}$ is surjective. Then we can simply replace the short exact sequence $0 \rightarrow H \rightarrow D \rightarrow G \rightarrow 0$ with

$$0 \rightarrow H_{\text{tors}} \rightarrow D_{\text{tors}} \rightarrow G_{\text{tors}} \rightarrow 0$$

and we have reduced to the previous case, since D divisible implies that D_{tors} is divisible, and $T_l A = T_l(A_{\text{tors}})$ for any abelian group A . \square

Remark 1.8. Given a regular homomorphism ϕ , using that surjective regular homomorphisms are surjective on torsion (see [ACMV20a, Rem. 3.3]), one can in fact show that $\ker(\phi)/N = \ker(\phi_{\text{tors}})/N$ for any non-zero integer N .

2. THE WALKER ABEL–JACOBI MAP

The aim of this section is to provide a new construction of the Walker Abel–Jacobi map (Theorem A), based on our general lifting Proposition 1.3.

2.1. The Bloch map and the coniveau filtration. Recall that, for any smooth projective variety X over an algebraically closed field and for any prime ℓ invertible in X , Bloch [Blo79] has defined a map $\lambda^p : \text{CH}^p(X)[\ell^\infty] \rightarrow H_{\text{ét}}^{2p-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p))$. In case X is a smooth projective complex variety, we obtain by comparison isomorphism a map $\lambda^p : \text{CH}^p(X)[\ell^\infty] \rightarrow H^{2p-1}(X^{\text{an}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p))$. When restricted to homologically trivial cycles, the Bloch map factors as (see, e.g., [ACMV20b, §A.5])

$$\text{CH}^p(X)_{\text{hom}}[\ell^\infty] \xrightarrow{\lambda^p} H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow H^{2p-1}(X^{\text{an}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(p)),$$

where the right-hand side arrow is the canonical inclusion coming from the universal coefficient theorem. The following lemma is due to Suzuki [Suz20b]:

Lemma 2.1. *Let X be a projective complex manifold. Then the restriction of the Bloch map λ^p to algebraically trivial cycles factors uniquely as:*

$$\begin{array}{ccccc} & & \mathbb{N}^{p-1} H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell & & \\ & \nearrow \lambda_W^p & \downarrow & & \\ A^p(X)[\ell^\infty] & \xrightarrow{\lambda^p} & H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell & \hookrightarrow & H^{2p-1}(X^{\text{an}}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(p)), \end{array}$$

where the vertical arrow is induced by the inclusion $\mathbb{N}^{p-1} H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \subseteq H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p))$.

Proof. The factorization in the bottom row was given above. The rest is obtained in the proof of [Suz20b, Lem. 2.2] as a consequence of [Ma17, Thm. 5.1]. The unicity of the factorization follows from the elementary Fact 1.1, together with the divisibility of $A^p(X)$ (e.g., [BO74, Lem. 7.10]) and the finiteness of torsion in $H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p))$. \square

2.2. The Abel–Jacobi map on torsion and the Bloch map. Let X be a projective complex manifold. We have the canonical identification

$$J^{2p-1}(X)[\ell^\infty] = H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \quad (2.1)$$

which comes from the classical identification $J(H)[N] = H_1(J(H), \mathbb{Z}/N\mathbb{Z}) = H_\tau \otimes \mathbb{Z}/N\mathbb{Z}$ for a pure integral Hodge structure H of weight -1 , and the elementary fact that the torsion-free quotient map $H \rightarrow H_\tau$ becomes an isomorphism after tensoring with a divisible group. After making the identification (2.1), the Bloch map coincides with the Abel–Jacobi map on torsion. Precisely:

Proposition 2.2 (Bloch [Blo79]). *On homologically trivial cycles of ℓ -primary torsion, the Bloch map coincides with the Abel–Jacobi map, i.e., the following diagram commutes:*

$$\begin{array}{ccc} & & J^{2p-1}(X)[\ell^\infty] \\ & \nearrow \text{AJ}[\ell^\infty] & \parallel (2.1) \\ \text{CH}^p(X)_{\text{hom}}[\ell^\infty] & \xrightarrow{\lambda^p} & H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell. \end{array}$$

Proof. This is due to Bloch [Blo79, Prop. 3.7] (see also [ACMV20b, §A.2.1]). \square

2.3. Proof of Theorem A. Let X be a projective complex manifold. As above in §2.2, we have a canonical identification

$$J_W^{2p-1}(X)[\ell^\infty] = \mathbb{N}^{p-1} H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell. \quad (2.2)$$

