ARITHMETIC OCCULT PERIOD MAPS

JEFFREY D. ACHTER

ABSTRACT. Several natural complex configuration spaces admit surprising uniformizations as arithmetic ball quotients, by identifying each parametrized object with the periods of some auxiliary object. In each case, the theory of canonical models of Shimura varieties gives the ball quotient the structure of a variety over the ring of integers of a cyclotomic field. We show that the (transcendentally-defined) period map actually respects these algebraic structures, and thus that occult period maps are arithmetic. As an intermediate tool, we develop an arithmetic theory of lattice-polarized K3 surfaces.

1. INTRODUCTION

It occasionally happens that complex varieties of a specified type are parametrized by an arithmetic quotient of a unit ball in a surprising way. We situate this remark by recalling some aspects of the primordial period map. Consider M_g, the moduli space of smooth projective curves of genus $g \geq 2$. Given a smooth projective curve of genus g, the possibilities for its period lattice are naturally parametrized by the quotient of $\mathbb{H}_g$, the Siegel upper-half space of dimension $g(g+1)/2$, by $\mathrm{Sp}_{2g}(\mathbb{Z})$. The classical Torelli theorem asserts that the corresponding map $\tau_{g,\mathbb{C}} : \mathsf{M}_g(\mathbb{C}) \to \mathbb{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z})$ is an inclusion. Even more is true. On one hand, M_g has a natural structure as a moduli space over $\mathbb{Z}$. On the other hand, let A_g be the moduli space of principally polarized abelian varieties of dimension g; it, too, is a space over $\mathbb{Z}$. Via the identification $\mathsf{A}_g(\mathbb{C}) \cong \mathbb{H}_g / \mathrm{Sp}_{2g}(\mathbb{Z})$, we endow the latter with a structure over $\mathbb{Z}$, as well. The key arithmetic fact about the Torelli map is that $\tau_{g,\mathbb{C}}$, *a priori* a transcendental map, descends to a morphism $\tau_g : \mathsf{M}_g \hookrightarrow \mathsf{A}_g$ over $\mathbb{Z}$ (e.g., [37, §7.4]). Still, as soon as $g > 3$, $\dim \mathsf{M}_g < \dim \mathsf{A}_g$. This means that many of the arithmetic structures on A_g, such as Hecke operators and modular forms, don't readily make sense for the moduli space of curves.

In the special case where $g = 4$, however, we have the intriguing observation of Kondō [25] that $\mathsf{M}_4(\mathbb{C})$ is very close to an arithmetic quotient of $\mathbb{B}^9$, the complex unit 9-dimensional ball. Slightly more precisely, let N_4 denote the (open, dense) locus of non-hyperelliptic curves. Kondō shows that there exist an arithmetic group Γ of automorphisms of $\mathbb{B}^9$ and an open immersion $\mathsf{N}_4(\mathbb{C}) \hookrightarrow \mathbb{B}^9/\Gamma$. (He even characterizes the image.) Instead of analyzing the periods of a non-hyperelliptic curve C, the construction of [25] proceeds by constructing an auxiliary variety Z associated to C, and analyzing *its* periods. Kudla and Rapoport [28] – who call such a period map *occult*, in recognition of its hidden nature – observe that the theory of canonical models of Shimura varieties produces a distinguished algebraic model of $\mathbb{B}^9/\Gamma$ over $\mathbb{Q}(\zeta_3)$. They prove that Kondō's occult period map actually respects the structures of N_4 and $\mathbb{B}^9/\Gamma$ as varieties over $\mathbb{Q}(\zeta_3)$, and conjecture that it extends to a map of integral canonical models over $\mathbb{Z}[\zeta_3, 1/3]$. (They also note certain stack-theoretic issues, which have since been resolved by Zheng [47]; see Remark 7.6 below.)

In fact, several different situations are known in which, for some moduli space V of low-complexity varieties, an occult period map yields an open immersion $\tau_{\mathsf{V},\mathbb{C}} : \mathsf{V}(\mathbb{C}) \hookrightarrow \mathbb{B}^{\dim V}/\Gamma_{\mathsf{V}}$; see, for instance, [17] and [28], or even §7 below, for examples. In each case known to the author, the theory of integral canonical models of Shimura varieties provides a distinguished model of

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$\mathsf{Sh}_{\Gamma_{\mathsf{V}}}(\mathbb{B}^{\dim \mathsf{V}})$ of $\mathbb{B}^{\dim \mathsf{V}}/\Gamma_{\mathsf{V}}$ over $\mathbb{Z}[\zeta_n, 1/n]$ for some $n = n(\mathsf{V})$. The goal of the present paper is to show that, in many cases, $\tau_{\mathsf{V},\mathbb{C}}$ descends to a morphism $\mathsf{V} \hookrightarrow \mathsf{Sh}_{\Gamma_{\mathsf{V}}}(\mathbb{B}^{\dim \mathsf{V}})$ over $\mathbb{Z}[\zeta_n, 1/2n]$.

Many of the original constructions involve somehow building a K3 surface out of the original variety, and then analyzing the periods of the corresponding K3 surface. Consequently, much of the work of the present paper is in analyzing moduli spaces $\mathsf{R}_{L,\underline{\chi}}$ of K3 surfaces polarized by the lattice L and equipped with a suitable action of $\boldsymbol{\mu}_n$. A representative result – the notation is defined later in the paper – is:

Proposition. *There are morphisms of stacks over $\mathbb{Z}[\zeta_3, 1/6]$:*

$$\begin{array}{ccc} \mathsf{R}^{\circ}_{L_4,\underline{\chi}_4} & \xrightarrow{\kappa_4} & \mathsf{N}_4 \\ \Big\downarrow{\scriptstyle \tau_{L_4,\underline{\chi}_4}} & & \\ \mathsf{Sh}^{(L_4,\underline{\chi}_4)} & & \end{array}$$

where κ_4 induces an isomorphism of coarse moduli spaces, and $\tau_{L_4,\underline{\chi}_4}$ induces an open immersion $\mathsf{R}^{\circ}_{L_4,\underline{\chi}_4}(\mathbb{C}) \hookrightarrow \mathsf{Sh}^{(L_4,\underline{\chi}_4)}(\mathbb{C})$.

The statement over $\mathbb{C}$ is, essentially, [25, Thm. 1]; taking fibers over $\mathbb{Q}(\zeta_3)$ recovers the descent result [28, Thm. 8.1].

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2. Notation on lattices

2.1. **Lattices.** Throughout this paper, a *lattice* is a free $\mathbb{Z}$-module L of finite rank, equipped with a nondegenerate, symmetric bilinear pairing $(\cdot,\cdot)$ (often notationally suppressed). For any nonzero n, we let $L(n)$ denote the lattice with the same underlying group structure as L and with pairing $(\cdot,\cdot)_{L(n)} = n(\cdot,\cdot)_L$. We follow the conventions of [19] for lattices. Lattices used here include:

- U the hyperbolic plane, which has rank 2 and pairing $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$;
- $\langle 1 \rangle$ the lattice of rank 1 and pairing (1);
- E_8 the unique positive definite unimodular lattice of rank 8;
- A_n, D_n the (positive) lattice associated to the Dynkin diagrams of type A_n and D_n, respectively (in particular, $A_1 \cong \langle 2 \rangle$);
- L_{K3} the lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, of signature $(3^+, 19^-)$;
- V the lattice of rank 2 and pairing $\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

The pairing induces an inclusion $L \hookrightarrow L^\vee$; the discriminant of L is the finite abelian group $\operatorname{disc}(L) = L^\vee/L$, and we set $\Delta_L = [L^\vee : L] = \#\operatorname{disc}(L)$. Finally, let

$$d(L) = \gcd(\{d \in \mathbb{N} : \exists \langle 2d \rangle \hookrightarrow L \text{ primitive}\}).$$

