

The p -rank strata of the moduli space of hyperelliptic curves

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ABSTRACT

We prove results about the intersection of the p -rank strata and the boundary of the moduli space of hyperelliptic curves in characteristic $p \geq 3$. Using this, we prove that the \mathbb{Z}/ℓ -monodromy of every irreducible component of the stratum \mathcal{H}_g^f of hyperelliptic curves of genus g and p -rank f is the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ if $g \geq 3$, $f \neq 0$, and $\ell \neq p$ is an odd prime. These results yield applications about the generic behavior of hyperelliptic curves of given genus and p -rank. The first application is that a generic hyperelliptic curve of genus $g \geq 3$ and p -rank 0 is not supersingular. Other applications are about absolutely simple Jacobians and the generic behavior of class groups and zeta functions of hyperelliptic curves of given genus and p -rank over finite fields.

1. Introduction

Suppose C is a smooth connected projective hyperelliptic curve of genus $g \geq 1$ over an algebraically closed field k of characteristic $p \geq 3$. The Jacobian $\mathrm{Pic}^0(C)$ is a principally polarized abelian variety of dimension g . The number of physical p -torsion points of $\mathrm{Pic}^0(C)$ is p^f for some integer f , called the p -rank of C , with $0 \leq f \leq g$.

Let \mathcal{H}_g be the moduli space over k of smooth connected projective hyperelliptic curves of genus g ; it is a smooth Deligne-Mumford stack over k . The p -rank induces a stratification $\mathcal{H}_g = \cup \mathcal{H}_g^f$ by locally closed reduced substacks \mathcal{H}_g^f , whose geometric points correspond to hyperelliptic curves of genus g and p -rank f .

In this paper, we prove two results about \mathcal{H}_g^f for $g \geq 3$. The first is about the boundary of \mathcal{H}_g^f , specifically that the boundary of every irreducible component of \mathcal{H}_g^f contains the moduli point of a singular curve which is a tree of elliptic curves and which has p -rank f . The second is that, for an odd prime number ℓ distinct from p , the ℓ -adic monodromy group of every irreducible component of \mathcal{H}_g^f is the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. Heuristically, this means that p -rank constraints alone do not force the existence of extra automorphisms (or other algebraic cycles) on a family of hyperelliptic curves.

This paper is a natural generalization of our paper [AP08], which is about \mathcal{M}_g^f , the p -rank strata of the moduli space of curves. The two papers share essential similarities, but there are several new phenomena for hyperelliptic p -rank strata which increase the difficulty of the proofs and influence the final results in this paper. First, the boundary component Δ_0 is more complicated for $\overline{\mathcal{H}}_g$ than for $\overline{\mathcal{M}}_g$. Second, for a singular hyperelliptic curve which is formed as a chain of two hyperelliptic curves of smaller genera, the set of possibilities for the location of the ordinary double point is

discrete. These two facts play a key role in the degeneration arguments which are part of the inductive step of the proofs. The third issue, which arises in a base case for the monodromy result, is that the stratum \mathcal{H}_3^0 is not as well understood as \mathcal{M}_3^0 .

We now state the results of this paper more precisely.

THEOREM 3.11(C). *Suppose p is an odd prime, $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f , the p -rank f stratum in \mathcal{H}_g . Then the closure \bar{S} of S in $\overline{\mathcal{H}}_g$ contains the moduli point of a tree of g elliptic curves, of which f are ordinary and $g - f$ are supersingular.*

Our proof does not yield much information on the structure of the tree in Theorem 3.11; however, once the tree's structure is fixed, we prove that any choice of labeling of f components as ordinary and $g - f$ components as supersingular will occur for some moduli point in \bar{S} .

As a consequence of Theorem 3.11, we give an application to Newton polygons of hyperelliptic curves. This relies on and generalizes [Oor91, Thm. 1.12], which is the case $g = 3$.

COROLLARY 3.16. *Suppose p is an odd prime and $g \geq 3$. Let S be an irreducible component of \mathcal{H}_g^0 , the p -rank 0 stratum in \mathcal{H}_g . Then S contains the moduli point of a curve whose Jacobian is not supersingular.*

The second result requires some notation. Let S be a connected stack over k , and let s be a geometric point of S . Let $C \rightarrow S$ be a relative smooth proper curve of genus g over S . Then $\text{Pic}^0(C)[\ell]$ is an étale cover of S with geometric fibers isomorphic to $(\mathbb{Z}/\ell)^{2g}$. The fundamental group $\pi_1(S, s)$ acts linearly on the fiber $\text{Pic}^0(C)[\ell]_s$, and the monodromy group $M_\ell(C \rightarrow S, s)$ is the image of $\pi_1(S, s)$ in $\text{Aut}(\text{Pic}^0(C)[\ell]_s)$. For the second result, we determine $M_\ell(S) := M_\ell(C \rightarrow S, s)$, where S is an irreducible component of \mathcal{H}_g^f and $C \rightarrow S$ is the tautological curve.

THEOREM 4.10-4.11. *Suppose p is an odd prime, $g \geq 1$, and $0 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f , the p -rank f stratum in \mathcal{H}_g .*

- (i) *If $1 \leq f \leq g$ and ℓ is an odd prime distinct from p , then $M_\ell(S) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$*
- (ii) *If $f = 0$ and if $g \geq 5$, then $M_\ell(S) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$ for all primes ℓ outside a finite set which depends only on p .*

We also prove that the ℓ -adic monodromy group is $\text{Sp}_{2g}(\mathbb{Z}_\ell)$ in the situation of Theorem 4.10-4.11. (Note that the case of ordinary hyperelliptic curves, i.e., when $f = g$, follows directly from previous work, see J.K. Yu [unpublished], [AP07, Thm. 3.4], or [Hal08, Thm. 5.1].) In addition, we determine the p -adic monodromy of components of \mathcal{H}_g^f when $f \geq 1$ (Proposition 4.13).

We give applications to class groups and zeta functions of hyperelliptic curves of given genus and p -rank over finite fields. These build upon [KS99, Thm. 9.7.13] and [Kow06, Thm. 6.1].

Applications: Let \mathbb{F} be a finite field of characteristic p . Under the hypotheses of Theorem 4.10-4.11:

- (i) there is an $\overline{\mathbb{F}}$ -curve C of genus g and p -rank f whose Jacobian is absolutely simple (Application 4.14);
- (ii) if $|\mathbb{F}| \equiv 1 \pmod{\ell}$, about $\ell/(\ell^2 - 1)$ of the \mathbb{F} -curves of genus g and p -rank f have a point of order ℓ on their Jacobian (Application 4.16);
- (iii) for most \mathbb{F} -curves C of genus g and p -rank f , the splitting field of the numerator of the zeta function of C has degree $2^g g!$ over \mathbb{Q} (Application 4.18).

The proof of Theorem 4.10-4.11 relies on Theorem 3.11 because it uses degeneration in order to proceed by induction on the genus. For the inductive step, Theorem 3.11 implies that the closure

of every component S of \mathcal{H}_g^f in $\overline{\mathcal{H}}_g$ intersects the component $\Delta_{1,1}$ of the boundary of \mathcal{H}_g (Corollary 3.14). As in [AP07], this implies that the monodromy group of S contains two non-identical copies of $\mathrm{Sp}_{2g-2}(\mathbb{Z}/\ell)$. The base cases use [Cha05, Prop. 4.4] for \mathcal{H}_2^f and an analysis of endomorphism algebras for \mathcal{H}_5^0 .

Here is an outline of the paper. Notation and background are found in Section 2. Section 3 contains the results about the boundary of the p -rank f stratum \mathcal{H}_g^f and the application to Newton polygons. This section ends with some open questions about the geometry of \mathcal{H}_g^f . For example, the number of irreducible components of \mathcal{H}_g^f is known only in special cases. The results about monodromy and the applications to class groups and zeta functions are in Section 4.

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2. Background

Let k be an algebraically closed field of characteristic $p \geq 3$. With the exception of Section 4.5, where we work over a finite field, all objects are defined on the category of k -schemes. Let ℓ be an odd prime distinct from p . We fix an isomorphism $\mu_\ell \simeq \mathbb{Z}/\ell$. Let $r \geq 1$ be a natural number.

2.1 Moduli spaces

Consider the following well-known categories, each of which is fibered in groupoids over the category of k -schemes in the étale topology:

- \mathcal{A}_g principally polarized abelian schemes of dimension g ;
- \mathcal{M}_g smooth connected proper relative curves of genus g ;
- \mathcal{H}_g smooth connected proper relative hyperelliptic curves of genus g ;
- $\overline{\mathcal{M}}_g$ stable curves of genus g ; and $\overline{\mathcal{M}}_{g,r}$ stable curves of genus g with r marked points.

Each of these is a smooth Deligne-Mumford stack and $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,r}$ are proper [DM69, Thm. 5.2]. Let \mathcal{C}_g be the tautological curve over $\overline{\mathcal{M}}_g$, and let $\mathcal{X}_g \rightarrow \overline{\mathcal{M}}_g$ be the semiabelian scheme which represents its Picard functor, so that $\mathcal{X}_{g,s} = \mathrm{Pic}^0(\mathcal{C}_{g,s})$. There is a natural inclusion $\mathcal{H}_g \rightarrow \mathcal{M}_g$. Let $\overline{\mathcal{H}}_g$ be the closure of \mathcal{H}_g in $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{H}}_{g,r} = \overline{\mathcal{H}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,r}$. Thus there are the following categories:

- $\overline{\mathcal{H}}_g$ stable hyperelliptic curves of genus g ;
- $\overline{\mathcal{H}}_{g,r}$ stable hyperelliptic curves of genus g with r marked points;
- $\tilde{\mathcal{H}}_g$ stable hyperelliptic curves of genus g , along with a labeling of the smooth ramification locus (see [AP07, Section 2.2]);
- $\tilde{\mathcal{H}}_{g,r}$ stable hyperelliptic curves of genus g with r marked points, along with a labeling of the smooth ramification locus.

Again, these are smooth proper Deligne-Mumford stacks [Eke95, Thm. 3.2], see e.g. [AP07, Lemma 2.2] for $\tilde{\mathcal{H}}_g$. If $S \subset \overline{\mathcal{H}}_g$, let \overline{S} be the closure of S in $\overline{\mathcal{H}}_g$.

The forgetful map $\phi_{g,r} : \overline{\mathcal{H}}_{g,r} \rightarrow \overline{\mathcal{H}}_g$ is proper, flat and surjective with connected fibers, and thus is a fibration. Let $\mathcal{H}_{g,r} = \overline{\mathcal{H}}_{g,r} \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$. The forgetful map $\omega_g : \tilde{\mathcal{H}}_g \rightarrow \overline{\mathcal{H}}_g$ is étale and Galois, with covering group $\Gamma_g \cong \mathrm{Sym}(2g+2)$.

2.2 Stratifications

Let X be a principally polarized abelian variety of dimension g defined over k . There is an integer $f(X)$, called the p -rank of X , such that $X[p](k) \cong (\mathbb{Z}/p)^{f(X)}$. More generally, if X/k is a semiabelian variety, then its p -rank is $\dim_{\mathbb{F}_p} \text{Hom}(\mu_p, X)$. The p -rank of a curve is that of its Jacobian. If $X \rightarrow S$ is a semiabelian scheme over a Deligne-Mumford stack, then there is a stratification $S = \cup S^f$ by locally closed substacks such that $s \in S^f(k)$ if and only if $f(X_s) = f$ (this follows from [Kat79, Thm. 2.3.1], see, e.g., [AP08, Lemma 2.1]). Thus, \mathcal{H}_g^f is the locus in \mathcal{H}_g parametrizing hyperelliptic curves of p -rank f .

Every component of \mathcal{H}_g^f has dimension $g - 1 + f$ [GP05, Thm. 1]. If ξ is a geometric generic point of \mathcal{H}_g^f (with $f \geq 1$ if $g \leq 2$), then $\text{Aut}(\mathcal{C}_{g,\xi}) = \{\pm 1\}$ [AGP08, Thm. 3.7].

We recall the definition of the Newton polygon of an abelian variety; see [Dem72, Chap. IV] for details. The isogeny class of a p -divisible group G/k is determined by $\nu(G)$, a lower-convex polygon in \mathbb{R}^2 connecting $(0, 0)$ to $(\text{height}(G), \dim(G))$ with slopes $\lambda \in \mathbb{Q} \cap [0, 1]$ and integral breakpoints. If X/k is an abelian variety, its Newton polygon is that of its p -divisible group $X[p^\infty]$. The Newton polygon is a finer invariant than the p -rank; indeed, the p -rank of X is exactly the length of the slope 0 part of the Newton polygon. For example, X is ordinary exactly when its Newton polygon only has slopes 0 and 1. By definition, X is supersingular if its Newton polygon only has slope $1/2$.