We are trying to construct a lift

$$\begin{array}{ccccc} & & J_W^{2p-1}(X)(\mathbb{C}) & & \\ & \nearrow \psi_W^p & \downarrow \alpha & & \\ A^p(X) & \xrightarrow{\psi^p} & J_a^{2p-1}(X)(\mathbb{C}) & \hookrightarrow & J^{2p-1}(X)(\mathbb{C}). \end{array}$$

From Proposition 1.3 it suffices to construct for all primes ℓ a lift

$$\begin{array}{ccccc} & & J_W^{2p-1}(X)[\ell^\infty] & & \\ & \nearrow \psi_\infty^p & \downarrow \alpha[\ell^\infty] & & \\ A^p(X)[\ell^\infty] & \xrightarrow{\psi^p[\ell^\infty]} & J_a^{2p-1}(X)[\ell^\infty] & \hookrightarrow & J^{2p-1}(X)[\ell^\infty]. \end{array}$$

Using the identifications (2.1) and (2.2), we then have a commutative diagram

$$\begin{array}{ccc} J_W^{2p-1}(X)[\ell^\infty] & \xlongequal{\quad} & \mathbb{N}^{p-1} H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell & (2.3) \\ \downarrow \alpha[\ell^\infty] & & \downarrow & \\ A^p(X)[\ell^\infty] \xrightarrow{\psi^p[\ell^\infty]} J_a^{2p-1}(X)[\ell^\infty] \hookrightarrow J^{2p-1}(X)[\ell^\infty] & \xlongequal{\quad} & H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \end{array}$$

where, by Proposition 2.2, the composition of the bottom row is the Bloch map, and the right vertical arrow is induced by the inclusion $\mathbb{N}^{p-1} H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p)) \subseteq H^{2p-1}(X^{\text{an}}, \mathbb{Z}(p))$. The desired lift on ℓ -power torsion is then an immediate consequence of Lemma 2.1, by taking torsion-free quotients, completing the proof of the theorem. \square

2.4. The Walker Abel–Jacobi map does not lift further along isogenies. The following result was communicated to us by Fumiaki Suzuki.

Theorem 2.3 (Suzuki). *Suppose X is a projective complex manifold. Then the kernel of the Walker Abel–Jacobi map $\psi_W^p : A^p(X) \rightarrow J_W^{2p-1}(X)$ is divisible. In particular, the Walker Abel–Jacobi map ψ_W^p is initial among all lifts of the Abel–Jacobi map $\psi^p : A^p(X) \rightarrow J_a^{2p-1}(X)$ along isogenies; i.e., if $\psi_W^p : A^p(X) \rightarrow J_W^{2p-1}(X)$ factors through an isogeny $f : A \rightarrow J_a^{2p-1}(X)$, then f is an isomorphism.*

Proof. If a regular homomorphism $\phi : A^p(X) \rightarrow A(\Omega)$ factors through an isogeny $f : A' \rightarrow A$, then $\ker f$ is a quotient of $\ker \phi$; in particular, if $\ker \phi$ is divisible, then since $\ker f$ is finite it must be zero. (Alternately, use Corollary 1.6.)

We now show that $\ker \psi_W^p$ is divisible. By Corollary 1.6 (or more simply by Remark 1.8), it is equivalent to show that the kernel of the restriction of ψ_W^p to ℓ -primary torsion is divisible for all primes ℓ . By the short exact sequence of [Suz20b, Lem. 2.2], $\ker(\psi_W^p[\ell^\infty])$ is a quotient of $K \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell$, where K is the kernel of the surjection

$$f^p : H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}(p))) \rightarrow \mathbb{N}^{p-1} H^{2p-1}(X, \mathbb{Z}(p)).$$

The divisibility of $\ker(\psi_W^p[\ell^\infty])$ then follows from the divisibility of $K \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell$. (For any abelian group A we have $A \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell$ is divisible.) \square

Remark 2.4. For a complex projective manifold X , the kernel of the Abel–Jacobi map $\psi^p : A^p(X) \rightarrow J_a^{2p-1}(X)$ is not divisible in general for $p > 2$, as shown by the construction of [OS20, Cor. 4.2].