For use in §7, we record the following elementary facts:

Lemma 2.1. (a) *If M is a primitive sublattice of L, then $d(L) | d(M)$.*
(b) *$d(U(n)) = n$, while $d(A_1 \oplus A_1(-1)^{\oplus 2}) = d(V \oplus A_4(-1)) = 1$.*

2.2. **Orthogonal groups.** To L we associate the orthogonal group O_L, with connected component of identity the special orthogonal group SO_L. Since we start with an integral model for SO as the automorphism group of the lattice L, we have a natural definition of $\mathrm{SO}_L(R)$ for any ring R. In particular, $\mathrm{SO}_L(\mathbb{Z}_p)$ is well-defined and, by definition, a hyperspecial subgroup of $\mathrm{SO}_L(\mathbb{Q}_p)$. (In Section 5.1, a choice of hyperspecial subgroup is necesary for the construction of a canonical integral model of the relevant Shimura variety. In fact, since SO_L is adjoint and split, $\mathrm{SO}_L(\mathbb{Q}_p)$ admits a unique $\mathrm{SO}_L(\mathbb{Q}_p)$-conjugacy class of hyperspecial subgroups.)

We will often have cause to work with a lattice L which comes equipped with a primitive embedding $\iota : L \hookrightarrow L_{\mathrm{K3}}$. With a slight abuse of notation, we will write $L^\perp$ for the orthogonal complement $\iota(L)^\perp$ of $\iota(L)$ in L_{K3}. Set $\mathrm{O}^L = \mathrm{O}_{L^\perp}$ and $\mathrm{SO}^L = \mathrm{SO}_{L^\perp}$. An element of $\mathrm{O}^L(\mathbb{Z})$ extends to an element of $\mathrm{O}_{L_{\mathrm{K3}}}(\mathbb{Z})$ acting trivially on L if and only if it acts trivially on $\mathrm{disc}(L)$ (e.g., [19, 14.2.6]). More generally, if R is flat over $\mathbb{Z}$, then an element of $\mathrm{O}^L(R)$ extends to an element of $\mathrm{O}_L(R)$ acting trivially on $L \otimes R$ if and only if it acts trivially on $\mathrm{disc}(L) \otimes R$ [33, Lemma 2.6]. The subgroup of admissible orthogonal automorphisms of $L^\perp$ is the group scheme $\widetilde{\mathrm{O}}^L$ fitting in the short exact sequence

$$1 \longrightarrow \widetilde{\mathrm{O}}^L \longrightarrow \mathrm{O}^L \longrightarrow \mathrm{Aut}(\mathrm{disc}(L)) \longrightarrow 1;$$

on points, we have

$$\begin{aligned}\widetilde{\mathrm{O}}^L(R) &= \{g|_{L^\perp \otimes R} : g \in \mathrm{O}_{L_{\mathrm{K3}}}(R)\} \\ &= \ker\Big(\mathrm{O}^L(R) \to \mathrm{Aut}(\mathrm{disc}(L))(R)\Big).\end{aligned}$$

If $\tilde{g} \in \widetilde{\mathrm{O}}^L(R)$, then there is a (necessarily unique) $g \in \mathrm{O}_{L_{\mathrm{K3}}}(R)$ such that $g|_{L^\perp} = \tilde{g}$ and $g|_L = \mathrm{id}_L$. In this way, $\widetilde{\mathrm{O}}^L(R)$ is naturally identified with a subgroup of $\mathrm{O}_{L_{\mathrm{K3}}}(R)$. The group scheme $\widetilde{\mathrm{SO}}^L := \widetilde{\mathrm{O}}^L \times_{\mathrm{O}^L} \mathrm{SO}^L$ represents admissible automorphisms of determinant one.

3. Families of K3 surfaces

3.1. **K3 surfaces.** Let k be an algebraically closed field. A K3 surface over k is a smooth, complete irreducible surface Z/k with trivial canonical bundle $\omega_Z := \Omega^2_{Z/k} \cong \mathcal{O}_Z$ and such that $H^1(Z, \mathcal{O}_Z) = 0$. Like any smooth complete surface, a K3 surface is projective, and its middle Hodge numbers are $(h^{2,0}, h^{1,1}, h^{0,2}) = (1, 20, 1)$. If $k = \mathbb{C}$, then the Betti cohomology group $H^2(Z, \mathbb{Z})$, endowed with the intersection pairing $(\cdot, \cdot)$, is isomorphic to L_{K3}. The natural pairing

$$\Omega_Z \times \Omega_Z \longrightarrow \omega_Z$$

induces an isomorphism of $\mathcal{T}_Z \cong \Omega_Z$. Since $H^1(Z, \mathcal{O}_Z)$ is trivial so is $\mathrm{Pic}^0(Z)$, and the Néron-Severi group $\mathrm{NS}(Z)$ coincides with the Picard group $\mathrm{Pic}(Z)$. As for any surface, $\mathrm{NS}(Z)$ is a free, finitely generated $\mathbb{Z}$-module, equipped with a symmetric, nondegenerate pairing

$$\mathrm{NS}(Z) \times \mathrm{NS}(Z) \xrightarrow{(\cdot,\cdot)} \mathbb{Z}.$$

The intersection pairing is even, nondegenerate, and of signature $(1, \mathrm{rank}(\mathrm{NS}(Z)) - 1)$.

Following Rizov [41] and successors, we say that a a relative K3 surface, or K3 space, over a scheme S is an algebraic space $Z \to S$ such that each geometric fiber is a K3 surface. If $Z \to S$ is a relative K3 surface, then $\mathcal{H}^2_{\mathrm{dR}}(Z/S)$ and $\mathcal{H}^{2,0}(Z/S)$ are locally free sheaves on S [13, Prop. 2.2], [32, §3.4] of respective ranks 22 and 1.

3.2. **Categories of K3 surfaces.** We will study three different sorts of moduli spaces of K3 surfaces.

Classically, one has R°_{2d}, the category of K3 surfaces equipped with a primitive ample polarization of degree $2d$. On points, $\mathsf{R}^\circ_{2d}(S)$ is the category of pairs $(Z \to S, \lambda)$, where $Z \to S$ is a K3 space, and $\lambda \in \mathrm{Pic}_{Z/S}(S)$ is étale-locally represented by an ample line bundle of self-intersection degree $2d$ which is not a nontrivial tensor power of any other line bundle. This is a subcategory of R_{2d}, the category of K3 surfaces equipped with a primitive quasi-ample polarization of degree $2d$. (A quasi-ample, or pseudo-ample, polarization is étale-locally a line bundle which is big and nef.)

Choose a generator e_0 for $\langle 2d\rangle$. To specify data $(Z \to S, \lambda)$ is to specify a K3 space $Z \to S$ and an embedding of lattices $\langle 2d\rangle \hookrightarrow \mathrm{Pic}_{Z/S}(S)$ which takes e_0 to the class of an ample line bundle. (Recall that if Z/k is a K3 surface over an algebraically closed field, and if $v \in \mathrm{Pic}_{Z/k}(k)$ satisfies $(v, v) > 0$, then exactly one of v and $-v$ and represents the class of an ample line bundle. The choice of a generator for $\langle 2d\rangle$ is equivalent to the choice of a "positive cone" in $\langle 2d\rangle \otimes \mathbb{R} \cong \mathbb{R}$.)