2.3 The boundary of the moduli space of hyperelliptic curves

The boundary of $\overline{\mathcal{H}}_g$ is $\partial\overline{\mathcal{H}}_g = \overline{\mathcal{H}}_g - \mathcal{H}_g$. The following description of $\partial\overline{\mathcal{H}}_g$ follows [CH88, Sec. 4(b)] closely. In fact, while [CH88] is written for the base field \mathbb{C} , the description of \mathcal{H}_g and $\overline{\mathcal{H}}_g$ is valid in any characteristic [Yam04]. Briefly, the irreducible components of $\partial\overline{\mathcal{H}}_g$ come from restriction of the components of the boundary of $\overline{\mathcal{M}}_g$, except that Δ_0 breaks into several components.

If $g \geq 2$, the boundary $\partial\overline{\mathcal{H}}_g$ is the union of components $\Delta_i = \Delta_i[\overline{\mathcal{H}}_g]$ for $1 \leq i \leq g - 1$ and $\Xi_i = \Xi_i[\overline{\mathcal{H}}_g]$ for $0 \leq i \leq g - 2$ by [Yam04, p.410]. Here Δ_i and Δ_{g-i} are the same substack of $\overline{\mathcal{H}}_g$ and Ξ_i and Ξ_{g-i-1} are the same substack of $\overline{\mathcal{H}}_g$. Each Δ_i and Ξ_i is an irreducible divisor in $\overline{\mathcal{H}}_g$.

For $1 \leq i \leq g - 1$, if η is the generic point of Δ_i , then the curve $\mathcal{C}_{g,\eta}$ is a chain of two smooth irreducible hyperelliptic curves Y_1 and Y_2 , of genera i and $g - i$, intersecting in one ordinary double point P . The hyperelliptic involution ι stabilizes each of Y_1 and Y_2 . The point P is a ramification point for the restriction of ι to each of Y_1 and Y_2 but is not part of the smooth ramification locus.

If η is the generic point of Ξ_0 , then the curve $\mathcal{C}_{g,\eta}$ is an irreducible hyperelliptic curve self-intersecting in an ordinary double point P . The normalization of $\mathcal{C}_{g,\eta}$ is a smooth hyperelliptic curve Y_1 of genus $g - 1$ and the inverse image of P in the normalization consists of an orbit under the hyperelliptic involution.

For $1 \leq i \leq g - 2$, if η is the generic point of Ξ_i , then the curve $\mathcal{C}_{g,\eta}$ has two components Y_1 and Y_2 , which are smooth irreducible hyperelliptic curves, of genera i and $g - 1 - i$, intersecting in two ordinary double points P and Q . The hyperelliptic involution ι stabilizes each of Y_1 and Y_2 . The points P and Q form an orbit of the restriction of ι to each of Y_1 and Y_2 .

One can associate to a stable curve C its dual graph, in which the vertices are in bijection with the irreducible components of C and in which there is an edge between two vertices exactly when the corresponding components intersect. A component of C is called terminal if the corresponding vertex is a leaf of the dual graph. A curve is called a tree if its dual graph is a tree. A curve is called a tree of elliptic curves if it is a tree and if each of its irreducible components is an elliptic curve.

A stable curve is a tree if and only if its Picard variety is represented by an abelian scheme; such a curve is also said to be of compact type. Let $\Delta_0 = \Delta_0[\overline{\mathcal{H}}_g]$ be the union of Ξ_i for $0 \leq i \leq$

$\lfloor (g-1)/2 \rfloor$. The moduli points of stable hyperelliptic curves which are not of compact type are exactly the points of $\Delta_0[\overline{\mathcal{H}}_g]$.

If S is a stack equipped with a map $S \rightarrow \overline{\mathcal{H}}_g$, let $\Delta_i[S]$ denote $S \times_{\overline{\mathcal{H}}_g} \Delta_i[\overline{\mathcal{H}}_g]$. In particular, $\Delta_i[\widetilde{\mathcal{H}}_g] = \widetilde{\mathcal{H}}_g \times_{\overline{\mathcal{H}}_g} \Delta_i$. Also define $\Delta_i[\overline{\mathcal{H}}_g]^f := (\Delta_i[\overline{\mathcal{H}}_g])^f$. Similar conventions are employed for Ξ_i .

2.4 Clutching maps

Here are three types of clutching maps for positive integers g_1 and g_2 :

$$\kappa_{g_1, g_2} : \widetilde{\mathcal{H}}_{g_1} \times \widetilde{\mathcal{H}}_{g_2} \xrightarrow{\widetilde{\kappa}_{g_1, g_2}} \widetilde{\mathcal{H}}_{g_1+g_2} \xrightarrow{\omega_{g_1+g_2}} \overline{\mathcal{H}}_{g_1+g_2};$$

$$\kappa_{g_1} : \overline{\mathcal{H}}_{g_1;1} \longrightarrow \overline{\mathcal{H}}_{g_1+1};$$

$$\lambda_{g_1, g_2} : \overline{\mathcal{H}}_{g_1;1} \times \overline{\mathcal{H}}_{g_2;1} \longrightarrow \overline{\mathcal{H}}_{g_1+g_2+1}.$$

Each of these clutching maps is the restriction of a finite, unramified morphism between moduli spaces of labeled curves [Knu83, Cor. 3.9]. We will describe these maps in terms of their images on T -points for an arbitrary k -scheme T .

2.4.1 Information about κ_{g_1, g_2} For $i = 1, 2$, suppose $s_i \in \widetilde{\mathcal{H}}_{g_i}(T)$ is the moduli point of a hyperelliptic curve Y_i with labeled smooth ramification locus. Then $\widetilde{\kappa}_{g_1, g_2}(s_1, s_2)$ is the moduli point of the labeled T -curve Y where Y has components Y_1 and Y_2 , and where the last ramification point of Y_1 and the first ramification point of Y_2 are identified in an ordinary double point. This nodal section is dropped from the labeling of the ramification points; the remaining smooth ramification sections of Y_1 maintain labels $\{1, \dots, 2g_1 + 1\}$ and the remaining ramification sections of Y_2 are relabeled $\{2g_1 + 2, \dots, 2(g_1 + g_2) + 2\}$. There is a unique hyperelliptic involution on Y which restricts to the hyperelliptic involution on Y_1 and Y_2 . Moreover, $\kappa_{g_1, g_2}(s_1, s_2)$ is the moduli point of the (unlabeled) hyperelliptic curve Y . The image of κ_{g_1, g_2} is $\Delta_{g_1}[\overline{\mathcal{H}}_{g_1+g_2}]$.

By [BLR90, Ex. 9.2.8],

$$\mathrm{Pic}^0(Y) \cong \mathrm{Pic}^0(Y_1) \times \mathrm{Pic}^0(Y_2). \quad (2.4.1)$$

Then the p -rank of Y is

$$f(Y) = f(Y_1) + f(Y_2). \quad (2.4.2)$$

2.4.2 Information about κ_{g_1} Suppose $s_1 \in \overline{\mathcal{H}}_{g_1;1}(T)$ is the moduli point of $(Y_1; P)$, a hyperelliptic curve with a marked section. Then $\kappa_{g_1}(s_1)$ is the moduli point of the T -curve Y , where Y is the stable model of the curve obtained by identifying the sections P and $\iota(P)$ in an ordinary double point P' . The hyperelliptic involution on Y_1 descends to a unique hyperelliptic involution on Y . The image of κ_{g_1} is $\Xi_0[\overline{\mathcal{H}}_{g_1+1}]$.

By [BLR90, Ex. 9.2.8], $\mathrm{Pic}^0(D)$ is an extension

$$0 \longrightarrow Z \longrightarrow \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(Y_1) \longrightarrow 0, \quad (2.4.3)$$

where Z is a one-dimensional torus. In particular, the toric rank of $\mathrm{Pic}^0(Y)$ is one greater than that

of $\text{Pic}^0(Y_1)$, and their maximal projective quotients are isomorphic, so that

$$f(Y) = f(Y_1) + 1. \quad (2.4.4)$$

2.4.3 Information about λ_{g_1, g_2} For $i = 1, 2$, suppose $s_i \in \overline{\mathcal{H}}_{g_i, 1}(T)$ is the moduli point of $(Y_i; P_i)$, a hyperelliptic curve with a marked section. Then $\lambda_{g_1, g_2}(s_1, s_2)$ is the moduli point of the stable model Y of the T -curve obtained by identifying P_1 and P_2 in an ordinary double point P and by identifying $\iota(P_1)$ and $\iota(P_2)$ in an ordinary double point Q . There is a unique hyperelliptic involution on Y that restricts to the hyperelliptic involution on Y_1 and Y_2 . The image of λ_{g_1, g_2} is $\Xi_{g_1}[\overline{\mathcal{H}}_{g_1+g_2+1}]$.

By [BLR90, Ex. 9.2.8], $\text{Pic}^0(Y)$ is an extension

$$0 \longrightarrow Z \longrightarrow \text{Pic}^0(Y) \longrightarrow \text{Pic}^0(Y_1) \times \text{Pic}^0(Y_2) \longrightarrow 0, \quad (2.4.5)$$

where Z is a one-dimensional torus. In particular,

$$f(D) = f(Y_1) + f(Y_2) + 1. \quad (2.4.6)$$

For later use, here is a description of the stable model Y when P_1 is a ramification point of Y_1 , but P_2 is not a ramification point of Y_2 . Then Y consists of three components, namely the strict transforms of Y_1 and Y_2 and an exceptional component W which is a projective line. Also Y_1 intersects W in an ordinary double point and Y_2 intersects W in two other points, which are also ordinary double points.

2.4.4 Clutching along trees The definition of $\tilde{\kappa}_{g_1, g_2}$ above relies on an arbitrary, although convenient, choice of sections along which to glue, and a labeling of the smooth ramification locus of the resulting curve. By considering morphisms of the form $\gamma_{g_1+g_2} \circ \tilde{\kappa}_{g_1, g_2} \circ (\gamma_{g_1} \times \gamma_{g_2})$, where $\gamma_{g_1+g_2} \in \Gamma_{g_1+g_2}$ and $\gamma_{g_1} \in \Gamma_{g_1}$ and $\gamma_{g_2} \in \Gamma_{g_2}$, it is possible to clutch along arbitrary sections, with complete control over the subsequent labeling.

This can be used to describe configurations of curves of compact type, as follows. A clutching tree is a finite tree Λ along with a choice of natural number g_v for each vertex $v \in \Lambda$ such that $\deg(v) \leq 2g_v + 2$. Such a tree is called a clutching tree of elliptic curves if $g_v = 1$ for all $v \in \Lambda$. Let $|\Lambda|$ be the number of vertices in Λ , and let $g(\Lambda) = \sum_{v \in \Lambda} g_v$.

Using a product of the clutching maps defined above, one can define a morphism

$$\kappa_\Lambda : \times_{v \in V} \tilde{\mathcal{H}}_{g_v} \xrightarrow{\tilde{\kappa}_\Lambda} \tilde{\mathcal{H}}_{g(\Lambda)} \xrightarrow{\omega_{g(\Lambda)}} \overline{\mathcal{H}}_{g(\Lambda)}.$$

Let Δ_Λ be the image of κ_Λ . If η is the generic point of Δ_Λ , then $\mathcal{C}_{g, \eta}$ is a hyperelliptic curve of compact type with dual graph isomorphic to Λ , such that the irreducible component of $\mathcal{C}_{g, \eta}$ corresponding to the vertex v has genus g_v .

Suppose v_1 and v_2 are adjacent vertices in a clutching tree Λ . Consider the tree Λ' obtained by identifying v_1 and v_2 in a new vertex, v , which is adjacent to all neighbors of v_1 or v_2 in Λ , with label $g_v = g_{v_1} + g_{v_2}$. Then κ_Λ factors through $\kappa_{\Lambda'}$ and $\Delta_\Lambda \subset \Delta_{\Lambda'}$. A tree Λ refines a tree Λ' if Λ' can be obtained from Λ through iterations of this construction.