2.5. First proof of Theorem B. Recall from [ACMV20a, Thm. A] that, given a smooth projective variety X over a subfield K of \mathbb{C} , the algebraic intermediate Jacobian $J_a^{2p-1}(X_{\mathbb{C}})$ admits a unique model over K such that the Abel–Jacobi map $\psi^p : A^p(X_{\mathbb{C}}) \rightarrow J_a^{2p-1}(X_{\mathbb{C}})$ is $\text{Aut}(\mathbb{C}/K)$ -equivariant. By Theorem 2.3, the Walker Abel–Jacobi map $\psi_W^p : A^p(X_{\mathbb{C}}) \rightarrow J_W^{2p-1}(X_{\mathbb{C}})$ is universal among lifts of the Abel–Jacobi map along isogenies. We can conclude from Theorem 1.5 that the Walker intermediate Jacobian $J_W^{2p-1}(X_{\mathbb{C}})$ admits a unique model over K such that the Walker Abel–Jacobi map $\psi_W^p : A^p(X_{\mathbb{C}}) \rightarrow J_W^{2p-1}(X_{\mathbb{C}})$ is $\text{Aut}(\mathbb{C}/K)$ -equivariant. \square

3. DESCENDING THE WALKER ABEL–JACOBI MAP

In this section we provide a second proof of Theorem B. It is based on a factorization of the Bloch map restricted to algebraically trivial cycles. This approach will prove crucial for our applications (Corollaries C and D). For that purpose, we start by recasting the results of §2 in the ℓ -adic setting.

3.1. The Bloch map and the coniveau filtration, ℓ -adically. For lack of a suitable reference, we start with a comparison between the analytic and ℓ -adic coniveau filtrations:

Lemma 3.1. *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. We have canonical identifications*

$$N^i H_{\text{ét}}^j(X_{\mathbb{C}}, \mathbb{Z}_{\ell}) = N^i H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}) \otimes \mathbb{Z}_{\ell}. \quad (3.1)$$

Moreover, the natural action of $\text{Aut}(\mathbb{C}/K)$ -action on $H_{\text{ét}}^j(X_{\mathbb{C}}, \mathbb{Z}_{\ell})$ induces an action on $N^i H_{\text{ét}}^j(X_{\mathbb{C}}, \mathbb{Z}_{\ell})$.

Proof. We have the following commutative diagram :

$$\begin{array}{ccccccc} 0 & \longrightarrow & N^i H_{\text{ét}}^j(X_{\mathbb{C}}, \mathbb{Z}_{\ell}) & \longrightarrow & H_{\text{ét}}^j(X_{\mathbb{C}}, \mathbb{Z}_{\ell}) & \longrightarrow & \varinjlim H_{\text{ét}}^j(X_{\mathbb{C}} \setminus Z, \mathbb{Z}_{\ell}) \\ & & & & \simeq \uparrow & & \simeq \uparrow \\ 0 & \longrightarrow & N^i H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}_{\ell}) & \longrightarrow & H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}_{\ell}) & \longrightarrow & \varinjlim H^j((X_{\mathbb{C}} \setminus Z)^{\text{an}}, \mathbb{Z}_{\ell}) \\ & & & & \simeq \uparrow & & \simeq \uparrow \\ 0 & \longrightarrow & N^i H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}) \otimes \mathbb{Z}_{\ell} & \longrightarrow & H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}) \otimes \mathbb{Z}_{\ell} & \longrightarrow & \varinjlim H^j((X_{\mathbb{C}} \setminus Z)^{\text{an}}, \mathbb{Z}) \otimes \mathbb{Z}_{\ell} \end{array}$$

Here the limits are taken over all closed subschemes Z of $X_{\mathbb{C}}$ of codimension $\leq i$. The top two rows are exact by definition of the coniveau filtration, while the third is also exact by flatness of the \mathbb{Z} -module \mathbb{Z}_{ℓ} . The bottom vertical arrows are isomorphisms by flatness of \mathbb{Z}_{ℓ} and the fact that \varinjlim commutes with \otimes . The top two vertical arrows are the isomorphisms provided by Artin's comparison theorem. Thus we obtain the desired identification.