More generally, we consider lattice-polarized K3 surfaces. Let L be a primitive sublattice of L_{K3} of signature $(1, r-1)$. The set

$$\{v \in L \otimes \mathbb{R} : (v, v) > 0\}$$

has two connected components. Choose one such, V^+, and, as in [15, §1] or [17, §10], define an abstract "ample cone" $\mathcal{C}(V^+)$, an open subset of V^+, and let $L^+ = L \cap \mathcal{C}(V)^+$. We suppress the choice of V^+ (and, thus, L^+) from the notation, and let R°_L be the category of ample L-polarized K3 surfaces. Objects in R°_L are isomorphism classes of pairs $(Z \to S, \alpha)$, where $Z \to S$ is a K3 space and $\alpha : L \hookrightarrow \mathrm{Pic}_{Z/S}(S)$ is a primitive embedding of lattices such that $\alpha(L^+)$ contains the class of an ample line bundle. (Since α is a primitive embedding, it is equivalent to ask that $\alpha(\mathcal{C}(V^+))$ contains such a class.) Here, we declare that two such data $(Z_i \to S, \alpha_i)$ are isomorphic if there is an isomorphism $f : Z_1 \to Z_2$ such that $f^*\alpha_2 = \alpha_1$. We define R_L, the category of L-polarized K3 surfaces, analogously except that it is only assumed that $\alpha(L^+)$ contains the class of a quasi-ample line bundle.

(Of course, one can also make the definition of an L-polarized K3 surface without keeping track of a positive cone, provided one is willing to identify α and $-\alpha$. For a Shimura-theoretic justification for this approach, see Remark 6.3 and, ultimately, [46, §5]. Consequently, the choice of L^+ is suppressed here.)

For a finite group scheme G, let $\mathsf{R}^*_{L,G}$ be the category of tuples $(Z \to S, \alpha, \rho)$ where $(Z \to S, \alpha) \in \mathsf{R}_L(S)$ and $\rho : G_S \hookrightarrow \mathrm{Aut}_S(Z \to S, \alpha)$ is a monomorphism of group schemes. If $\#G$ is invertible on S – equivalently, if the cardinality of G is relatively prime to the characteristic exponent of all residue fields of points of S – then representations of G on $\mathcal{O}_S$-modules are rigid, and thus $\mathcal{H}^2_{\mathrm{dR}}(Z/S)$ and $\mathcal{H}^{2,0}(Z/S)$ are locally free sheaves of $\mathcal{O}_S[G]$-modules.

We now specialize to the case $G = \boldsymbol{\mu}_n$, and restrict to the category of schemes over $\mathbb{Z}[1/2\Delta_L n]$. Let χ^ω be a faithful character of $\boldsymbol{\mu}_n$, χ_0 the trivial character, and χ be an arbitrary character; let $m(\chi^\omega) = m_\chi(\chi^\omega)$ and $m(\chi_0) = m_\chi(\chi_0)$ be the multiplicities of, respectively, χ_0 and χ^ω in χ. Let $\mathsf{R}_{L,\boldsymbol{\mu}_n,\chi^\omega,\chi}$ be the open and closed substack of $\mathsf{R}^*_{L,\boldsymbol{\mu}_n}$ parametrizing those $(Z \to S, \alpha, \rho)$ such that

- $\boldsymbol{\mu}_n$ acts on $\mathcal{H}^{2,0}(Z/S)$ via χ^ω;
- $\boldsymbol{\mu}_n$ acts on $\mathcal{H}^2(Z/S)$ via χ; and
- $m_\chi(\chi_0) = \mathrm{rank}(L)$.

In particular, the action of $\boldsymbol{\mu}_n$ is purely non-symplectic, in the sense that no nontrivial section of $\boldsymbol{\mu}_n$ fixes a nonzero holomorphic 2-form.

Suppose S is irreducible and $\bar{s}$ is a geometric point of S. Because $2n$ is invertible on S, representations of $\boldsymbol{\mu}_n$ on $\mathcal{O}_S$-modules are rigid. In particular, the character of the action of $\boldsymbol{\mu}_n$ on $\mathcal{H}^2(Z/S)$ is determined by the action on $\mathcal{H}^2_{\mathrm{dR}}(Z_{\bar{s}})$. Moreover, it is equivalent to specify this character in terms of the action of $\boldsymbol{\mu}_n$ on $H^2_{\mathrm{cris}}(Z_{\bar{s}})$, or any of the étale cohomology groups $H^2(Z_{\bar{s}}, \mathbb{Q}_\ell)$ [21, Thm 2.2], or (since K3 surfaces have torsion-free cohomology) $H^2(Z_{\bar{s}}, \mathbb{Z}_\ell)$.

We will often use the symbol $\underline{\chi}$ to denote the collection of data $(\boldsymbol{\mu}_n, \chi^\omega, \chi)$, and thus write $\mathsf{R}_{L,\underline{\chi}}$ for $\mathsf{R}_{L,\boldsymbol{\mu}_n,\chi^\omega,\chi}$, etc.

The possibilities for data $(L, \underline{\chi})$ such that $\mathsf{R}_{L,\underline{\chi}}(\mathbb{C})$ is nonempty are reasonably well-understood. On one hand, it is not hard to see that a purely non-symplectic group of automorphisms is finite and cyclic; the possible orders of such a group are also known [31]. On the other hand, starting with the work of Nikulin, one has a good classification of primitive sublattices of L_{K3} [38]. In §7 we will see a number of explicit examples of naturally occurring families of lattice-polarized K3 surfaces.

3.3. **Stacks of K3 surfaces.** It turns out that each R_{2d}, R_L, and $\mathsf{R}_{L,\underline{\chi}}$ is a Deligne-Mumford stack. Indeed, Rizov proves that R°_{2d} is a Deligne-Mumford stack [41, Thm. 4.3.3], and Beauville essentially proves the same of R_L in [8]. The partial compactification R_{2d} of R°_{2d} is also Deligne-Mumford and even smooth over $\mathbb{Z}[1/2d]$ [35, Prop. 2.1], albeit no longer separated (e.g., [19, 5.1.4]). Rather than working *ab ovo* to study R_L and $\mathsf{R}^*_{L,G}$, we find it expedient to bootstrap from Rizov's work.

It is convenient to make at the outset a few (arbitrary) choices; the final claims are intrinsic, and independent of these choices. Let $e_1, \cdots, e_r$ be a $\mathbb{Z}$-basis for L. Fix some $\lambda \in L^+$.

Lemma 3.1. *The category* R_L *is a stack over* $\operatorname{Spec}\mathbb{Z}$.

Proof. We must show that the diagonal $\mathsf{R}_L \to \mathsf{R}_L \times \mathsf{R}_L$ is representable, and that étale descent in the category R_L is effective.

For the first claim, it suffices to show that if $(Z_1 \to S, \alpha_1)$ and $(Z_2 \to S, \alpha_2)$ are elements of $\mathsf{R}_L(S)$, then
$$\mathsf{Isom}((Z_1, \alpha_1), (Z_2, \alpha_2))$$
is representable by a scheme over S. The functor $\mathsf{Isom}(Z_1, Z_2)$ is represented by a scheme over S. (In fact, it is open in $\mathrm{Hilb}(Z_1 \times Z_2)$.) Pullback by isomorphisms gives a pairing $\mathsf{Isom}(Z_1, Z_2) \times_S \mathrm{Pic}_{Z_2/S} \to \mathrm{Pic}_{Z_1/S}$. Consider some i between 1 and r. Pulling back the pairing by the section $\alpha_2(e_i) : S \to \mathrm{Pic}_{Z_2/S}$ induces a morphism $\beta_i : \mathsf{Isom}(Z_1, Z_2) \to \mathrm{Pic}_{Z_1/S}$. Then
$$\mathsf{Isom}((Z_1, \alpha_1(e_i)), (Z_2, \alpha_2(e_i))) := \mathsf{Isom}(Z_1, Z_2) \times_{\mathrm{Pic}_{Z_1/S}, \alpha_1(e_i)} S$$
is the sub-scheme of $\mathsf{Isom}(Z_1, Z_2)$ parametrizing those isomorphisms which take $\alpha_2(e_i)$ to $\alpha_1(e_i)$. Insofar as $\mathsf{Isom}((Z_1, \alpha_1), (Z_2, \alpha_2))$ is the fiber product over $\mathsf{Isom}(Z_1, Z_2)$ of the r different schemes $\mathsf{Isom}((Z_1, \alpha_1(e_i)), (Z_2, \alpha_2(e_i)))$, it too is represented by a scheme.