2.4.5 *One more clutching map* In the special case where Λ is a tree on three vertices, where the leaves v_1 and v_3 have $g_{v_1} = g_{v_3} = 1$, and where $g_{v_2} = g - 2$, we obtain the following diagram:

$$\begin{array}{ccc}
 \tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_{g-2} \times \tilde{\mathcal{H}}_1 & \longrightarrow & \tilde{\mathcal{H}}_{g-1} \times \tilde{\mathcal{H}}_1 \\
 \downarrow & \searrow^{\kappa_{1,g-2,1}} & \downarrow \\
 \tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_{g-1} & \longrightarrow & \tilde{\mathcal{H}}_g.
 \end{array} \tag{2.4.7}$$

Let $\Delta_{1,1}[\overline{\mathcal{H}}_g]$ be the image of $\kappa_{1,g-2,1} = \omega_g \circ \tilde{\kappa}_{1,g-2,1}$; it is an irreducible component of the self-intersection locus of $\Delta_1[\overline{\mathcal{H}}_g]$. If η is the generic point of $\Delta_{1,1}[\overline{\mathcal{H}}_g]$, then the curve $\mathcal{C}_{g,\eta}$ is a chain of three irreducible smooth hyperelliptic curves Y_1, Y_2, Y_3 with $g_{Y_1} = g_{Y_3} = 1$ and $g_{Y_2} = g - 2$. For $i \in \{1, 3\}$, the curves Y_i and Y_2 intersect in a point P_i which is an ordinary double point.

3. Boundary of the hyperelliptic p -rank strata

3.1 Preliminary intersection results

Suppose $p \geq 3$, and $g \geq 1$ and $0 \leq f \leq g$. The p -rank strata of the boundary of $\overline{\mathcal{H}}_g$ are easy to describe using the clutching maps. First, if $1 \leq i \leq g - 1$, then (2.4.2) implies that $\Delta_i[\overline{\mathcal{H}}_g]^f$ is the union of the images of $\tilde{\mathcal{H}}_i^{f_1} \times \tilde{\mathcal{H}}_{g-i}^{f_2}$ under $\kappa_{i,g-i}$ as (f_1, f_2) ranges over all pairs such that

$$0 \leq f_1 \leq i, 0 \leq f_2 \leq g - i \text{ and } f_1 + f_2 = f. \tag{3.1.1}$$

Second, if $f \geq 1$, then $\Xi_0[\overline{\mathcal{H}}_g]^f$ is the image of $\overline{\mathcal{H}}_{g-1,1}^{f-1}$ under κ_{g-1} by (2.4.4). Third, if $f \geq 1$ and $1 \leq i \leq g - 2$, then (2.4.6) implies that $\Xi_i[\overline{\mathcal{H}}_g]^f$ is the image of $\overline{\mathcal{H}}_{i,1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i,1}^{f_2}$ under $\lambda_{i,g-1-i}$ as (f_1, f_2) ranges over all pairs such that

$$0 \leq f_1 \leq i, 0 \leq f_2 \leq g - 1 - i \text{ and } f_1 + f_2 = f - 1. \tag{3.1.2}$$

LEMMA 3.1. *Suppose $g \geq 2$ and $0 \leq f \leq g$.*

- (a) *If $1 \leq i \leq g - 1$, then every component of $\Delta_i[\overline{\mathcal{H}}_g]^f$ and of $\Delta_i[\tilde{\mathcal{H}}_g]^f$ has dimension $g - 2 + f$.*
- (b) *If $f \geq 1$ and $0 \leq i \leq g - 2$, then every component of $\Xi_i[\overline{\mathcal{H}}_g]^f$ and of $\Xi_i[\tilde{\mathcal{H}}_g]^f$ has dimension $g - 2 + f$.*

Proof. For parts (a) and (b), the claims for $\overline{\mathcal{H}}_g$ and for $\tilde{\mathcal{H}}_g$ are equivalent, since ω_g is a finite map which preserves the p -rank stratification. Recall that $\overline{\mathcal{H}}_g^f$ and $\tilde{\mathcal{H}}_g^f$ are pure of dimension $g - 1 + f$ [GP05, Thm. 1].

For part (a), suppose $0 \leq f \leq g$, $1 \leq i \leq g - 1$, and that (f_1, f_2) is a pair which satisfies (3.1.1). Then $\tilde{\mathcal{H}}_i^{f_1} \times \tilde{\mathcal{H}}_{g-i}^{f_2}$ is pure of dimension $\dim(\tilde{\mathcal{H}}_i^{f_1}) + \dim(\tilde{\mathcal{H}}_{g-i}^{f_2}) = g - 2 + f$. Since $\kappa_{i,g-i}$ is finite, $\Delta_i[\overline{\mathcal{H}}_g]^f$ is pure of dimension $g - 2 + f$ as well.

For part (b), first suppose $i = 0$. Then $\overline{\mathcal{H}}_{g-1,1}^{f-1}$ is pure of dimension $\dim(\overline{\mathcal{H}}_{g-1}^{f-1}) + 1 = g - 2 + f$. Since κ_{g-1} is finite, $\Xi_0[\overline{\mathcal{H}}_g]^f$ is pure of dimension $g - 2 + f$ as well.

To finish part (b), suppose $1 \leq i \leq g - 2$, and that (f_1, f_2) is a pair which satisfies (3.1.2). Then $\overline{\mathcal{H}}_{i,1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i,1}^{f_2}$ is pure of dimension $\dim(\overline{\mathcal{H}}_i^{f_1}) + \dim(\overline{\mathcal{H}}_{g-1-i}^{f_2}) + 2 = g - 2 + f$. Since $\lambda_{i,g-1-i}$ is finite, $\Xi_i[\overline{\mathcal{H}}_g]^f$ is pure of dimension $g - 2 + f$ as well. \square

The next lemma shows that if η is a generic point of $\overline{\mathcal{H}}_g^f$, then the curve $\mathcal{C}_{g,\eta}$ is smooth. Thus no component of $\overline{\mathcal{H}}_g^f$ is contained in the boundary $\partial\overline{\mathcal{H}}_g$.

LEMMA 3.2. *Suppose $g \geq 1$ and $0 \leq f \leq g$.*

- (a) *Then \mathcal{H}_g^f is open and dense in $\overline{\mathcal{H}}_g^f$ and $\widetilde{\mathcal{H}}_g^f \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$ is open and dense in $\widetilde{\mathcal{H}}_g^f$.*
- (b) *If $r \geq 1$, then $\mathcal{H}_{g;r}^f$ is open and dense in $\overline{\mathcal{H}}_{g;r}^f$ and $\widetilde{\mathcal{H}}_{g;r}^f \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$ is open and dense in $\widetilde{\mathcal{H}}_{g;r}^f$.*

Proof. Part (a) is well-known for $g = 1$. For $g \geq 2$, part (a) follows from Lemma 3.1 since $\overline{\mathcal{H}}_g^f$ and $\widetilde{\mathcal{H}}_g^f$ are pure of dimension $g - 1 + f$ [GP05, Thm. 1]. Part (b) follows from part (a) since the p -rank of a labeled curve depends only on the underlying curve, so that $\overline{\mathcal{H}}_{g;r}^f = \overline{\mathcal{H}}_{g;r} \times_{\overline{\mathcal{H}}_g} \overline{\mathcal{H}}_g^f$. \square

LEMMA 3.3. *Let S be an irreducible component of \mathcal{H}_g^f . If \overline{S} intersects a component Γ of $\Delta_i[\overline{\mathcal{H}}_g]^f$ then \overline{S} contains Γ .*

Proof. A smooth proper stack has the same intersection-theoretic properties as a smooth proper scheme [Vis89, p. 614]. In particular, if two closed substacks of $\overline{\mathcal{H}}_g$ intersect then the codimension of their intersection is at most the sum of their codimensions. Now \overline{S} and $\Delta_i[\overline{\mathcal{H}}_g]$ are closed substacks of $\overline{\mathcal{H}}_g$ and $\overline{S} \not\subset \Delta_i[\overline{\mathcal{H}}_g]$. Thus the intersection of \overline{S} with the divisor $\Delta_i[\overline{\mathcal{H}}_g]$ has pure dimension $\dim \overline{S} - 1$, which equals $\dim(\Delta_i[\overline{\mathcal{H}}_g]^f)$ by Lemma 3.1. Thus if \overline{S} intersects a component Γ of $\Delta_i[\overline{\mathcal{H}}_g]^f$ then it must contain the full component Γ . \square

LEMMA 3.4. *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f .*

- (a) *Then \overline{S} intersects $\Delta_0[\overline{\mathcal{H}}_g]$ if and only if $f \geq 1$.*
- (b) *If $f \geq 1$, then each irreducible component of $\Delta_0[\overline{S}]$ contains either (i) the image of a component of $\overline{\mathcal{H}}_{g-1}^{f-1}$ under κ_{g-1} or (ii) the image of a component of $\overline{\mathcal{H}}_{i-1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i}^{f_2}$ under $\lambda_{i,g-1-i}$ for some $1 \leq i \leq g-2$ and some pair (f_1, f_2) which satisfies (3.1.2).*
- (c) *If $f = 0$, then \overline{S} contains the image of a component of $\widetilde{\mathcal{H}}_i^0 \times \widetilde{\mathcal{H}}_{g-i}^0$ under $\kappa_{i,g-i}$ for some $1 \leq i \leq g-1$.*

The containment in Lemma 3.4(b) may be strict since $\Xi_i[\overline{S}]$ may contain points s such that $\mathcal{C}_{g,s}$ has p -rank strictly less than f .

Proof. If $f = 0$, then (2.4.4) implies that \overline{S} does not intersect $\Delta_0[\overline{\mathcal{H}}_g]$. If $f \geq 1$, then \overline{S} is a complete substack of dimension greater than $g - 1$. By [FvdG04, Lemma 2.6], a complete substack of $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$ has dimension at most $g - 1$. Therefore, \overline{S} intersects $\Delta_0[\overline{\mathcal{H}}_g]$ nontrivially. This completes part (a).

For part (b), each irreducible component of $\Delta_0[\overline{S}]$ intersects either $\kappa_{g-1}(\overline{\mathcal{H}}_{g-1}^{f-1}) \subset \Xi_0[\overline{\mathcal{H}}_g]$ or $\lambda_{i,g-1-i}(\overline{\mathcal{H}}_{i-1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i}^{f_2}) \subset \Xi_i[\overline{\mathcal{H}}_g]$ for some $1 \leq i \leq g-2$ and some pair (f_1, f_2) which satisfies (3.1.2). The result then follows from Lemma 3.3.

For part (c), recall that \mathcal{H}_g contains no complete substacks of positive dimension (e.g., [Yam04, Cor. 1.9]). Thus \overline{S} intersects Δ_i for some $0 \leq i \leq g-1$. By part (a), $i \neq 0$. The result follows from Lemma 3.3. \square

3.2 Complements on trees

Many of the results for Δ_i for positive i have analogues for Δ_Λ . For a clutching tree Λ and a non-negative integer f , define an index set by

$$\mathcal{F}(\Lambda, f) = \left\{ \{f_v : v \in \Lambda\} : 0 \leq f_v \leq g_v, \sum_v f_v = f \right\}. \quad (3.2.1)$$

LEMMA 3.5. Let Λ be a clutching tree with $g(\Lambda) = g$.

(a) The p -rank strata of $\Delta_\Lambda[\overline{\mathcal{H}}_g]$ are given by

$$\Delta_\Lambda[\overline{\mathcal{H}}_g]^f = \bigcup_{\{f_v\} \in \mathcal{F}(\Lambda, f)} \kappa_\Lambda(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v}). \quad (3.2.2)$$

(b) Every component of $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$ has dimension $g + f - |\Lambda|$.

Proof. Part (a) follows from (3.1.1) and induction on $|\Lambda|$. Part (b) follows from this and the calculation that, for $\{f_v\} \in \mathcal{F}(\Lambda, f)$,

$$\dim(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v}) = \sum_{v \in \Lambda} (g_v + f_v - 1) = g + f - |\Lambda|.$$

□

LEMMA 3.6. Let S be an irreducible component of \mathcal{H}_g^f . Let Λ be a clutching tree with $g(\Lambda) = g$. If \overline{S} intersects a component Γ of $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$, then \overline{S} contains Γ .

Proof. The proof is similar to that of Lemma 3.3. Note that $\dim \Gamma \geq \dim \overline{S} + \dim \Delta_\Lambda - \dim \overline{\mathcal{H}}_g$. By Lemma 3.5(b), this equals $g + f - |\Lambda| = \dim \Delta_\Lambda[\overline{\mathcal{H}}_g]^f$. □

3.3 Adjusting marked points and trees

The next lemma shows that one can adjust the marked points of an r -marked hyperelliptic curve of genus g and p -rank f without leaving the irreducible component of $\overline{\mathcal{H}}_{g;r}^f$ to which its moduli point belongs.

LEMMA 3.7. Let S be an irreducible component of $\mathcal{H}_{g;r}^f$, and let \overline{S} be the closure of S in $\overline{\mathcal{H}}_{g;r}^f$. Then $\overline{S} = \phi_{g;r}^{-1}(\phi_{g;r}(\overline{S}))$. Equivalently, if T is a k -scheme, if $(C; P_1, \dots, P_r) \in \overline{S}(T)$, and if (Q_1, \dots, Q_r) is any other marking of C , then $(C; Q_1, \dots, Q_r) \in \overline{S}(T)$.