The action of $\text{Aut}(\mathbb{C}/K)$ on $N^i H_{\text{ét}}^j(X_{\mathbb{C}}, \mathbb{Z}_{\ell})$ comes from the fact that the coniveau filtration on $X_{\mathbb{C}}$ can be obtained using subvarieties defined over K (as can be seen by spreading out and by using smooth base-change, followed by taking Galois-orbits). \square

As an immediate consequence of Lemma 3.1, we obtain :

Lemma 3.2. *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. Then the restriction of the $\text{Aut}(\mathbb{C}/K)$ -equivariant Bloch map $\lambda^p : \text{CH}^p(X_{\mathbb{C}})[\ell^{\infty}] \rightarrow H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(p))$ to algebraically trivial cycles factors uniquely into the following commutative diagram of $\text{Aut}(\mathbb{C}/K)$ -modules :*

$$\begin{array}{ccccc} & & N^{p-1} H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p)) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell} & & \\ & \nearrow \lambda_W^p & \downarrow & & \\ A^p(X_{\mathbb{C}})[\ell^{\infty}] & \xrightarrow{\lambda^p} & H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p)) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell} & \longrightarrow & H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(p)), \end{array}$$

where the vertical arrow is induced by the inclusion $N^{p-1} H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p)) \subseteq H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p))$.

Proof. The factorization as groups follows directly from Lemma 2.1 together with the identification (3.1). Now, since $H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p))$ has finite rank and finite torsion, the elementary Fact 1.1 shows that the lift λ_W^p is uniquely determined by λ^p . In addition, still by Fact 1.1, since both λ^p and the inclusion $N^{p-1} H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p)) \subseteq H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(p))$ are $\text{Aut}(\mathbb{C}/K)$ -equivariant, then so is λ_W^p . \square

Remark 3.3. In forthcoming work, we will give a new proof of Lemma 3.2 and extend it to positive characteristic.

3.2. The Walker Abel–Jacobi map on torsion and the Bloch map, ℓ -adically. From the identification (2.2) and Lemma 3.1, we obtain the following canonical identification of abelian groups :

$$J_W^{2p-1}(X_C)[\ell^\infty] = N^{p-1} H_{\text{ét}}^{2p-1}(X_C, \mathbb{Z}_\ell(p)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell / \mathbb{Z}_\ell, \quad (3.2)$$

which is the ℓ -adic analogue of the identification (2.2). In addition, by the comparison isomorphism in cohomology, Proposition 2.2 provides a commutative diagram :

$$\begin{array}{ccc} & & J^{2p-1}(X)[\ell^\infty] \\ & \nearrow \text{AJ}[\ell^\infty] & \parallel \\ \text{CH}^p(X)_{\text{hom}}[\ell^\infty] & \xrightarrow{\lambda^p} & H_{\text{ét}}^{2p-1}(X_C, \mathbb{Z}_\ell(p)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell / \mathbb{Z}_\ell. \end{array}$$

The following lemma will play a crucial role in the proof of Theorem B. It shows that, *via* the identification (3.2), the restriction of the Walker Abel–Jacobi map to ℓ -primary torsion coincides with the factorization of the Bloch map given in Lemma 3.2.

Lemma 3.4. *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. On algebraically trivial cycles of ℓ -primary torsion, the map λ_W^p coincides with the Walker Abel–Jacobi map ψ_W^p , i.e., the following diagram commutes :*

$$\begin{array}{ccc} & & J_W^{2p-1}(X_C)[\ell^\infty] \\ & \nearrow \psi_W^p[\ell^\infty] & \parallel (3.2) \\ A^p(X_C)[\ell^\infty] & \xrightarrow{\lambda_W^p} & N^{p-1} H_{\text{ét}}^{2p-1}(X_C, \mathbb{Z}_\ell(p)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell / \mathbb{Z}_\ell. \end{array}$$

Proof. This follows directly from restricting the previous diagram to algebraically trivial cycles and from the fact that λ_W^p (resp. $\psi_W^p[\ell^\infty]$) are the unique lifts of λ^p (resp. $\psi^p[\ell^\infty]$). \square

3.3. Second proof of Theorem B. Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. Recall that we showed in [ACMV20a, Thm. A] (see also [ACMV19a, Thm. 9.1]) that $J_a^{2p-1}(X_C)$ admits a unique model over K such that the Abel–Jacobi map $\psi^p : A^p(X_C) \rightarrow J_a^{2p-1}(X_C)$ is $\text{Aut}(\mathbb{C}/K)$ -equivariant. We are going to show that α descends uniquely to K with respect to the above K -structure on $J_a^{2p-1}(X_C)$. The $\text{Aut}(\mathbb{C}/K)$ -equivariance of $\psi_W^p : A^p(X) \rightarrow J_W^{2p-1}(X)$ then follows from the unicity of ψ_W^p .