For the second, let $T \to S$ be étale and let $(\widetilde{Z} \to T, \widetilde{\alpha}) \in \mathsf{R}_L(T)$ be equipped with T/S descent data. In [41, Lemma 4.3.7], the author shows that in the ample case $(\widetilde{Z} \to T, \alpha(\lambda))$ descends, as a polarized K3 space, to S; the quasi-polarized case follows from [35, §2]. Since $\mathrm{Pic}_{Z/S}$ is a sheaf in the étale topology, each $\widetilde{\alpha}(e_i)$ descends to Z/S. □

Lemma 3.2. *The category* R_L *is a Deligne-Mumford stack over* $\operatorname{Spec}\mathbb{Z}$.

Proof. Because $\mathsf{R}_{2d(\lambda)}$ is known to be a Deligne-Mumford stack, it suffices to show that the forgetful morphism
$$\begin{array}{ccc} \mathsf{R}_L & \xrightarrow{\phi_\lambda} & \mathsf{R}_{2d(\lambda)} \\ & & \\ (Z \to S, \alpha) & \longmapsto & (Z \to S, \alpha(\lambda)) \end{array}$$
is relatively representable [29, Prop. 4.5.(ii)]. Now proceed as in [8]. Let $H_\lambda \subset \mathrm{O}_L(\mathbb{Z})$ be the subgroup which stabilizes λ. Since $\lambda^\perp$ is negative definite, H_λ is finite. Given an S-point $S \to \mathsf{R}_{2d(\lambda)}$,
$$\mathsf{R}_L \times_{\phi_\lambda, \mathsf{R}_{2d(\lambda)}} S,$$
if nonempty, is a torsor under H_λ, and in particular is representable. □

Proposition 3.3. *The category* R_L *is a smooth Deligne-Mumford stack over* $\operatorname{Spec}\mathbb{Z}[1/2\Delta_L]$ *of relative dimension* $20-r$.

Proof. Given Lemma 3.2, it suffices to show that the local deformation space of a quasi-ample L-polarized K3 surface is smooth. In characteristic zero, this is asserted in [15, Prop. 2.1], and details are provided in [8, Prop. 1.4]. In positive characteristic, this follows from Deligne and Illusie's deformation theory; see Proposition 3.8 below. □

Now consider the moduli spaces of lattice-polarized K3 surfaces with group action.

Lemma 3.4. *Suppose* $(Z \to S, \alpha) \in \mathsf{R}_L(S)$. *Then* $\operatorname{Aut}(Z \to S, \alpha)$ *is represented by a proper finite group scheme over* S.

Proof. The automorphism functor $\operatorname{Aut}_S(Z)$ is represented by a separated, unramified group scheme over S [41, Thm. 3.3.1]. Moreover, $\operatorname{Aut}_S(Z \to S, \alpha(\lambda))$ is a closed, finite, subgroup scheme of $\operatorname{Aut}_S(Z)$ (see [41, Prop. 3.3.3] for the polarized case; the extension to quasi-polarizations follows from [35, p.2369]). As in the proof of Lemma 3.1, $\operatorname{Aut}_S(Z \to S, \alpha)$ is a sub-S-group scheme of $\operatorname{Aut}_S(Z \to S, \alpha(\lambda))$. The claimed properness follows from [34, Thm. 2]. □

Lemma 3.5. *The forgetful morphism* $\mathsf{R}^*_{L,G} \to \mathsf{R}_L$ *is finite, and* $\mathsf{R}^*_{L,G}$ *is a Deligne-Mumford stack.*

Proof. First, the forgetful functor $\mathsf{R}^*_{L,G} \to \mathsf{R}_L$ is relatively representable. Indeed, for any affine scheme S and any $(Z \to S, \alpha) \in \mathsf{R}_L(S)$, both G_S and $\operatorname{Aut}_S(Z \to S, \alpha)$ are relatively representable, and thus $\operatorname{Hom}(G_S, \operatorname{Aut}_S(Z \to S, \alpha))$ is representable, too; and the condition that a homomorphism be injective is open. Therefore, $\mathsf{R}^*_{L,G}$ is also a Deligne-Mumford stack. The properness (and finitude) in Lemma 3.4 imply that $\mathsf{R}^*_{L,G} \to \mathsf{R}_L$ is proper and quasifinite, thus finite. □

Proposition 3.6. *The category* $\mathsf{R}_{L,\underline{\chi}}$ *is a smooth Deligne-Mumford stack over* $\mathbb{Z}[\zeta_n, 1/6\Delta_L n]$ *of relative dimension* $m(\chi^\omega) - 1$.

Proof. All that needs to be checked is smoothness; this is done in Lemma 3.9. □

3.4. **Local calculations.** If $Z/\mathbb{C}$ is a complex K3 surface, then (the local Torelli theorem asserts that) the deformation theory of Z is well-captured by its Hodge theory. In particular, let $\operatorname{Def}(Z)$ be the deformation functor of Z, with base point s. Then there is a canonical isomorphism

$$T_s \operatorname{Def}(Z) \xrightarrow{\sim} \operatorname{Hom}(H^{2,0}(Z), H^{2,0}(Z)^\perp / H^{2,0}(Z)).$$

There is a parallel deformation theory for K3 surfaces in arbitrary characteristic, which we review here. Let k be an algebraically closed field of characteristic $p > 0$, with ring of Witt vectors $W = W(k)$.

Let Z/k be a K3 surface. The deformation functor $\operatorname{Def}(Z)$ is formally smooth over $\operatorname{Spf} W$ of relative dimension 20; $\operatorname{Def}(Z)$ is pro-represented by a formal scheme noncanonically isomorphic to $\operatorname{Spf} W[\![t_1, \cdots, t_{20}]\!]$ [13, Cor. 1.2], [41, 4.1.1]. Let s be the base point of $\operatorname{Def}(Z)$, corresponding to Z/k itself.

Lemma 3.7. *There is a canonical isomorphism of* k*-vector spaces*

$$\mathrm{T}_s \operatorname{Def}(Z) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Fil}^2 H^2_{\mathrm{dR}}(Z/k), \operatorname{Gr}^1 H^2_{\mathrm{dR}}(Z/k)).$$

Sketch. See [13, 2.4] [39, 5.2], [40, 5.1]. Briefly, let A be a nilpotent extension of k with a divided power structure. The intersection pairing $(\cdot,\cdot)$ extends to a pairing on the crystal $H^2_{\mathrm{cris}}(Z)$. To give a deformation of Z to A is to lift $\operatorname{Fil}^2 H^2_{\mathrm{cris}}(Z)(k)$ to an isotropic direct summand of $H^2_{\mathrm{cris}}(Z)(A)$. Now use the fact [13, (2.3.7)] that the orthogonal complement to $\operatorname{Fil}^2 H^2_{\mathrm{cris}}(Z)(k)$ is $\operatorname{Fil}^1 H^2_{\mathrm{cris}}(Z)(k)$. □

If $2d$ is invertible in k and $(Z, \lambda) \in \mathsf{R}_{2d}(k)$, then $\mathrm{Def}(Z, \lambda) \subset \mathrm{Def}(Z)$ is prorepresentable, and formally smooth of dimension 19 over $\operatorname{Spf} W$ [13, 1.5 and 1.6], [32, 3.8], [41, 4.1.3]. More generally, we have:

Proposition 3.8. *Let L be a lattice of rank r and discriminant Δ_L, and suppose that Δ_L is invertible in k. Let $(Z/k, \alpha) \in \mathsf{R}_L(k)$ be an L-polarized K3 surface. Then* $\mathrm{Def}(Z, \alpha)$ *is prorepresentable and formally smooth of dimension* $20 - r$ *over* $\operatorname{Spf} W$.