Proof. Let S be an irreducible component of $\mathcal{H}_{g;r}^f$, with closure \overline{S} . It suffices to show that $\phi_{g;r}^{-1}(\phi_{g;r}(\overline{S})) \subseteq \overline{S}$. Note that \overline{S} is the largest irreducible substack of $\overline{\mathcal{H}}_{g;r}^f$ which contains S . The fibers of $\phi_{g;r}|_{\overline{S}}$ are irreducible, so $\phi_{g;r}^{-1}(\phi_{g;r}(S))$ is also an irreducible substack of $\overline{\mathcal{H}}_{g;r}^f$ which contains S . Thus $\phi_{g;r}^{-1}(\phi_{g;r}(S)) \subset \overline{S}$. This shows that $\phi_{g;r}^{-1}(\phi_{g;r}(S)) = S$.

To finish the proof, it suffices to show that the T -points of \overline{S} and $\phi_{g;r}^{-1}(\phi_{g;r}(\overline{S}))$ coincide for an arbitrary k -scheme T . To this end, let $\alpha = (C; P_1, \dots, P_r) \in \overline{S}(T)$, and let $\beta = (C; Q_1, \dots, Q_r) \in \overline{\mathcal{H}}_{g;r}^f(T)$. Note that $\phi_{g;r}(\beta) = \phi_{g;r}(\alpha)$, and $\phi_{g;r}(\alpha)$ is supported in the closure of $\phi_{g;r}(S)$ in $\overline{\mathcal{H}}_g^f$. Because $\mathcal{H}_{g;r}$ is dense in $\overline{\mathcal{H}}_{g;r}$, it follows that β is supported in the closure of $\phi_{g;r}^{-1}(\phi_{g;r}(S))$ in $\overline{\mathcal{H}}_{g;r}^f$, which is \overline{S} . □

It is not clear whether one can change the labeling of the smooth ramification locus of a hyperelliptic curve without changing the irreducible component of $\overline{\mathcal{H}}_g^f$ to which its moduli point belongs. To circumvent this issue, the following lemma about hyperelliptic curves of genus 2 and p -rank 1 will be useful.

LEMMA 3.8. (a) First, $\overline{\mathcal{H}}_2^1$ is irreducible and intersects $\kappa_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$.

(b) Second, let \tilde{S} be an irreducible component of $\tilde{\mathcal{H}}_2^1$. If \tilde{S} intersects $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$, then \tilde{S} also intersects $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^1)$.

Proof. For part (a), recall that the Torelli morphism $\mathcal{H}_2 \rightarrow \mathcal{A}_2$ is an inclusion [OS80, Lemma 1.11]. Since $\dim(\mathcal{H}_2^1) = \dim(\mathcal{A}_2^1)$, and since \mathcal{A}_2^1 is irreducible (e.g., [EvdG09, Ex. 11.6]), it follows that $\overline{\mathcal{H}_2^1}$ is irreducible. Consider a chain Y of two elliptic curves, one ordinary and one supersingular, intersecting in an ordinary double point, which is a fixed point of the hyperelliptic involution on each elliptic curve. The moduli point of Y is in the intersection of $\kappa_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$ and $\overline{\mathcal{H}_2^1}$.

For part (b), let S^* be the closure of \tilde{S} in $\tilde{\mathcal{H}}_2$. The reason for working with S^* is that it can contain the moduli points of curves with p -rank 0. By hypothesis and Lemma 3.3, \tilde{S} contains a component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$. Every component of \tilde{H}_1^1 contains a component of $\tilde{\mathcal{H}}_1^0$ in its closure since any nonisotrivial proper family of curves of genus one has supersingular fibers. It follows that S^* contains a component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^0)$, and thus intersects the closure of a component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^1)$. By Lemma 3.3, S^* contains a component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^1)$, which then implies the same for \tilde{S} . \square

REMARK 3.9. Note that a genus two curve has six ramification points and thus there are potentially up to $6! = |\Gamma_2|$ irreducible components of $\tilde{\mathcal{H}}_2^1$. In particular, the fact from Lemma 3.8(a) that $\overline{\mathcal{H}_2^1}$ intersects $\kappa_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$ does not imply the hypothesis in part (b) that \tilde{S} intersects $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_1^0)$.

LEMMA 3.10. *Let S be an irreducible component of \mathcal{H}_g^f . Suppose Λ is a clutching tree of elliptic curves with $g(\Lambda) = g$. If \bar{S} intersects $\Delta_\Lambda[\overline{\mathcal{H}}_g]$, then for any choice of $\{f_v\} \in \mathcal{F}(\Lambda, f)$, \bar{S} contains an irreducible component of $\kappa_\Lambda(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v})$.*

Proof. By Lemma 3.6, there exist a choice of data $\{f_v^*\} \in \mathcal{F}(\Lambda, f)$ and components $\tilde{T}_v \subset \tilde{\mathcal{H}}_{g_v}^{f_v^*}$ such that \bar{S} contains $\kappa_\Lambda(\times_{v \in \Lambda} \tilde{T}_v)$. One immediately reduces to the case in which v_1 and v_2 are adjacent vertices in Λ with $f_{v_1}^* = 1$ and $f_{v_2}^* = 0$, and $\{f_v\} \in \mathcal{F}(\Lambda, f)$ is given by

$$f_v = \begin{cases} f_v^* & v \notin \{v_1, v_2\} \\ 1 - f_v^* & v \in \{v_1, v_2\}. \end{cases}$$

Let Λ' be the tree obtained by identifying v_1 and v_2 in a new vertex v_{12} with $g_{v_{12}} = 2$. By Lemma 3.6, there is a component $\tilde{T}_{v_{12}} \subset \tilde{\mathcal{H}}_2^1$ such that \bar{S} contains

$$\kappa_{\Lambda'}(\tilde{T}_{v_{12}} \times (\times_{v \in \Lambda', v \neq v_{12}} \tilde{T}_v)).$$

Now, $\tilde{T}_{v_{12}}$ contains a component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \mathcal{H}_1^0)$. By Lemma 3.8(b), $\tilde{T}_{v_{12}}$ contains a component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_1^1)$ as well. Then \bar{S} contains a component of $\kappa_\Lambda(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v})$. \square

3.4 Main intersection results

In this section, we prove that the closure of each irreducible component S of \mathcal{H}_g^f contains the moduli point of a singular curve which is a tree of elliptic curves and has p -rank f .

THEOREM 3.11. *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f .*

- There exists a clutching tree of elliptic curves Λ with $g(\Lambda) = g$ such that \bar{S} contains an irreducible component of $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$.*
- For any choice of $\{f_v\} \in \mathcal{F}(\Lambda, f)$, \bar{S} contains an irreducible component of $\kappa_\Lambda(\times_{v \in \Lambda} \tilde{\mathcal{H}}_{g_v}^{f_v})$.*
- In particular, \bar{S} contains the moduli point of a tree of elliptic curves, of which f are ordinary and $g - f$ are supersingular.*

Proof. It suffices to prove part (c) since parts (a) and (c) are equivalent by Lemma 3.6 and since parts (a) and (b) are equivalent by Lemma 3.10.

First suppose $g = 2$. If $f = 2$, then \mathcal{H}_2^2 is irreducible and affine and $\overline{\mathcal{H}}_2^2$ contains the moduli point of a tree of 2 ordinary elliptic curves. If $f = 1$ (resp. $f = 0$), the result is true by Lemma 3.8(a) (resp. Lemma 3.4(c)). Now suppose $g \geq 3$ and $0 \leq f \leq g$ and suppose as an inductive hypothesis that the result is true when $2 \leq g' < g$. Let S be an irreducible component of \mathcal{H}_g^f .

CLAIM 3.12. *To complete the proof, it suffices to show that \overline{S} intersects $\Delta_i[\overline{\mathcal{H}}_g]^f$ for some $1 \leq i \leq g - 1$.*

Proof of claim. Suppose \overline{S} intersects $\Delta_i[\overline{\mathcal{H}}_g]^f$ for some $1 \leq i \leq g - 1$. Let $g_1 = i$ and $g_2 = g - i$. By Lemma 3.3, \overline{S} contains a component of $\Delta_{g_1}[\overline{\mathcal{H}}_g]^f$. In other words, there exist a pair (f_1, f_2) satisfying (3.1.1) and, for $j = 1, 2$, components \tilde{V}_j of $\overline{\mathcal{H}}_{g_j}^{f_j}$ such that \overline{S} contains $\kappa_{g_1, g_2}(\tilde{V}_1 \times \tilde{V}_2)$. Then $\overline{V}_j = \omega_{g_j}(\tilde{V}_j)$ is a component of $\overline{\mathcal{H}}_{g_j}^{f_j}$.

By the inductive hypothesis, \overline{V}_j contains the moduli point s_j of a tree Y_j of g_j elliptic curves, of which f_j are ordinary and $g_j - f_j$ are supersingular. Let Λ_j be the dual graph of Y_j . Let $\tilde{s}_j \in \tilde{V}_j$ be such that $\omega_{g_j}(\tilde{s}_j) = s_j$. In other words, \tilde{s}_j is the moduli point of Y_j along with the data of a choice of labeling of the smooth ramification locus. Then $\kappa_{g_1, g_2}(\tilde{s}_1, \tilde{s}_2)$ is the moduli point of a curve C whose dual graph is obtained by connecting a vertex of Λ_1 with a vertex of Λ_2 . Since C is a tree, $\kappa_{g_1, g_2}(\tilde{s}_1, \tilde{s}_2)$ is the moduli point of a tree of g elliptic curves, of which $f = f_1 + f_2$ are ordinary and $g - f$ are supersingular. This completes the proof of the claim since $\kappa_{g_1, g_2}(\tilde{s}_1, \tilde{s}_2)$ is in \overline{S} . \square

Continuing the proof of Theorem 3.11, first suppose $f = 0$. By Lemma 3.4(c), \overline{S} intersects $\Delta_i[\overline{\mathcal{H}}_g]^f$ for some $1 \leq i \leq g - 1$. By Claim 3.12, this completes the proof when $f = 0$.

Now suppose $f > 0$. By Lemma 3.4(a), \overline{S} intersects $\Delta_0[\overline{\mathcal{H}}_g]^f$.

Case (i): \overline{S} intersects Ξ_0 .

By Lemma 3.4(b), \overline{S} contains the image of a component \overline{V}' of $\overline{\mathcal{H}}_{g-1,1}^{f-1}$ under κ_{g-1} . Consider $\overline{V} = \phi_{g-1,1}(\overline{V}')$ which is a component of $\overline{\mathcal{H}}_{g-1}^{f-1}$. By the inductive hypothesis, \overline{V} contains the moduli point of a curve Y_1 which is a tree of $g - 1$ elliptic curves, of which $f - 1$ are ordinary and $g - f$ are supersingular. Let E be a terminal component of Y_1 and let Y'_1 be the closure of $Y_1 - E$ in C . Let R be the point of intersection of E and Y'_1 . Since the quotient of Y_1 by the hyperelliptic involution ι has genus 0, the elliptic curve E is stabilized by ι . Let $P \neq R$ be a point of E which is not a ramification point of ι . By Lemma 3.7, the moduli point t of $(Y_1; P)$ is in \overline{V}' .

Let Z be the singular irreducible hyperelliptic curve of genus two with exactly one ordinary double point P' such that the normalization of Z is the elliptic curve E and the pre-image of P' consists of the points P and $\iota(P)$. In other words, the moduli point of Z is the image of the moduli point of $(E; P)$ under κ_1 .

Consider the point $s = \kappa_{g-1}(t)$ of \overline{S} . The curve C_s has components Z and Y'_1 which intersect in exactly one ordinary double point R . The p -rank of C_s is $f(E) + f(Y'_1) + 1 = f$. Since $g \geq 3$, there is a terminal component of Y'_1 not containing R which is an elliptic curve. Thus s is in $\Delta_1[\overline{\mathcal{H}}_g]^f$. (In fact, s is also in $\Delta_2[\overline{\mathcal{H}}_g]^f$ because of the component Z .) By Claim 3.12, this completes the proof in Case (i).

Case (ii): \overline{S} intersects Ξ_i for some $1 \leq i \leq g - 2$.