To that end, let C be a K -pointed, geometrically integral, smooth projective curve over K , together with a correspondence $\Gamma \in \text{CH}^p(C \times_K X)$ such that the induced homomorphism $J(C_C) \rightarrow J_a^{2p-1}(X_C)$ is surjective. The existence of such a C and Γ is provided by [ACMV20a, Prop. 1.1]. We thus obtain a commutative diagram

$$\begin{array}{ccccc} A^1(C_C) & \xrightarrow{\psi^1 = \psi_W^1} & J(C_C)(\mathbb{C}) & & \\ \Gamma_* \downarrow & & \gamma \downarrow & \searrow & \\ A^p(X_C) & \xrightarrow{\psi_W^p} & J_W^{2p-1}(X_C)(\mathbb{C}) & \xrightarrow{\alpha} & J_a^{2p-1}(X_C)(\mathbb{C}), \end{array} \quad (3.3)$$

where the homomorphism γ , which is defined by the fact that the Jacobian of a curve together with the Abel map is a universal regular homomorphism, is also induced by the correspondence $\Gamma_* : H^1(C_C^{\text{an}}, \mathbb{Z}(1)) \rightarrow H^{2p-1}(X_C^{\text{an}}, \mathbb{Z}(p))$ (which factors through $N^{p-1} H^{2p-1}(X_C^{\text{an}}, \mathbb{Z}(p))$;

see e.g., [ACMV20b, Prop. 1.1]). We then show that, with respect to the K -structure on $J(C_C)$ given by the Jacobian $J(C)$ of C , the surjective homomorphism γ descends to K . (That $\alpha \circ \gamma$ descends to K was established in [ACMV20a, §2].) For that purpose, by the elementary [ACMV20a, Lem. 2.3], it suffices to show that, for all primes ℓ , the ℓ -primary torsion in

$$P := \ker(\gamma : J(C_C) \rightarrow J_W^{2p-1}(X_C))$$

is stable under the action of $\text{Aut}(C/K)$ on $J(C)(C)$.

For this we take ℓ -primary torsion in the commutative diagram (3.3), then use the compatibility of the Bloch map with the Walker Abel–Jacobi map (Lemma 3.4) to obtain the commutative diagram

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 & & & & P[\ell^\infty] \\
 & & & & \downarrow \\
 & & \psi^1[\ell^\infty] & \searrow & \\
 A^1(C_C)[\ell^\infty] & \xrightarrow[\sim]{\lambda^1[\ell^\infty]} & H_{\text{ét}}^1(C_C, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \xlongequal{\quad} & J(C_C)[\ell^\infty] \\
 \downarrow \Gamma_* & & \downarrow \Gamma_* & & \downarrow \gamma \\
 A^p(X_C)[\ell^\infty] & \xrightarrow{\lambda_W^p[\ell^\infty]} & N^{p-1} H_{\text{ét}}^{2p-1}(X_C, \mathbb{Z}_\ell(p)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & = & J_W^{2p-1}(X_C)[\ell^\infty] \\
 & & \psi_W^p[\ell^\infty] & \swarrow & \downarrow \\
 & & & & 0
 \end{array}$$

The only things that needs explaining is the middle vertical map : here we are using the fact that the Bloch map is compatible with correspondences, and the fact mentioned above that the correspondence $\Gamma_* : H_{\text{ét}}^1(C_C, \mathbb{Z}_\ell(1)) \rightarrow H_{\text{ét}}^{2p-1}(X_C, \mathbb{Z}_\ell(p))$ factors through $N^{p-1} H_{\text{ét}}^{2p-1}(X_C, \mathbb{Z}_\ell(p))$. (Although we do not strictly need it for the argument, we note that the left hand square is $\text{Aut}(C/K)$ -equivariant due to Lemma 3.1.)

Therefore $P[\ell^\infty]$ is identified with the kernel of

$$\Gamma_* : H_{\text{ét}}^1(C_C, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow N^{p-1} H_{\text{ét}}^{2p-1}(X_C, \mathbb{Z}_\ell(p)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

which, since Γ is defined over K , is stable under the action of $\text{Aut}(C/K)$. We have thus showed that γ descends to K . Combined with the fact [ACMV20a, §2] that $\alpha \circ \gamma$ also descends to K with respect to the K -structure of $J(C_C)$ given by $J(C_C) = J(C)_C$, we readily obtain that α descends to K (e.g., by the elementary [ACMV20a, Lem. 2.4]). \square

3.4. Further remarks.

Remark 3.5 (Base change of field). If X is a smooth projective variety over a field $K \subseteq L \subseteq \mathbb{C}$, then there is a canonical identification $J_{W, X_L/L}^{2p-1} = (J_{W, X/K}^{2p-1})_L$.