Proof. As in 3.3, let $e_1, \cdots, e_r$ be a $\mathbb{Z}$-basis for L, and let $\mathcal{L}_i = \alpha(e_i) \in \mathrm{Pic}(Z)$. Then $\mathrm{Def}(Z, \alpha) = \mathrm{Def}(Z, \{\mathcal{L}_1, \cdots, \mathcal{L}_r\})$ is the largest formal subscheme of $\mathrm{Def}(Z)$ to which each of the line bundles $\mathcal{L}_i$ extends. Thus, $\mathrm{Def}(Z, \alpha)$ is the scheme theoretic intersection of the $\mathrm{Def}(Z, \mathcal{L}_i)$. Now, each $\mathrm{Def}(Z, \mathcal{L}_i)$ is the vanishing locus in $\mathrm{Def}(Z)$ of a single function f_i [13, 1.5]. So (any component of) $\mathrm{Def}(Z, \alpha)$ has codimension at most r in $\mathrm{Def}(Z, \alpha)$, and it suffices to show that the dimension of the tangent space of $\mathrm{Def}(Z, \alpha)$ at the base point is exactly $20 - r$.

Since Z is smooth and proper, there is a crystalline Chern class map $c_1 : \mathrm{NS}(Z) \to H^2_{\mathrm{cris}}(Z/W)$. Moreover, since the crystalline cohomology of Z is torsion-free and the Hodge to deRham spectral sequence for $H^\bullet(Z/W)$ degenerates at E_1, the Chern class map yields an inclusion $\bar{c}_1 : \mathrm{NS}(Z)/p\,\mathrm{NS}(Z) \hookrightarrow H^2_{\mathrm{cris}}(Z/k) \cong H^2_{\mathrm{dR}}(Z/k)$ [13, Rem. 3.5].

Let $\bar{c}_1(\alpha) \subset H^2_{\mathrm{dR}}(Z/k)$ be the span of $\bar{c}_1(\alpha(e_1)), \cdots, \bar{c}_1(\alpha(e_r))$; it is actually a subspace of $\mathrm{Fil}^1 H^2_{\mathrm{dR}}(Z/k)$ (e.g., [32, 3.4]).

Since Δ_L is invertible in k, $\mathcal{L}_1, \cdots, \mathcal{L}_r$ are linearly independent in $\mathrm{NS}(Z)/p\,\mathrm{NS}(Z)$, and thus $\dim \bar{c}_1(\alpha) = r$. Now use the fact (e.g., [32, Thm. 3.8(3)], modelled after [40, 5.1.2]) that a line bundle $\mathcal{L}$ extends to a given deformation $\widetilde{Z}/A$ if and only if $c_1(\mathcal{L})$ is orthogonal to the corresponding lift $\mathrm{Fil}^2 H^2_{\mathrm{cris}}(\widetilde{Z})(A)$. Under the isomorphism of Lemma 3.7, we see that the tangent space $\mathrm{T}_s \mathrm{Def}(Z, \alpha)$ corresponds to homomorphisms from $\mathrm{Fil}^2 H^2_{\mathrm{dR}}(Z/k)$ into the orthogonal complement of $\bar{c}_1(\mathcal{L}_1), \cdots, \bar{c}_1(\mathcal{L}_r)$ in $\mathrm{Gr}^1 H^2_{\mathrm{dR}}(Z/k)$. Because these Chern classes are linearly independent over k and $(\cdot, \cdot)$ is nondegenerate, the codimension of $\mathrm{T}_s \mathrm{Def}(Z, \alpha)$ in $T_s \mathrm{Def}(Z)$ is r. □

We now suppose that data $\underline{\chi} = (\boldsymbol{\mu}_n, \chi^\omega, \chi)$ is chosen so that $\mathsf{R}_{L,\underline{\chi}}$ is nonempty, and further assume that n is invertible in k.

Lemma 3.9. *Suppose $(Z, \alpha, \rho) \in \mathsf{R}_{L,\underline{\chi}}(k)$ and $\mathrm{char}(k) \nmid 2\Delta_L n$. The (equicharacteristic) tangent space to $\mathsf{R}_{L,\underline{\chi}}$ at (Z, α, ρ) has dimension $m(\chi^\omega) - 1$.*

Proof. By the crystalline local Torelli theorem ([9, Rem. 3.23] and [20, Lemma 3.1]; see also [39, Thm. 5.3 and Rem. (5.3.1)] and [40, 5.1.2] for characteristic at least 5), it suffices to identify the sublocus of $\mathrm{Def}(Z, \alpha)$ to which the G-action on $H^2_{\mathrm{cris}}(Z)$ extends. Thus, let $\bar{c}_1(\alpha)^\perp$ be the orthogonal complement of $\bar{c}_1(\alpha)$, and consider the inclusions of formal deformation spaces $\mathrm{Def}(Z, \alpha, \rho) \subset \mathrm{Def}(Z, \alpha) \subset \mathrm{Def}(Z)$. Computing equicharacteristic tangent spaces at the base point s, we have

$$\begin{tikzcd}
\mathrm{T}_s \mathrm{Def}(Z) & \mathrm{Hom}(\mathrm{Fil}^2 H^2_{\mathrm{dR}}(Z/k), \mathrm{Gr}^1 H^2_{\mathrm{dR}}(Z/k)) \arrow[l, "\sim"'] \\
\mathrm{T}_s \mathrm{Def}(Z, \alpha) \arrow[u, hook] & \mathrm{Hom}(\mathrm{Fil}^2 H^2_{\mathrm{dR}}(Z/k), c_1(\alpha)^\perp / \mathrm{Fil}^2 H^2_{\mathrm{dR}}(Z/k)) \arrow[l, "\sim"'] \arrow[u, hook] \\
\mathrm{T}_s \mathrm{Def}(Z, \alpha, \rho) \arrow[u, hook] & \mathrm{Hom}_G(\mathrm{Fil}^2 H^2_{\mathrm{dR}}(Z/k), c_1(\alpha)^\perp / \mathrm{Fil}^2 H^2_{\mathrm{dR}}(Z/k)) \arrow[l, "\sim"'] \arrow[u, hook]
\end{tikzcd}$$

By definition of $\mathsf{R}_{L,\underline{\chi}}$, the χ-eigenspace of $\mathrm{Fil}^1 H^2_{\mathrm{dR}}(Z/k)$ is fully contained in $c_1(\alpha)^\perp$; the result now follows. □

4. Period maps for complex K3 surfaces

4.1. **Period maps.** The global complex Torelli theorem for K3 surfaces asserts that the isomorphism class of a K3 surface $Z/\mathbb{C}$ is determined by the isomorphism class of $H^2(Z,\mathbb{Z})$ as a polarized Hodge structure. Via Hodge theory, one thus obtains a good global understanding of the moduli of complex K3 surfaces, as follows.