Let $g_1 = i$ and $g_2 = g - 1 - i$. By Lemma 3.4(b), \overline{S} contains a component of $\Xi_i[\overline{\mathcal{H}}_g]^f$. In other words, there exists a pair (f_1, f_2) satisfying (3.1.2) and, for $j = 1, 2$, there exist components \overline{V}'_j of $\overline{\mathcal{H}}_{g_j,1}^{f_j}$ such that \overline{S} contains $\lambda_{g_1, g_2}(\overline{V}'_1 \times \overline{V}'_2)$. Then $\overline{V}_j = \phi_{g_j,1}(\overline{V}'_j)$ is a component of $\overline{\mathcal{H}}_{g_j}^{f_j}$.

By the inductive hypothesis, \bar{V}_j contains the moduli point s_j of a tree Y_j of g_j elliptic curves, of which f_j are ordinary and $g_j - f_j$ are supersingular. Let E_j be a terminal component of Y_j and let Y'_j be the closure of $Y_j - E_j$ in C . Let R_j be the point of intersection of E_j and Y'_j . Since the quotient of Y_j by the hyperelliptic involution ι_j has genus 0, the elliptic curve E_j is stabilized by ι_j . Let $P_1 \neq R_1$ be a point of E_1 which is a ramification point of ι_1 . Let $P_2 \neq R_2$ be a point of E_2 which is not a ramification point of ι_2 .

By Lemma 3.7, the moduli point s'_j of $(Y_j; P_j)$ is in \bar{V}'_j . Consider $s = \lambda_{g_1, g_2}(s'_1, s'_2)$ which is a point of \bar{S} . By Section 2.4.3, the components of the stable model C_s are the strict transforms of Y_1 and Y_2 and an exceptional component W which is a projective line. Moreover, Y_1 intersects W in an ordinary double point and Y_2 intersects W in two other points, which are also ordinary double points. The p -rank of C_s is $f(Y_1) + f(Y_2) + 1 = f$.

The curve C_s has a terminal component E'_1 of genus 1. To see this, let $E'_1 = E_1$ when $i = 1$ (e.g., when $g = 3$) and when $i > 1$, let $E'_1 \neq E_1$ be another terminal component of Y_1 . It follows that s is in $\Delta_1[\bar{\mathcal{H}}_g]^f$. (Also s is in $\Delta_i[\mathcal{H}_g]^f$ because of the component Y_1 .) By Claim 3.12, this completes the proof in Case (ii). \square

Here are several consequences of Theorem 3.11 which will be used later in the paper.

In the setting of Theorem 3.11, one can deduce that \bar{S} intersects Δ_i nontrivially only when Λ has an edge whose removal yields two (connected) trees of size i and $g - i$. This is only guaranteed when $i = 1$. Luckily, the following information on degeneration to Δ_1 is sufficient for the later applications in the paper.

COROLLARY 3.13. *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f . Then \bar{S} intersects $\Delta_1[\bar{\mathcal{H}}_g]^f$. Furthermore:*

- (a) *if $f \leq g - 1$, then \bar{S} contains an irreducible component of $\kappa_{1, g-1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_{g-1}^f)$; and*
- (b) *if $f \geq 1$, then \bar{S} contains an irreducible component of $\kappa_{1, g-1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-1}^{f-1})$.*

Proof. By Theorem 3.11(c), \bar{S} contains the moduli point of a tree of elliptic curves, of which f are ordinary and $g - f$ are supersingular. Every tree has a leaf; by Theorem 3.11(b), that leaf can be chosen to be ordinary or supersingular if the obvious necessary constraint is satisfied. The result follows by Lemma 3.6. \square

The ℓ -adic and p -adic monodromy proofs in Section 4 rely on degeneration to $\Delta_{1,1}$. We label the four possibilities for (f_1, f_2, f_3) such that $f_1 + f_2 + f_3 = f$ and $f_1, f_3 \in \{0, 1\}$ as follows: (A) $(1, f - 2, 1)$; (B) $(0, f - 1, 1)$; (B') $(1, f - 1, 0)$; and (C) $(0, f, 0)$.

COROLLARY 3.14. *Suppose $g \geq 3$ and $0 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f .*

- (a) *Then \bar{S} intersects $\Delta_{1,1}[\bar{\mathcal{H}}_g]^f$.*
- (b) *There is an irreducible component \tilde{S} of $\bar{S} \times \tilde{\mathcal{H}}_g$, and a choice of (f_1, f_2, f_3) from cases (A)–(C); and there are irreducible components S_1 of $\tilde{\mathcal{H}}_1^{f_1}$ and S_2 of $\tilde{\mathcal{H}}_{g-2}^{f_2}$ and S_3 of $\tilde{\mathcal{H}}_1^{f_3}$; and there are irreducible components S_R of $\tilde{\mathcal{H}}_{g-1}^{f_2+f_3}$ and S_L of $\tilde{\mathcal{H}}_{g-1}^{f_1+f_2}$; such that the restriction of the clutching maps of 2.4.7 yields a commutative diagram*

$$\begin{array}{ccc}
 S_1 \times S_2 \times S_3 & \longrightarrow & S_1 \times S_R \\
 \downarrow & \searrow^{\kappa_{1, g-2, 1}} & \downarrow \\
 S_L \times S_3 & \longrightarrow & \tilde{S} \cap \Delta_{1,1}[\tilde{\mathcal{H}}_g].
 \end{array} \tag{3.4.1}$$

- (c) Furthermore, case (A) occurs as long as $f \geq 2$, case (B) or (B') occurs as long as $1 \leq f \leq g - 1$, and case (C) occurs as long as $f \leq g - 2$.

Proof. By Theorem 3.11(a), there is a clutching tree of elliptic curves Λ such that \bar{S} contains a component of $\Delta_\Lambda[\tilde{\mathcal{H}}_g]^f$. Let v_1 and v_3 be two leaves of Λ ; using Theorem 3.11(b), one can assume that $f_{v_i} = f_i$ where (f_1, f_2, f_3) is chosen as in part (c). Let Λ' be the tree obtained by coalescing all vertices of Λ except for v_1 and v_3 . Let v_2 denote this new vertex and let $f_{v_2} = f_2$. Since Λ refines Λ' , then \bar{S} intersects $\Delta_{\Lambda'}[\tilde{\mathcal{H}}_g]^f$ which completes part (a). Moreover, there is an irreducible component \tilde{S} of $\bar{S} \times_{\tilde{\mathcal{H}}_g} \tilde{\mathcal{H}}_g$ such that \tilde{S} intersects $\Delta_{1,1}[\tilde{\mathcal{H}}_g]^f$. Part (b) follows from the definition of $\Delta_{1,1}[\tilde{\mathcal{H}}_g]^f$ and Lemma 3.6. \square

COROLLARY 3.15. *Let S be an irreducible component of \mathcal{H}_g^f . For each $0 \leq f' < f$, there exists an irreducible component T of $\mathcal{H}_g^{f'}$ such that \bar{S} contains T .*

Proof. It suffices to prove the result for $f' = f - 1$. Let S^* be the closure of S in $\bar{\mathcal{H}}_g - \Delta_0[\mathcal{H}_g]$. A purity result [Oor74, Lemma 1.6] shows that $S^* - (S^*)^f$, if nonempty, is pure of dimension $\dim S^* - 1$. In particular, let $Z = (S^*)^{f-1}$; then Z , if nonempty, is pure of dimension $g - 2 + f$.

By Corollary 3.13(b), S^* contains an irreducible component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-1}^{f-1})$. Since $\tilde{\mathcal{H}}_1^1$ is dense in $\tilde{\mathcal{H}}_1$, its closure contains the moduli points of supersingular elliptic curves (with labeled smooth ramification locus). Therefore, S^* contains an irreducible component of $\tilde{\kappa}_{1,1}(\tilde{\mathcal{H}}_1^0 \times \tilde{\mathcal{H}}_{g-1}^{f-1})$, and Z is nonempty. Then $\dim Z = g - 2 + f = \dim \bar{\mathcal{H}}_g^{f-1}$, and so Z contains a component of $\bar{\mathcal{H}}_g^{f-1}$. By Lemma 3.2 Z , and thus \bar{S} , contains a component of \mathcal{H}_g^f . \square

3.5 Application to Newton polygons

Recall that a stable curve C of compact type is supersingular if all the slopes of the Newton polygon of its Jacobian equal $1/2$. This is equivalent to the condition that the Jacobian of C is isogenous to a product of g supersingular elliptic curves. Note that a supersingular curve necessarily has p -rank zero. An abelian variety of p -rank zero is necessarily supersingular only when $g \in \{1, 2\}$.

In this section, we prove that the Newton polygon of a generic hyperelliptic curve of p -rank 0 is not supersingular when $g \geq 3$. The result generalizes [Oor91, Thm. 1.12] which is the case $g = 3$.

Newton polygons have the following semicontinuity property: let $S = \text{Spec}(R)$ be the spectrum of a local ring, with geometric generic point $\bar{\eta}$ and geometric closed point \bar{s} ; if G is a p -divisible group over S , then $\nu(G_{\bar{\eta}})$ either equals or lies below $\nu(G_{\bar{s}})$. (The latter condition means that $\nu(G_{\bar{\eta}})$ and $\nu(G_{\bar{s}})$ have the same endpoints and all points of $\nu(G_{\bar{\eta}})$ lie below $\nu(G_{\bar{s}})$.)

COROLLARY 3.16. *Suppose p is an odd prime and $g \geq 3$. Let η be a generic point of \mathcal{H}_g^0 . Then $\mathcal{C}_{g,\eta}$ is not supersingular. In particular, there exists a smooth hyperelliptic curve of genus g and p -rank 0 which is not supersingular.*

Proof. When $g = 3$, this follows from [Oor91, Thm. 1.12]. For $g \geq 4$, the proof proceeds by induction. Let S be the closure of η in \mathcal{H}_g^0 . By Corollary 3.13, \bar{S} contains a component of $\Delta_1[\tilde{\mathcal{H}}_g]^0$. Thus there are components \tilde{V}_1 of $\tilde{\mathcal{H}}_1^0$ and \tilde{V}_2 of $\tilde{\mathcal{H}}_{g-1}^0$ such that \bar{S} contains $\kappa_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$. By the inductive hypothesis, the Newton polygon of the generic point of \tilde{V}_2 is not supersingular; in particular, it has a slope λ such that $0 < \lambda < 1/2$. The same is then true of the generic point of $\kappa_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$. By semicontinuity [Kat79, Thm. 2.3.1], the generic Newton polygon of \bar{S} (and thus of S) either equals or lies below that of $\kappa_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$. In particular, it has a slope $\lambda' < 1/2$. Thus $\mathcal{C}_{g,\eta}$ is not supersingular. \square

REMARK 3.17. When $p = 2$, there are some results about the slopes of Newton polygons of hyperelliptic curves of p -rank 0, see e.g. [SZ02].

3.6 Open questions about the geometry of the hyperelliptic p -rank strata

QUESTION 3.18. Does the closure of each component of \mathcal{H}_g^f contain the moduli point of a chain of elliptic curves with p -rank f ?

If the answer to Question 3.18 is affirmative then Lemma 3.10 implies that every ordering of f ordinary and $g - f$ supersingular elliptic curves occurs for such a chain. In [AP08, Cor. 3.6], the authors show the analogous question has a positive answer for every component of \mathcal{M}_g^f . The difference for \mathcal{M}_g^f is that the clutching morphism identifies two curves at an arbitrary point of each, rather than a ramification point of each. The location of these points can then be changed using Lemma 3.7.

QUESTION 3.19. For $2 \leq i \leq g - 2$, does the closure of each component of \mathcal{H}_g^f intersect $\Delta_i[\mathcal{H}_g^f]$?

In [AP08, Prop. 3.4], the authors show that the analogous question has a positive answer for every component of \mathcal{M}_g^f , also with control over the arrangement of p -ranks. An affirmative answer to Question 3.18 would imply an affirmative answer to Question 3.19.

QUESTION 3.20. How many irreducible components does \mathcal{H}_g^f have?

One knows that \mathcal{H}_g^f is irreducible for all p when $f = g$ or when $g = 2$ and $f = 1$. If $g \geq 3$, then \mathcal{A}_g^f is irreducible by [Cha05, Remark 4.7]. If \mathcal{H}_g^f is irreducible, then there is a very short proof that Questions 3.18 and 3.19 have affirmative answers.

4. Monodromy

In this section, we determine the ℓ -adic monodromy of components of \mathcal{H}_g^f for odd primes ℓ using an inductive process. The first base case, when $g = 2$ and $1 \leq f \leq 2$, relies on a special case of [Cha05, Prop. 4.4]. Unfortunately, the monodromy group of \mathcal{H}_2^0 is small, since supersingular families of abelian varieties have finite ℓ -adic monodromy groups; and the methods of [Cha05] do not apply to \mathcal{H}_g^0 for $g \geq 3$, because the hyperelliptic Torelli locus is not Hecke-stable. Thus the proof requires another base case when $f = 0$. For lack of a strategy to calculate the ℓ -adic monodromy of \mathcal{H}_3^0 , we analyze the case when $g = 5$ and $f = 0$ in Section 4.2 by studying endomorphism algebras.