Remark 3.6 (Independence of embedding of K in \mathbb{C}). Let X be a smooth *complex* projective variety. For a smooth projective complex variety Z and an automorphism $\sigma \in \text{Aut}(\mathbb{C})$, we denote $Z^\sigma := Z \otimes_\sigma \mathbb{C}$ the base-change of Z along σ . Arguing as in the proof of [ACMV19b, Prop. 3.1] shows the following extension of Theorem B : for all $\sigma \in \text{Aut}(\mathbb{C})$ there is a canonical identification

$$J_W^{2p-1}(X^\sigma) = J_W^{2p-1}(X)^\sigma$$

and a commutative diagram

$$\begin{array}{ccc} \mathbf{A}^p(X) & \xrightarrow{\psi_{W,X}^p} & J_W^{2p-1}(X) \\ \sigma^* \downarrow & & \downarrow \sigma^* \\ \mathbf{A}^p(X^\sigma) & \xrightarrow{\psi_{W,X^\sigma}^p} & J_W^{2p-1}(X^\sigma). \end{array}$$

As a consequence, for a smooth projective variety X over a field K of characteristic 0, the kernel of the Walker Abel–Jacobi map associated to X and an embedding of K into \mathbb{C} is independent of that embedding.

4. APPLICATIONS REGARDING THE CONIVEAU FILTRATION

4.1. Modeling coniveau – on a question of Mazur. In this paragraph, we show Corollary C stating that the model of the Walker intermediate Jacobian over K from Theorem B models the torsion-free quotient of $N^{p-1} H_{\text{ét}}^{2p-1}(X_{\overline{K}}, \mathbb{Z}_\ell)$.

Proof of Corollary C. Let $J_{W,X/K}^{2p-1}$ be the model over K , provided by Theorem B, of the Walker intermediate Jacobian $J_W^{2p-1}(X_{\mathbb{C}})$ making ψ_W^p an $\text{Aut}(\mathbb{C}/K)$ -equivariant homomorphism. By the very construction of $J_{W,X/K}^{2p-1}$, and the identification (3.2), we have for all primes ℓ an $\text{Aut}(\mathbb{C}/K)$ -equivariant identification $T_\ell J_{W,X/K}^{2p-1} = N^{p-1} H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_\ell)_\tau$, thereby concluding the proof of Corollary C. \square

Remark 4.1. Following on Remark 3.6, for X a smooth complex projective variety, the identifications of Corollary C more generally fit in the commutative diagram

$$\begin{array}{ccc} T_\ell J_W^{2p-1}(X) & \xlongequal{\quad} & N^{p-1} H_{\text{ét}}^{2p-1}(X, \mathbb{Z}_\ell)_\tau \\ \sigma^* \downarrow & & \downarrow \sigma^* \\ T_\ell J_W^{2p-1}(X^\sigma) & \xlongequal{\quad} & N^{p-1} H_{\text{ét}}^{2p-1}(X^\sigma, \mathbb{Z}_\ell)_\tau \end{array}$$

for every $\sigma \in \text{Aut}(\mathbb{C})$.

4.2. The image of the ℓ -adic Bloch map. Following up on [ACMV20b], Corollary D determines exactly the image of the ℓ -adic Bloch map in case the base field K has zero characteristic. Here is a proof of that corollary :

Proof of Corollary D. By the Lefschetz principle we may and do assume $K \subseteq \mathbb{C}$. By rigidity, it suffices to establish the proposition after base-change to \mathbb{C} . Taking Tate modules in the commutative diagram of Lemma 3.4, we obtain a commutative diagram

$$\begin{array}{ccc} & & T_\ell J_W^{2p-1}(X_{\mathbb{C}}) \\ & \nearrow T_\ell \psi_W^p & \parallel \\ T_\ell \mathbf{A}^p(X_{\mathbb{C}}) & \xrightarrow{T_\ell \lambda_W^p} & N^{p-1} H_{\text{ét}}^{2p-1}(X_{\mathbb{C}}, \mathbb{Z}_\ell(p))_\tau. \end{array}$$

The proposition then follows from the fact that the Walker Abel–Jacobi map does not lift along non-trivial isogenies (Theorem 2.3) and from the equivalence of (3) and (4) in Corollary 1.6. \square

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