Let Z be a marked K3 surface, i.e., a K3 surface $Z/\mathbb{C}$ equipped with an isometry $\phi : H^2(Z,\mathbb{Z}) \xrightarrow{\sim} L_{\mathrm{K3}}$. The image of $\phi_{\mathbb{C}}(H^{2,0}(Z))$ determines an element of the period domain

$$\mathbb{X}_{L_{\mathrm{K3}}} = \{[\sigma] \in \mathbb{P}(L_{\mathrm{K3}} \otimes \mathbb{C}) : (\sigma,\sigma) = 0, (\sigma,\bar\sigma) > 0\},$$

and the isomorphism class of Z is determined by the class of $\phi_{\mathbb{C}}(H^{2,0}(Z))$ in $\mathrm{O}(L_{\mathrm{K3}})\backslash\mathbb{X}_{L_{\mathrm{K3}}}$. (We recall a useful description of $\mathbb{X}_{L_{\mathrm{K3}}}$ below in 4.2.) We remind the reader that the action of the orthogonal group $\mathrm{O}(L_{\mathrm{K3}})$ on $\mathbb{X}_{L_{\mathrm{K3}}}$ is not properly discontinuous, and thus the quotient space $\mathrm{O}(L_{\mathrm{K3}})\backslash\mathbb{X}_{L_{\mathrm{K3}}}$ is not even Hausdorff.

Now suppose that Z is equipped with a polarization λ of degree $2d$. Recall that we have fixed an embedding $\iota : \langle 2d\rangle \hookrightarrow L_{\mathrm{K3}}$. A marking ϕ of Z induces an identification of the primitive cohomology $P^2_\lambda(Z,\mathbb{Z})$ with $\langle 2d\rangle^\perp \subset L_{\mathrm{K3}}$, and thus $\phi_{\mathbb{C}}(H^{2,0}(Z))$ lies in

$$\mathbb{X}^{\langle 2d\rangle} := \mathbb{X}_{\langle 2d\rangle^\perp} = \{[\sigma] \in \mathbb{P}(\langle 2d\rangle^\perp \otimes \mathbb{C}) : (\sigma,\sigma) = 0, (\sigma,\bar\sigma) > 0\} \subset \mathbb{X}_{L_{\mathrm{K3}}}.$$

Recall (Section 2) that $\widetilde{\mathrm{O}}^{\langle 2d\rangle}(\mathbb{Z})$ consists of those orthogonal automorphisms of $\langle 2d\rangle^\perp$ which admit an extension to L_{K3} fixing $\langle 2d\rangle$. We thus have a natural inclusion

$$\widetilde{\mathrm{O}}^{\langle 2d\rangle}(\mathbb{Z})\backslash\mathbb{X}^{\langle 2d\rangle} \hookrightarrow \mathrm{O}_{L_{\mathrm{K3}}}(\mathbb{Z})\backslash\mathbb{X}_{L_{\mathrm{K3}}}.$$

The strong Torelli theorem for polarized K3 surfaces implies that there is an open immersion

$$\mathsf{R}^\circ_{2d,\mathbb{C}} \xhookrightarrow{\tau_{2d,\mathbb{C}}} \widetilde{\mathrm{O}}^{\langle 2d\rangle}(\mathbb{Z})\backslash\mathbb{X}^{\langle 2d\rangle}.$$

e.g., [19, Thm. 6.3.4] which extends to an isomorphism of coarse moduli spaces $\mathsf{R}_{2d,\mathbb{C}} \to \widetilde{\mathrm{O}}^{\langle 2d\rangle}(\mathbb{Z})\backslash\mathbb{X}^{\langle 2d\rangle}$ [19, Rem. 6.4.5].

More generally, for a primitive sublattice $L \subset L_{\mathrm{K3}}$ of signature $(1, r-1)$, we set

$$\mathbb{X}^L = \mathbb{X}_{L^\perp} = \{[\sigma] \in \mathbb{P}(L^\perp \otimes \mathbb{C}) : (\sigma,\sigma) = 0, (\sigma,\bar\sigma) > 0\}$$

and obtain an open immersion

$$\mathsf{R}_{L,\mathbb{C}} \xhookrightarrow{\tau_{L,\mathbb{C}}} \widetilde{\mathrm{O}}^{L}(\mathbb{Z})\backslash\mathbb{X}^{L};$$

see [15, §3] and [17, §11] for more details.

Finally, fix data $\underline{\chi} = (\boldsymbol{\mu}_n, \chi^\omega, \chi)$ and suppose that $(Z,\alpha,\rho) \in \mathsf{R}_{L,\underline{\chi}}(\mathbb{C})$. A choice of marking ϕ on Z induces an action of $\boldsymbol{\mu}_n$ on L_{K3} with character χ. The period point $\phi_{\mathbb{C}}(H^{2,0}(Z))$ then lies in

$$\mathbb{X}^{L,\underline{\chi}} = \{[\sigma] \in \mathbb{P}((L^\perp_{\mathbb{C}})(\chi^\omega)) : (\sigma,\sigma) = 0, (\sigma,\bar\sigma) > 0\}$$

where we single out an eigenspace for the action of $\boldsymbol{\mu}_n$ on $L^\perp$ by

$$L^\perp_{\mathbb{C}}(\chi^\omega) = \{v \in L^\perp \otimes \mathbb{C} : \forall \zeta \in \boldsymbol{\mu}_n(\mathbb{C}), \zeta v = \chi^\omega(\zeta) v\}.$$

Let $\mathrm{O}^{L,\underline{\chi}}$ be the group of automorphisms of $L^\perp$ which commute with the action of $\boldsymbol{\mu}_n$; if R is a ring over $\mathbb{Z}[\zeta_n]$, then

$$\mathrm{O}^{L,\underline{\chi}}(R) = \{g \in \mathrm{O}^L(R) : \forall \zeta \in \boldsymbol{\mu}_n(R), g\zeta v = \zeta g v\}.$$

Let $\widetilde{\mathrm{O}}^{L,\underline{\chi}}$ be the subgroup of admissible automorphisms of $L^\perp$. Then we again have an open immersion

$$\mathsf{R}_{L,\underline{\chi},\mathbb{C}} \hookrightarrow \widetilde{\mathrm{O}}^{L,\underline{\chi}}(\mathbb{Z})\backslash \mathbb{X}^{L,\underline{\chi}}.$$

Recall that $m_{\underline{\chi}}(\chi^\omega) = m(\chi^\omega) = \dim L^\perp_{\mathbb{C}}(\chi^\omega)$ is the multiplicity of the faithful character χ^ω in the representation χ, and that $L \subset L_{\mathrm{K3}}$ is the module of $\boldsymbol{\mu}_n$-invariants.

If $n \geq 3$, then χ^ω is imaginary, and

$$\mathbb{X}^{L,\underline{\chi}} \cong \mathbb{B}^{m_{\underline{\chi}}(\chi^\omega)-1},$$

the complex unit ball of dimension $m_{\underline{\chi}}(\chi^\omega) - 1$.

If $n = 2$, then χ^ω is real; $L^\perp_{\mathbb{C}}(\chi^\omega) = L^\perp_{\mathbb{R}}(\chi^\omega) \otimes \mathbb{C}$; and $\mathbb{X}^{L,\underline{\chi}}$ is a type IV Hermitian symmetric space of dimension $m_{\underline{\chi}}(\chi^\omega) - 1$.