4.1 Integral monodromy

We summarize the discussion in [AP07, Sec. 3.1] about \mathbb{Z}/ℓ - and \mathbb{Z}_ℓ -monodromy. Let S be a connected k -scheme on which the prime ℓ is invertible. Let $\pi : C \rightarrow S$ be a relative curve of compact type whose fibres have genus g . Then $R^1\pi_*(\mu_\ell)$, or equivalently $\text{Pic}^0(C)[\ell]$, is an étale sheaf of \mathbb{Z}/ℓ -modules. If s is a geometric point of S , then $R^1\pi_*(\mu_\ell)$ is equivalent to a linear representation

$$\rho_{C \rightarrow S, \mathbb{Z}/\ell} : \pi_1(S, s) \rightarrow \text{Aut}((R^1\pi_*(\mu_\ell))_s) \cong \text{GL}_{2g}(\mathbb{Z}/\ell).$$

Let $M_\ell(C \rightarrow S, s)$ be the image of $\rho_{C \rightarrow S, \mathbb{Z}/\ell}$ and let $M_\ell(S)$ be the isomorphism class of this image as an abstract group. If the family $C \rightarrow S$ is clear from context, these will be denoted $M_\ell(S, s)$ and $M_\ell(S)$, respectively. There is a canonical polarization on $\text{Pic}^0(C)$, and thus (after a choice of ℓ^{th} root of unity on S) there is a symplectic pairing on $\text{Pic}^0(C)[\ell]$. Therefore, there is an inclusion of groups $M_\ell(S) \subseteq \text{Sp}_{2g}(\mathbb{Z}/\ell)$. Similarly, for each natural number n there is a representation

$$\rho_{C \rightarrow S, \mathbb{Z}/\ell^n} : \pi_1(S, s) \rightarrow \text{Aut}((R^1\pi_*(\mu_{\ell^n}))_s).$$

Let $M_{\mathbb{Z}_\ell}(C \rightarrow S, s) = \lim_{\leftarrow n} \rho_{C \rightarrow S, \mathbb{Z}/\ell^n}(\pi_1(S, s))$. One can employ an analogous formalism to define the monodromy group of a relative curve over a stack [Noo04].

4.2 A base case for p -rank zero

In this section we show that if S is an irreducible component of \mathcal{H}_5^0 , then $M_\ell(S) \cong \mathrm{Sp}_{10}(\mathbb{Z}/\ell)$ for all primes ℓ outside some finite set. The set of exceptional primes may depend on the characteristic p of the base field, so that our results are valid for $\ell \gg_p 0$. Recall that $\mathcal{X}_g \rightarrow \overline{\mathcal{H}}_g$ represents the relative Picard functor.

To ease notation, if X is an abelian variety, let $\mathbf{E}(X) = \mathrm{End}(X) \otimes \mathbb{Q}$, and let $\mathbf{E}_\ell(X) = \mathrm{End}(X) \otimes \mathbb{Z}_\ell$; then $\mathbf{E}_\ell(X)$ acts on the Tate module $T_\ell(X)$. If X is simple, then the center of $\mathbf{E}(X)$ is either a totally real or totally imaginary number field.

LEMMA 4.1. *Let ξ be a geometric generic point of \mathcal{H}_3^0 . Then either $\mathbf{E}(\mathcal{X}_{3,\xi}) \cong \mathbb{Q}$ or $\mathbf{E}(\mathcal{X}_{3,\xi})$ is a totally real cubic field.*

Proof. Suppose there is an isogeny $\mathcal{X}_{3,\xi} \sim A_1 \oplus A_2$ for abelian varieties A_1 and A_2 of dimensions 1 and 2. Then A_1 and A_2 each have p -rank 0 and are thus supersingular. Then $\mathcal{X}_{3,\xi}$ is supersingular, which contradicts the fact that the Newton polygon of $\mathcal{X}_{3,\xi}$ has slopes $1/3$ and $2/3$ [Oor91, Thm. 1.12]. Thus $\mathcal{X}_{3,\xi}$ is simple.

From the classification of simple endomorphism algebras of abelian varieties of prime dimension (e.g., [Oor88, 7.2]), to complete the proof it suffices to show that neither a complex multiplication field of degree six nor a quadratic imaginary field acts on $\mathcal{X}_{3,\xi}$. Let S be the closure of ξ in \mathcal{H}_3^0 . Since $\dim S = 2 > 0$ but abelian varieties with complex multiplication are rigid, $\mathbf{E}(\mathcal{X}_{3,\xi})$ is not a complex multiplication field of degree 6.

To address the possibility of an action by a quadratic imaginary field, suppose to the contrary that there is a subring of $\mathrm{End}(\mathcal{X}_{3,\xi})$ isomorphic to an order \mathcal{O}_K in a quadratic imaginary field K . Then $\mathcal{O}_K \otimes \mathbb{Z}_p$ acts on the p -divisible group $\mathcal{X}_{3,\xi}[p^\infty]$. There is an inclusion $K \otimes \mathbb{Q}_p \hookrightarrow \mathrm{End}(\mathcal{X}_{3,\xi}[p^\infty]) \otimes \mathbb{Q}_p \cong D_{1/3} \oplus D_{2/3}$. (Here, D_λ denotes the central simple \mathbb{Q}_p -algebra with Brauer invariant λ .) Since every maximal subfield of $D_{1/3}$ or $D_{2/3}$ is a cubic extension of \mathbb{Q}_p , $K \otimes \mathbb{Q}_p$ cannot be a field. In particular, $\mathcal{X}_{3,\xi}$ does not admit an action by a quadratic imaginary field inert or ramified at p .

Finally, suppose $\mathcal{X}_{3,\xi}$ admits an action by a quadratic imaginary field which splits at p . Let (r, s) be the signature of the action of \mathcal{O}_K on $\mathrm{Lie}(\mathcal{X}_{3,\xi})$; the dimensions r and s are nonnegative and $r + s = 3$. Consider the moduli space $Sh_{\mathcal{O}_K;(r,s)}$ of abelian threefolds with an action by \mathcal{O}_K of signature (r, s) . The Torelli morphism τ restricts to a finite morphism from S to a component of $Sh_{\mathcal{O}_K;(r,s)}$. Since $\dim S = 2$ and $\dim Sh_{\mathcal{O}_K;(r,s)} = r \cdot s$, then (r, s) is either $(1, 2)$ or $(2, 1)$. Thus $\tau(S)$ is dense in $Sh_{\mathcal{O}_K;(r,s)}$. This gives a contradiction since $\mathcal{C}_{3,\xi}$ has p -rank zero but the generic member of $Sh_{\mathcal{O}_K;(r,s)}$ is ordinary [Wed99, Thm. 1.6.2]. \square

REMARK 4.2. In the situation of Lemma 4.1, using the description of the p -rank strata of Hilbert modular threefolds, one can show that if $\mathbf{E}(\mathcal{X}_{3,\xi})$ is a cubic field, then it is either (totally) inert or ramified at p . This will not be used in the sequel.

LEMMA 4.3. *Let Y be a simple abelian variety whose dimension g is relatively prime to 3. If there exists a geometric generic point ξ_3 of \mathcal{H}_3^0 for which there is a nontrivial homomorphism $\mathrm{End}(Y) \rightarrow \mathbf{E}(\mathcal{X}_{3,\xi_3})$, then $\mathrm{End}(Y) \cong \mathbb{Z}$.*

Proof. If Z is a simple abelian variety, let $\mathbf{E}_0(Z)$ be the subring of $\mathbf{E}(Z)$ fixed by the Rosati involution, and let $e_0(Z) = [\mathbf{E}_0(Z) : \mathbb{Q}]$. Then $e_0(Z) \mid \dim(Z)$.

By Lemma 4.1, $\mathbf{E}(\mathcal{X}_{3,\xi_3})$ is a totally real field of dimension 1 or 3 over \mathbb{Q} . On one hand, the existence of a nontrivial homomorphism $\mathrm{End}(Y) \rightarrow \mathbf{E}(\mathcal{X}_{3,\xi_3})$ forces $e_0(Y)$ to divide $e_0(\mathcal{X}_{3,\xi_3})$, and

thus $e_0(Y)|3$. On the other hand, $e_0(Y)|g$. Therefore, $e_0(Y) = 1$ and $E_0(Y) \cong \mathbb{Q}$. Neither a noncommutative algebra nor a totally imaginary field admits a nontrivial homomorphism to $\mathbf{E}(\mathcal{X}_{3,\xi_3})$, and thus $\mathbf{E}(Y) = \mathbf{E}_0(Y)$ and $\text{End}(Y) \cong \mathbb{Z}$. \square

LEMMA 4.4. *Let $X \rightarrow S$ be a polarized abelian scheme over a reduced irreducible noetherian stack. Let η be the generic point of S , and let $s \in S$ be any point. Then there exists an inclusion $\text{End}(X_{\bar{\eta}}) \hookrightarrow \text{End}(X_{\bar{s}})$.*

Proof. By introducing a rigidifying structure on $X \rightarrow S$, such as coordinates on the space of sections of the third power of the given ample line bundle, we may and do assume S is an integral noetherian scheme. Since the absolute endomorphism ring of an abelian variety is attained after a finite of the base field, it suffices to show the existence of an inclusion $\text{End}(X_{\eta}) \rightarrow \text{End}(X_s)$. If S is normal, then $\text{End}(X_{\eta})$ extends uniquely to $\text{End}(X_s)$, and in particular to $\text{End}(X_s)$ [FC90, I.2.7]. In general, let $\nu : S' \rightarrow S$ be the normalization map; let η' be the generic point of S' , and let s' be a point of S' over s . The desired result follows from the canonical map $\text{End}((\nu^*X)_{\eta'}) \rightarrow \text{End}((\nu^*X)_{s'})$ and the isomorphisms of abelian varieties $(\nu^*X)_{\eta'} \cong X_{\eta}$ and $(\nu^*X)_{s'} \cong X_s$. \square

PROPOSITION 4.5. *If ξ_4 is a geometric generic point of \mathcal{H}_4^0 , then $\text{End}(\mathcal{X}_{4,\xi_4}) \cong \mathbb{Z}$.*

Proof. Let \bar{S}_4 be the closure of ξ_4 in \mathcal{H}_4^0 .

The first step is to show that \mathcal{X}_{4,ξ_4} is simple. Suppose there is an isogeny $\mathcal{X}_{4,\xi_4} \sim A_1 \oplus A_2$ for two abelian varieties A_1 and A_2 . If A_1 and A_2 each have dimension 2, then they are supersingular since they have p -rank 0. Then \mathcal{X}_{4,ξ_4} is supersingular, which contradicts Corollary 3.16. If A_1 has dimension 1 and A_2 has dimension 3, then there is a curve W of genus 3 such that $\text{Jac}(W) \simeq A_2$. The inclusion of A_2 into \mathcal{X}_{4,ξ_4} yields a cover $\psi : C_{4,\xi_4} \rightarrow W$. By the Riemann-Hurwitz formula $6 \geq 4\deg(\psi)$ which is impossible since $\deg(\psi) \geq 2$. Thus \mathcal{X}_{4,ξ_4} is simple.

By Corollary 3.13(a) there exist components $\tilde{V}_1 \subset \tilde{\mathcal{H}}_1^0$ and $\tilde{V}_2 \subset \mathcal{H}_3^0$ such that \bar{S} contains $\kappa_{1,3}(\tilde{V}_1 \times \tilde{V}_2)$. Let ξ_1 and ξ_3 be geometric generic points of \tilde{V}_1 and \tilde{V}_2 , respectively, and let $\eta = \kappa_{1,3}(\xi_1, \xi_3)$. Since \mathcal{X}_{1,ξ_1} is an elliptic curve and since \mathcal{X}_{3,ξ_3} is simple, there are no nontrivial homomorphisms between \mathcal{X}_{3,ξ_3} and \mathcal{X}_{1,ξ_1} . Since \mathcal{X}_{4,ξ_4} is simple, this yields an isomorphism

$$\mathbf{E}(\mathcal{X}_{4,\eta}) \cong \mathbf{E}(\mathcal{X}_{1,\xi_1}) \oplus \mathbf{E}(\mathcal{X}_{3,\xi_3}).$$

By Lemma 4.4, there is an inclusion $\mathbf{E}(\mathcal{X}_{4,\xi_4}) \hookrightarrow \mathbf{E}(\mathcal{X}_{3,\xi_3})$. Then $\text{End}(\mathcal{X}_{4,\xi_4}) \cong \mathbb{Z}$ by Lemma 4.3. \square

PROPOSITION 4.6. *If $g = 5$ and ξ_5 is a geometric generic point of \mathcal{H}_5^0 , then $\text{End}(\mathcal{X}_{5,\xi_5}) \cong \mathbb{Z}$.*

Proof. Let ξ_5 be a geometric generic point of \mathcal{H}_5^0 , with closure \bar{S}_5 in $\bar{\mathcal{H}}_5^0$.