4.2. **Period spaces.** Since L has signature $(1, r-1)$, $L^\perp$ has signature $(2, 19-(r-1))$. It is traditional in the K3 literature to describe the relevant period space as

$$\mathbb{X}^L = \mathbb{X}_{L^\perp} \cong \frac{\mathrm{O}^L(\mathbb{R})}{\mathrm{SO}_2(\mathbb{R}) \times \mathrm{O}_{20-r}(\mathbb{R})}$$

To facilitate comparison with the Shimura variety literature, we prefer to recall that the special orthogonal group $\mathrm{SO}^L(\mathbb{R})$ already acts transitively on $\mathbb{X}^L$, and we in fact have

$$\mathbb{X}^L \cong \frac{\mathrm{SO}^L(\mathbb{R})}{\mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_{20-r}(\mathbb{R})}.$$

It is perhaps worth noting that the special orthogonal group of a definite form is connected, while $\mathrm{SO}_{2,20-r}(\mathbb{R})$ has two topological components, indexed by the two classes of the spinor norm. In particular, $\mathbb{X}^L$ consists of two connected components, say $\mathbb{X}^{L+}$ and $\mathbb{X}^{L-}$; these components are stabilized by the component $\mathrm{SO}^L(\mathbb{R})^+ \subset \mathrm{SO}^L(\mathbb{R})$ of elements with trivial spinor norm.

Let $\Gamma \subset \mathrm{O}^L(\mathbb{R})$ be any arithmetic group. Then Γ has finite covolume, and in particular meets every topological component of $\mathrm{O}^L(\mathbb{R})$. We have isomorphisms of complex analytic spaces

$$\Gamma\backslash\mathbb{X}^L \cong (\Gamma \cap \mathrm{O}^L(\mathbb{R})^+)\backslash\mathbb{X}^{L+} \cong (\Gamma \cap \mathrm{SO}^L(\mathbb{R}))\backslash\mathbb{X}^L.$$

In particular, the period map is an open immersion

$$\mathsf{R}_{L,\mathbb{C}} \xhookrightarrow{\tau_L} \widetilde{\mathrm{SO}}^L(\mathbb{Z})\backslash\mathbb{X}^L.$$

5. Shimura varieties

5.1. **Integral canonical models.** We review some basic concepts concerning Shimura varieties, referring the reader to [10] for foundational material, [12] for canonical models, [23] for integral canonical models, and [46] for stack-theoretic issues. All Shimura data are assumed to be of abelian type, so that the cited references suffice.

Let $(G, \mathbb{X})$ be a Shimura datum, consisting of a reductive group $G/\mathbb{Q}$ and a conjugacy class $\mathbb{X}$ of homomorphisms $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ of $\mathbb{R}$-groups, subject to the usual axioms. Further assume that $(G, \mathbb{X})$ is of abelian type.

Let $\mathbb{K} \subset G(\mathbb{A}_f)$ be a neat compact open subgroup of the finite adelic points. The holomorphic analytic quotient stack $\mathsf{Sh}^{\mathrm{an}}_{\mathbb{K}}[G, \mathbb{X}] := [G(\mathbb{Q})\backslash(\mathbb{X} \times G(\mathbb{A}_f)/\mathbb{K})]$ is represented by the analytification of a smooth complex quasiprojective variety $\mathsf{Sh}_{\mathbb{K}}(G, \mathbb{X})$. The variety $\mathsf{Sh}_{\mathbb{K}}(G, \mathbb{X})$ and the stack $\mathsf{Sh}_{\mathbb{K}}[G, \mathbb{X}]$ both descend to the reflex field $E(G, \mathbb{X})$.

More generally, let $\mathbb{K} \subset G(\mathbb{A}_f)$ be an open compact subgroup, and let $\mathbb{K}_0 \subset \mathbb{K}$ be a neat subgroup of finite index. Define the corresponding Shimura stack by $\mathsf{Sh}_{\mathbb{K}} = [\mathsf{Sh}_{\mathbb{K}_0}[G,\mathbb{X}]/(\mathbb{K}/\mathbb{K}_0)]$; it is independent of the choice of $\mathbb{K}_0$.

Fix a prime p, let v be a prime of $E(G,\mathbb{X})$ lying over p, and let $\mathbb{K}_p \subset G(\mathbb{Q}_p)$ be hyperspecial. Then the pro-variety

$$\mathsf{Sh}_{\mathbb{K}_p} := \varprojlim_{\mathbb{K}^p \subset G(\mathbb{A}_f^p)} \mathsf{Sh}_{\mathbb{K}_p\mathbb{K}^p}(G,\mathbb{X})$$

admits an extension to $\mathcal{O}_{E(G,\mathbb{X}),v}$ as a pro-scheme with continuous $G(\mathbb{A}_f^p)$-action, which we continue to denote $\mathsf{Sh}_{\mathbb{K}_p}$. What makes this model the *integral canonical model* is the following extension property: If T is any regular, formally smooth (pro-)scheme over $\mathcal{O}_{E(G,\mathbb{X}),v}$, then any morphism $T_{E(G,\mathbb{X})} \to \mathsf{Sh}_{\mathbb{K}_p}$ extends to all of T (e.g., [23, §(2.3.7)].

Consequently, for any $\mathbb{K} \subset G(\mathbb{A}_f)$ hyperspecial at p, $\mathsf{Sh}_{\mathbb{K}}[G,\mathbb{X}]$ extends canonically to a smooth Deligne-Mumford stack over $\mathcal{O}_{E(G,\mathbb{X}),v}$. (If necessary, one can start with the canonical integral model of $\mathsf{Sh}_{\mathbb{K}_0}[G,\mathbb{X}]$ for some neat compact open subgroup $\mathbb{K}_0 \subset \mathbb{K}$, and then pass to the quotient by the action of $\mathbb{K}/\mathbb{K}_0$.)

In fact, let $\mathbb{K} \subset G(\mathbb{A}_f)$ be an open compact subgroup, and let $M = M(\mathbb{K})$ be the (finite) product of all primes p such that the component $\mathbb{K}_p$ is *not* hyperspecial. Using [30, Thm. 2.2.1], we see that $\mathsf{Sh}_{\mathbb{K}}[G,\mathbb{X}]$ admits a canonical integral model over $\mathcal{O}_{E(G,\mathbb{X})}[1/M]$.

A morphism $f : (G_1,\mathbb{X}_1) \to (G_2,\mathbb{X}_2)$ of Shimura data is a morphism $G_1 \to G_2$ of algebraic groups which induces a morphism $\mathbb{X}_1 \to \mathbb{X}_2$. For future use, we collect some standard functorialities for morphisms of Shimura varieties.

Lemma 5.1. *Let $f : (G_1,\mathbb{X}_1) \to (G_2,\mathbb{X}_2)$ be a morphism of Shimura data. Let $\mathbb{K}_1 \subset G_1(\mathbb{A}_f)$ and $\mathbb{K}_2 \subset G_2(\mathbb{A}_f)$ be compact open subgroups such that $f(\mathbb{K}_1) \subseteq f(\mathbb{K}_2)$.*

(a) *Then f induces a morphism $f_{\mathbb{K}_1,\mathbb{K}_2} : \mathsf{Sh}_{\mathbb{K}_1}[G_1,\mathbb{X}_1] \to \mathsf{Sh}_{\mathbb{K}_2}[G_2,\mathbb{X}_2]$ of Shimura stacks over E.*

(b) *If $\mathbb{K}_1$ and $\mathbb{K}_2$ are hyperspecial at all $p \nmid M$, then $f_{\mathbb{K}_1,\mathbb{K}_2}$ extends to a morphism of Shimura stacks over $\mathcal{O}_E[1/M]$.*

(c) *If $f : G_1 \to G_2$ is injective, then $f_{\mathbb{K}_1,\mathbb{K}_2}$ is a closed morphism of Shimura stacks. If $\mathbb{K}_2$ is a sufficiently small compact open subgroup of $G_2(\mathbb{A}_f)$ which contains $\mathbb{K}_1$, then the generic fiber of $f_{\mathbb{K}_1,\mathbb{K}_2}$ is a closed embedding.*

Proof. Part (a) and (the generic fiber of) part (c) are due to Deligne [10, 1.15]; see also [36, 5.16 and 13.8]. The extension to integral models follows from the extension property and the smoothness of the integral model of $\varprojlim_{\mathbb{K}_1} \mathsf{Sh}_{\mathbb{K}_1}[G_1,\mathbb{X}_1]$. □

5.2. **Orthogonal Shimura varieties.** Fix a nondegenerate lattice L of signature $(2,n_-)$, and let $G_L = \mathrm{SO}_{L\otimes\mathbb{Q}}$ be the associated special orthogonal group. Let $\mathbb{X}_L$ be the corresponding Hermitian symmetric space (§4.2).