By Corollary 3.14, \bar{S}_5 intersects $\Delta_{1,1}[\bar{\mathcal{H}}_5]^0$ and there is an irreducible component \tilde{S} of $\bar{S}_5 \times \tilde{\mathcal{H}}_5$, and there are irreducible components S_1 of $\tilde{\mathcal{H}}_1^0$ and S_2 of $\tilde{\mathcal{H}}_3^0$ and S_3 of $\tilde{\mathcal{H}}_1^0$; and there are irreducible components S_R of $\tilde{\mathcal{H}}_4^0$ and S_L of $\tilde{\mathcal{H}}_4^0$; such that the restriction of the clutching maps yields a commutative diagram

$$\begin{array}{ccc} S_1 \times S_2 \times S_3 & \longrightarrow & S_1 \times S_R \\ \downarrow & \searrow^{\tilde{\kappa}_{1,3,1}} & \downarrow \\ S_L \times S_3 & \longrightarrow & \tilde{S} \cap \Delta_{1,1}[\tilde{\mathcal{H}}_5]. \end{array} \quad (4.2.1)$$

Let η_i be the generic point of S_i for $i \in \{1, 2, 3\}$; similarly, let η_L be the generic point of S_L , and η_R that of S_R . Let $s = \tilde{\kappa}_{1,3,1}(\eta_1, \eta_2, \eta_3)$.

Choose a prime ℓ such that for any supersingular elliptic curve E/k , $\mathbf{E}_{\ell}(E) \cong \text{Mat}_2(\mathbb{Z}_{\ell})$. There is a canonical isomorphism $T_{\ell}(\mathcal{X}_{5,s}) \cong T_{\ell}(\mathcal{X}_{1,\eta_1}) \times T_{\ell}(\mathcal{X}_{3,\eta_2}) \times T_{\ell}(\mathcal{X}_{1,\eta_3})$. Choose coordinates on the

10-dimensional \mathbb{Z}_ℓ -module $T_\ell(\mathcal{X}_{5,s})$ compatible with this decomposition. By Lemma 4.4 and Proposition 4.5, $\mathbf{E}_\ell(\mathcal{X}_{5,\tilde{\kappa}_{4,1}(\eta_L \times \eta_3)})$ acts on $T_\ell(\mathcal{X}_{5,s})$ as $\text{diag}_8(\mathbb{Z}_\ell) \oplus \text{Mat}_2(\mathbb{Z}_\ell)$. Similarly, $\mathbf{E}_\ell(\mathcal{X}_{5,\tilde{\kappa}_{1,4}(\eta_1 \times \eta_R)})$ acts as $\text{Mat}_2(\mathbb{Z}_\ell) \oplus \text{diag}_8(\mathbb{Z}_\ell)$. Invoking Lemma 4.4 twice shows that

$$\mathbf{E}_\ell(\mathcal{X}_{5,\xi_5}) \subseteq \mathbf{E}_\ell(\mathcal{X}_{5,\tilde{\kappa}_{4,1}(\eta_L \times \eta_3)}) \cap \mathbf{E}_\ell(\mathcal{X}_{5,\tilde{\kappa}_{1,4}(\eta_1 \times \eta_R)}) \subseteq \text{End}(T_\ell(\mathcal{X}_{5,s})).$$

Therefore, $\mathbf{E}_\ell(\mathcal{X}_{5,\xi_5}) \cong \mathbb{Z}_\ell$ and $\text{End}(\mathcal{X}_{5,\xi_5}) \cong \mathbb{Z}$. \square

REMARK 4.7. One can use the same argument to show inductively that if ξ is a generic point of \mathcal{H}_g^0 for any $g \geq 4$, then $\text{End}(\mathcal{X}_\xi)$. Instead, we will deduce this in Application 4.14 as a consequence of a monodromy calculation.

COROLLARY 4.8. *For $\ell \gg_p 0$, if S is an irreducible component of \mathcal{H}_5^0 then $M_\ell(S) \cong \text{Sp}_{10}(\mathbb{Z}/\ell)$.*

Proof. Since \mathcal{H}_5^0 has only finitely many irreducible components, it suffices to prove the statement for $\ell \gg_p 0$ for each component S . This follows from Proposition 4.6 and Serre's open image theorem [Ser00, Thm. 3]. (In fact, Serre's result is stated for abelian varieties of odd dimension over number fields; but his argument, which involves only the fundamental group of the base and the structure of symplectic groups, works in this generality [Ser00, 8.2].) \square

4.3 Monodromy of the hyperelliptic p -rank strata

In this section, let p and ℓ be distinct odd primes. We find the integral monodromy of the p -rank strata \mathcal{H}_g^f when $1 \leq f \leq g$. We also find the integral monodromy of \mathcal{H}_g^0 for $g \geq 5$ for all but finitely many ℓ . The integral monodromy of \mathcal{H}_g , which is the same as the case $f = g$, already appears in [AP07, Thm. 3.4] (see also unpublished work of J.-K. Yu, and [Hal08, Thm. 5.1]).

The following argument shows that to determine the monodromy of families of hyperelliptic curves, one may work with either $\overline{\mathcal{H}}_g$ or $\tilde{\mathcal{H}}_g$.

LEMMA 4.9. *Let p and ℓ be distinct odd primes and suppose $g \geq 2$. Let $S \subset \mathcal{H}_g$ be irreducible and let \tilde{S} be an irreducible component of $\overline{S} \times_{\overline{\mathcal{H}}_g} \tilde{\mathcal{H}}_g$. Then $M_\ell(\tilde{S}) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$ if and only if $M_\ell(S) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$.*

Proof. Since $\tilde{S} \rightarrow S$ is finite, $M_\ell(\tilde{S})$ is a subgroup of $M_\ell(\overline{S})$. If $M_\ell(\tilde{S}) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$ then $M_\ell(\tilde{S})$ is maximal, and thus so are $M_\ell(\overline{S})$ and $M_\ell(S)$.

Conversely, suppose $M_\ell(S) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$. Since ϖ_g is étale with Galois group $\Gamma_g \cong \text{Sym}(2g + 2)$, the cover $\tilde{S} \rightarrow \overline{S}$ is Galois with covering group G a subgroup of Γ_g . Consider the finite simple group $\text{PSp}_{2g}(\mathbb{Z}/\ell)$. The smallest integer N for which there exists an embedding $\text{PSp}_{2g}(\mathbb{Z}/\ell) \hookrightarrow \text{Sym}(N)$ is $N = (\ell^{2g} - 1)/(\ell - 1)$ [Gre03, Thm. 3]. Thus G and $\text{Sp}_{2g}(\mathbb{Z}/\ell)$ have no common non-trivial quotient, and $M_\ell(\tilde{S}) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$ by the argument of [AP07, Lemma 3.3]. \square

THEOREM 4.10. *Let p and ℓ be distinct odd primes and suppose $g \geq 1$. Suppose $1 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f , the p -rank f stratum in \mathcal{H}_g . Then $M_\ell(S) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$.*

Proof. The proof is by induction on g . The base cases involve the monodromy of \mathcal{H}_2^2 and \mathcal{H}_2^1 , which follow from [Cha05, Prop. 4.4]; see [AP08, Thm. 4.5].

Now suppose $g \geq 3$ and $1 \leq f \leq g$. As an inductive hypothesis assume, for all pairs (g', f') where $1 \leq f' \leq g' < g$, that $M_\ell(S') \cong \text{Sp}_{2g'}(\mathbb{Z}/\ell)$ for every irreducible component S' of $\mathcal{H}_{g'}^{f'}$.

Let S be an irreducible component of \mathcal{H}_g^f . Recall the degeneration types identified immediately before Corollary 3.14. If $f = g$, let $(f_1, f_2, f_3) = (1, g - 2, 1)$ as in case (A), if $f = g - 1$, let

$(f_1, f_2, f_3) = (0, g - 2, 1)$ as in case (B), and if $1 \leq f \leq g - 2$, let $(f_1, f_2, f_3) = (0, f, 0)$ as in case (C). Then by Corollary 3.14, there are irreducible components \tilde{S} of $\bar{S} \times_{\overline{\mathcal{H}}_g} \tilde{\mathcal{H}}_g$, \tilde{S}_1 of $\tilde{\mathcal{H}}_1^{f_1}$, \tilde{S}_2 of $\tilde{\mathcal{H}}_{g-2}^{f_2}$ and \tilde{S}_3 of $\tilde{\mathcal{H}}_1^{f_3}$; and there are irreducible components \tilde{S}_R of $\tilde{\mathcal{H}}_{g-1}^{f_2+f_3}$ and \tilde{S}_L of $\tilde{\mathcal{H}}_{g-1}^{f_1+f_2}$; such that the restriction of the clutching morphisms gives a commutative diagram

$$\begin{array}{ccc} \tilde{S}_1 \times \tilde{S}_2 \times \tilde{S}_3 & \longrightarrow & \tilde{S}_1 \times \tilde{S}_R \\ \downarrow & \searrow^{\tilde{\kappa}_{1,g-2,1}} & \downarrow \\ \tilde{S}_L \times \tilde{S}_3 & \longrightarrow & \tilde{S} \cap \Delta_{1,1}[\tilde{\mathcal{H}}_g]. \end{array} \quad (4.3.1)$$

In all cases $f_1 + f_2$ and $f_2 + f_3$ are positive, and so \tilde{S}_L and \tilde{S}_R have monodromy $\mathrm{Sp}_{2(g-1)}(\mathbb{Z}/\ell)$, by induction and Lemma 4.9.

The rest of the proof is identical to that of [AP07, Thm. 3.4]. Briefly, one calculates the monodromy group of \tilde{S} at a point s in the image of $\tilde{S}_1 \times \tilde{S}_2 \times \tilde{S}_3$ under $\tilde{\kappa}_{1,g-2,1}$. On one hand, there is an a priori inclusion $M_\ell(\tilde{S}, s) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. On the other hand, we have shown that $M_\ell(\tilde{S}, s)$ contains two distinct subgroups isomorphic to $\mathrm{Sp}_{2(g-1)}(\mathbb{Z}/\ell)$. A group-theoretic result shows that $M_\ell(\tilde{S}, s) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. The result then follows from Lemma 4.9. \square

In the following result, note that the exceptional set of primes depends on p , but not on g .

THEOREM 4.11. *If $\ell \gg_p 0$, if $g \geq 5$ and if S is an irreducible component of \mathcal{H}_g^0 , then $M_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.*

Proof. Suppose ℓ is not in the finite set of exceptional primes from Corollary 4.8. The proof is by induction on g with the case $g = 5$ supplied by Corollary 4.8. The rest of the proof is identical to that of Theorem 4.10, with Corollary 3.14 being used to degenerate to an irreducible component of $\tilde{\kappa}_{1,g-2,1}(\mathcal{H}_1^0 \times \mathcal{H}_{g-2}^0 \times \mathcal{H}_1^0)$. \square

4.4 A p -adic complement

In this section we determine the p -adic monodromy of components of the p -rank strata \mathcal{H}_g^f .

Let S be a connected scheme of characteristic p with geometric point s , and let $X \rightarrow S$ be an abelian scheme with constant p -rank f . The group scheme $X[p]$ and p -divisible group $X[p^\infty]$ admit largest étale quotients, $X[p]^\text{ét}$ and $X[p^\infty]^\text{ét}$. These are respectively classified by homomorphisms $\pi_1(S, s) \rightarrow \mathrm{Aut}(X[p]^\text{ét})_s \cong \mathrm{GL}_f(\mathbb{Z}/p)$ and $\pi_1(S, s) \rightarrow \mathrm{Aut}(X[p^\infty]^\text{ét})_s \cong \mathrm{GL}_f(\mathbb{Z}_p)$, whose images we denote $M_p(X \rightarrow S)$ and $M_{\mathbb{Z}_p}(X \rightarrow S)$, or simply $M_p(S)$ and $M_{\mathbb{Z}_p}(S)$.

LEMMA 4.12. *Suppose $g \geq 1$ and $1 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f , and let \tilde{S} be an irreducible component of $S \times_{\overline{\mathcal{H}}_g} \tilde{\mathcal{H}}_g$. Then $M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ if and only if $M_p(\tilde{S}) \cong \mathrm{GL}_f(\mathbb{Z}/p)$.*

Proof. Since $\tilde{S} \rightarrow \bar{S}$ is finite, $M_p(\tilde{S})$ is a subgroup of $M_p(\bar{S})$. If $M_p(\tilde{S}) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ then $M_p(\bar{S})$ is maximal, and thus so are $M_p(\bar{S})$ and $M_p(S)$.

Conversely, suppose $M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$. Let S^* be the closure of S in $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$ and let \tilde{S}^* be the closure of \tilde{S} in $\tilde{\mathcal{H}}_g - \Delta_0[\tilde{\mathcal{H}}_g]$. Let $T = S^* - (S^*)^f$ be the locus with p -rank smaller than f . Then T is nonempty by Corollary 3.15. On one hand, $\tilde{S}^* \rightarrow S^*$ is étale since p is odd and the cover $\tilde{S}^* \rightarrow S^*$ is tantamount to a partial level-two structure. On the other hand, the $\mathrm{GL}_f(\mathbb{Z}/p)$ -cover $I_f := \mathrm{Isom}_S((\mathbb{Z}/p)^f, \mathrm{Jac}(\mathcal{C}_g)[p]^\text{ét}) \rightarrow S$ is ramified along T . Therefore, the covers $I_f \rightarrow S$ and $\tilde{S} \rightarrow S$ are disjoint, and $M_p(\tilde{S}) = M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$. \square

PROPOSITION 4.13. *Suppose $g \geq 2$ and $1 \leq f \leq g$. Let S be an irreducible component of \mathcal{H}_g^f . Then $M_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ and $M_{\mathbb{Z}_p}(S) \cong \mathrm{GL}_f(\mathbb{Z}_p)$.*

Proof. First suppose $f = g$. When $g = 2$, the result for \mathcal{H}_2^2 , or equivalently \mathcal{M}_2^2 , is a special case of [Eke91, Thm. 2.1]. For $g \geq 3$, suppose as an inductive hypothesis that $M_p(\mathcal{H}_{g-1}^{g-1}) \cong \mathrm{GL}_{g-1}(\mathbb{Z}/p)$. By Lemma 4.12, $M_p(\tilde{\mathcal{H}}_{g-1}^{g-1}) \cong \mathrm{GL}_{g-1}(\mathbb{Z}/p)$. By Corollary 3.14, there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-2}^{g-2} \times \tilde{\mathcal{H}}_1^1 & \longrightarrow & \tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-1}^{g-1} \\ \downarrow & \searrow^{\tilde{\kappa}_{1,g-2,1}} & \downarrow \\ \tilde{\mathcal{H}}_{g-1}^{g-1} \times \tilde{\mathcal{H}}_1^1 & \longrightarrow & \Delta_{1,1}[\tilde{\mathcal{H}}_g]. \end{array} \quad (4.4.1)$$

Consider a geometric point $s \in \tilde{\mathcal{H}}_g^g$ in the image of $\tilde{\mathcal{H}}_1^1 \times \tilde{\mathcal{H}}_{g-2}^{g-2} \times \tilde{\mathcal{H}}_1^1$ under $\tilde{\kappa}_{1,g-2,1}$. By the inductive hypothesis and Lemma 4.12, $M_p(\tilde{\mathcal{H}}_g, s)$ contains two distinct copies of $\mathrm{GL}_{g-1}(\mathbb{Z}/p)$ and thus equals $\mathrm{GL}_g(\mathbb{Z}/p)$ by the argument of [AP07, Lemma 3.2]).

Now suppose $1 \leq f \leq g - 1$. By Corollary 3.13(a), there are irreducible components $\tilde{V}_1 \subset \tilde{\mathcal{H}}_1^0$ and $\tilde{V}_2 \subset \tilde{\mathcal{H}}_{g-1}^f$ such that \tilde{s} contains $\tilde{\kappa}_{1,g-1}(\tilde{V}_1 \times \tilde{V}_2)$. By the inductive hypothesis and Lemma 4.12, $M_p(\tilde{V}_2) \cong \mathrm{GL}_f(\mathbb{Z}/p)$, and thus $M_p \cong \mathrm{GL}_f(\mathbb{Z}/p)$ as well.

The proof that $M_{\mathbb{Z}_p}(S) \cong \mathrm{GL}_f(\mathbb{Z}_p)$ is identical. \square

4.5 Arithmetic applications

The results of the previous section about the monodromy of components of \mathcal{H}_g^f have arithmetic applications involving hyperelliptic curves over finite fields. For example, they imply that there exist hyperelliptic curves of given genus and p -rank with absolutely simple Jacobian (Application 4.14). Moreover, they give estimates for the proportion of hyperelliptic curves with a given genus and p -rank that have a rational point of order ℓ on the Jacobian (Application 4.16) or for which the numerator of the zeta function has large splitting field (Application 4.18).

Throughout this section, \mathbb{F} denotes a finite extension of \mathbb{F}_p .

4.5.1 Technical context We do not include proofs in this section, since they are very similar to those found in [AP08, Section 5]. Here is a brief description of the main ideas involved. One first defines \mathcal{H}_g over the category of \mathbb{F}_p -schemes and defines the arithmetic monodromy group of a substack of \mathcal{H}_g . For a relative curve $\pi : C \rightarrow S/\mathbb{F}$ of genus $g \geq 2$ defined over a finite field, one shows that if $M_\ell^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$, then $M_{\mathbb{Z}_\ell}^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$; and $M_{\mathbb{Z}_\ell}^{\mathrm{arith}}(S)$ has finite index in $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$; and $M_{\mathbb{Q}_\ell}^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$ [AP08, Lemma 5.1].

Secondly, in order to use Chebotarev arguments for curves over finite fields, it is necessary to add rigidifying data, such as the data of a tricanonical structure, so that the corresponding moduli problems are representable by schemes. Recall that $\Omega_{C/S}^{\otimes 3}$ is very ample, that $\pi_*(\Omega_{C/S}^{\otimes 3})$ is a locally free \mathcal{O}_S -module of rank $5g - 5$, and that sections of this bundle define a closed embedding $C \hookrightarrow \mathbb{P}_S^{5g-5}$. A tricanonical (3K) structure on $\pi : C \rightarrow S$ is a choice of isomorphism $\mathcal{O}_S^{\oplus 5g-5} \cong \pi_*(\Omega_{C/S}^{\otimes 3})$, and the only automorphisms of a hyperelliptic curve with 3K-structure are the identity and the hyperelliptic involution. The moduli space $\mathcal{H}_{g,3K}$ of smooth hyperelliptic curves of genus g equipped with a 3K-structure is representable by a scheme [KS99, 10.6.5], [MFK94, Prop. 5.1].

Third, since \mathcal{H}_g may be constructed as the quotient of $\mathcal{H}_{g,3K}$ by GL_{5g-5} , the forgetful functor $\psi_g : \mathcal{H}_{g,3K} \rightarrow \mathcal{H}_g$ is open [MFK94, p. 6] and a fibration with connected fibers [Noo04, Thm. A.12].

This allows one to show, if $S \subset \mathcal{H}_g$ is a connected substack and $S_{3K} = S \times_{\mathcal{H}_g} \mathcal{H}_{g,3K}$, then $M_\ell(S_{3K}) \cong M_\ell(S)$ [AP08, Lemma 5.2]. Since the data of a tricanonical structure exists Zariski-locally on the base (as opposed to that of a Jacobi level structure which only exists étale-locally on the base), one can relate point counts on $\mathcal{H}_{g,3K}(\mathbb{F})$ to those on $\mathcal{H}_g(\mathbb{F})$. Specifically, if $s \in \mathcal{H}_g^f(\mathbb{F})$ is such that $\text{Aut}(\mathcal{C}_{g,s}) \cong \{\pm 1\}$, then the fiber of $\mathcal{H}_{g,3K}(\mathbb{F})$ over s consists of $|\text{GL}_{2g}(\mathbb{F})|/2$ points [KS99, 10.6.8].

4.5.2 Application to simple Jacobians Using the \mathbb{Q}_ℓ -monodromy of \mathcal{H}_g^f , we deduce that there exist hyperelliptic curves of genus g and p -rank f with absolutely simple Jacobian.

APPLICATION 4.14. *Suppose $g \geq 1$ and $0 \leq f \leq g$ with $f \neq 0$ if $g \leq 3$. Let S be an irreducible component of \mathcal{H}_g^f .*

- (a) *Then there exists $s \in S(\overline{\mathbb{F}})$ such that the Jacobian of \mathcal{C}_s is absolutely simple.*
- (b) *If η is a geometric generic point of S , then $\text{End}(\text{Jac}(\mathcal{C}_{5,\eta})) \cong \mathbb{Z}$.*

Proof. The proof of (a) is very similar to that of [AP08, Application 5.7] and uses Theorems 4.10-4.11 (and also Proposition 4.5 when $g = 4$ and $f = 0$) and [Cha05, Cor. 4.3]. Statement (b) follows immediately from Theorems 4.10-4.11. \square

REMARK 4.15. Under the hypotheses of Application 4.14, if η is a generic point of S , one can deduce that $\text{Aut}(\text{Jac}(\mathcal{C}_{g,\eta})) = \{\pm 1\}$. This yields a new proof of [AGP08, Thm. 3.7].

4.5.3 Application to class groups Recall that if $s \in \mathcal{H}_g(\mathbb{F})$, then $\text{Pic}^0(\mathcal{C}_{g,s})(\mathbb{F})$ is isomorphic to the class group of the function field $\mathbb{F}(\mathcal{C}_{g,s})$. The size of the class group is divisible by ℓ exactly when there is a point of order ℓ on the Jacobian. Roughly speaking, Application 4.16 shows that among all curves over \mathbb{F} of specified genus and p -rank, slightly more than $1/\ell$ of them have an \mathbb{F} -rational point of order ℓ on their Jacobian.

APPLICATION 4.16. *Suppose $\ell \neq p$ are odd primes and $g \geq 3$ and $1 \leq f \leq g$. Suppose S is an irreducible component of \mathcal{H}_g^f such that $S(\mathbb{F}) \neq \emptyset$. Let m be the image of $|\mathbb{F}|$ in $(\mathbb{Z}/\ell)^\times$. There exists a rational function $\alpha_{g,m}(T) \in \mathbb{Q}(T)$ and a constant $B = B(p, g, \ell)$ such that*

$$\left| \frac{\#\{s \in S(\mathbb{F}) : \ell \mid \#\text{Pic}^0(\mathcal{C}_{g,s})(\mathbb{F})\}}{\#S(\mathbb{F})} - \alpha_{g,m}(\ell) \right| < \frac{B}{\sqrt{q}}. \quad (4.5.1)$$

If $f = 0$ and $g \geq 5$, the same result is true for all $\ell \gg_p 0$.

REMARK 4.17. For ℓ odd, one knows that $\alpha_{g,1}(\ell) = \frac{\ell}{\ell^2-1} + \mathcal{O}(1/\ell^3)$, while $\alpha_{g,m}(\ell) = \frac{1}{\ell-1} + \mathcal{O}(1/\ell^3)$ if $m \neq 1$. A formula for $\alpha_{g,1}(\ell)$ is given in [Ach06].

Proof of 4.16. The proof is very similar to that of [AP08, Application 5.9] and uses Theorems 4.10-4.11 and [KS99, 9.7.13]. \square

4.5.4 Application to zeta functions If C/\mathbb{F} is a smooth projective curve of genus g , its zeta function has the form $L_{C/\mathbb{F}}(T)/(1-T)(1-qT)$, where $L_{C/\mathbb{F}}(T) \in \mathbb{Z}[T]$ is a polynomial of degree $2g$. The principal polarization on the Jacobian of C forces a symmetry among the roots of $L_{C/\mathbb{F}}(T)$; the largest possible Galois group for the splitting field over \mathbb{Q} of $L_{C/\mathbb{F}}(T)$ is the Weyl group of Sp_{2g} which is a group of size $g!2^g$.

APPLICATION 4.18. *Suppose $g \geq 3$, $p > 2g + 1$, and $0 \leq f \leq g$. Suppose S is an irreducible component of \mathcal{H}_g^f such that $S(\mathbb{F}) \neq \emptyset$. There exists a constant $\gamma = \gamma(g) > 0$ and a constant*

$E = E(p, g)$ such that

$$\frac{\#\{s \in S(\mathbb{F}) : L_{C_{g,s}/\mathbb{F}}(T) \text{ is reducible, or has splitting field with degree } < 2^s g!\}}{\#S(\mathbb{F})} < Eq^{-\gamma}. \quad (4.5.2)$$

Proof. The proof is very similar to that of [AP08, Application 5.11] and uses Theorems 4.10-4.11 and [Kow06, Thm. 6.1 and Remark 3.2.(4)]. \square

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