Inside $G_L(\mathbb{A}_f)$ we single out the admissible integral automorphisms:

$$\mathbb{K}_L := \ker G_L(\widehat{\mathbb{Z}}) \to \mathrm{Aut}(\mathrm{disc}(L))(\widehat{\mathbb{Z}}).$$

The local component at p, $\mathbb{K}_{L,p}$, is hyperspecial if $p \nmid \Delta_L$. Consequently, we have an integral canonical model

$$\mathsf{Sh}_L := \mathsf{Sh}_{\mathbb{K}_L}[G_L,\mathbb{X}_L]$$

over $\mathbb{Z}[1/2\Delta_L]$. (By inverting 2, we sidestep the intricacies of orthogonal groups and Shimura varieties in even characteristic.)

Note that $\mathbb{K} \cap \mathrm{SO}_L(\mathbb{R})^+ = \widetilde{\mathrm{SO}}_L(\mathbb{Z})$, and thus [12, 2.1.2]

$$\mathsf{Sh}_{L,\mathbb{C}} \cong \mathrm{SO}_L(\mathbb{Z})^+\backslash\mathbb{X}_L^+ \cong \widetilde{\mathrm{SO}}_L(\mathbb{Z})\backslash\mathbb{X}_L. \tag{5.2.1}$$

If $\mathbb{K} \subset \mathbb{K}_L \subset G_L(\mathbb{A}_f)$ is any compact open subgroup, then there is a surjection $\mathsf{Sh}_{\mathbb{K}}[G_L, \mathbb{X}_L] \to \mathsf{Sh}_L$ of stacks over $\mathbb{Z}[1/2\Delta_L N(\mathbb{K})]$. In particular, let $\mathbb{K}_{L,N} = \ker(\mathbb{K}_L \to G_L(\mathbb{Z}/N))$, and let $\mathsf{Sh}_{L,N} = \mathsf{Sh}_{\mathbb{K}_N}[G_L, \mathbb{X}_L]$, a stack over $\mathbb{Z}[1/2\Delta_L N]$. There is a surjection $\mathsf{Sh}_{L,N} \to \mathsf{Sh}_L$, with each geometric fiber a torsor under $\mathbb{K}_L/\mathbb{K}_{L,N} \cong G_L(\mathbb{Z}/N)$.

Now fix a primitive embedding of lattices $L_1 \hookrightarrow L_2$, with respective signatures $(2, n_{1,-})$ and $(2, n_{2,-})$.

Lemma 5.2. *There is a closed morphism*

$$\mathsf{Sh}_{L_1} \xrightarrow{\psi_{L_1,L_2}} \mathsf{Sh}_{L_2}$$

of Shimura stacks over $\mathbb{Z}[1/(2\Delta_{L_1}\Delta_{L_2})]$ whose generic fiber is a closed embedding.

Proof. The chosen embedding gives an inclusion $G_{L_1} \to G_{L_2}$ of groups over $\mathbb{Q}$, which induces $\mathbb{X}_{L_1} \hookrightarrow \mathbb{X}_{L_2}$. Because of the admissibility condition, we have an inclusion $\mathbb{K}_{L_1} \hookrightarrow \mathbb{K}_{L_2}$, whence (Lemma 5.1) a morphism $\psi_{L_1,L_2} : \mathsf{Sh}_{L_1} \to \mathsf{Sh}_{L_2}$ over $\mathbb{Z}[1/(2\Delta_{L_1}\Delta_{L_2})]$. To verify that $\psi_{L_1,L_2,\mathbb{Q}}$ is a closed embedding, it suffices to check that $\psi_{L_1,L_2,\mathbb{C}}$ is an inclusion. This last claim follows from the description (5.2.1) and the fact that $\widetilde{\mathrm{SO}}_{L_1}(\mathbb{Z})$ consists of those orthogonal transformations of determinant one which lift to automorphisms of L_2. □

Similarly, for each positive integer N, there is a closed morphism $\mathsf{Sh}_{L_1,N} \to \mathsf{Sh}_{L_2,N}$ whose generic fiber is a closed embedding.

5.3. **Unitary Shimura varieties.** Let K be a quadratic imaginary field. Let L be a free $\mathcal{O}_K$-module of rank r, equipped with a nondegenerate Hermitian form $h(\cdot,\cdot)$ of signature $(1, r-1)$. Attached to this is a Shimura datum $(G_{\mathcal{O}_K,L}, \mathbb{X}_{\mathcal{O}_K,L})$, where $G_{\mathcal{O}_K,L} = \mathrm{U}(L,h)$ is the group of $\mathcal{O}_K$-linear automorphisms of L which preserve h, and $\mathbb{X}_{\mathcal{O}_K,L} \cong \mathbb{B}^{r-1}$, the unit complex ball of dimension $r-1$. Let $\mathbb{K}_{\mathcal{O}_K,L}$ be the stabilizer in $G_{\mathcal{O}_K,L}(\mathbb{A}_f)$ of L. Let $\mathsf{Sh}_{\mathcal{O}_K,L} = \mathsf{Sh}_{\mathbb{K}_{\mathcal{O}_K,L}}[G_{\mathcal{O}_K,L}, \mathbb{X}_{\mathcal{O}_K,L}]$; it's the moduli space of abelian varieties of dimension r equipped with an action by $\mathcal{O}_K$ of signature $(1, r-1)$ and a polarization λ with $\ker(\lambda) \cong \mathrm{disc}(L)$. (More precisely, the relevant Shimura datum is $(\mathrm{U}(L \otimes \mathbb{Q}, h), \mathbb{X}_{(L\otimes\mathbb{Q},h)})$; the choice of lattice L inside the $\mathbb{Q}$-vector space $L \otimes \mathbb{Q}$ defines the integral structure on $G_{\mathcal{O}_K,L}(\mathbb{A}^f)$.)

More generally, suppose K is a CM field, with maximal totally real subfield K^+, and again let L be a free $\mathcal{O}_K$-module of rank r, equipped with a nondegenerate Hermitian form h. The archimedean signature of (L,h) is determined by data

$$\{(m_\sigma, n_\sigma)\}_{\sigma: K^+ \hookrightarrow \mathbb{R}}. \tag{5.3.1}$$

Let $G_{\mathcal{O}_K,L} = \mathrm{U}(L,h)$; the associated Hermitian symmetric domain $\mathbb{X}_{\mathcal{O}_K,L}$ has dimension $\sum m_\sigma n_\sigma$. If there exists some σ_0 such that $(m_{\sigma_0}, n_{\sigma_0}) = (1, r-1)$, and if $m_\sigma n_\sigma = 0$ for $\sigma \neq \sigma_0$, then we again have $\mathbb{X}_{\mathcal{O}_K,L} \cong \mathbb{B}^{r-1}$. Again, let $\mathsf{Sh}_{\mathcal{O}_K,L} = \mathsf{Sh}_{\mathbb{K}_{\mathcal{O}_K,L}}[G_{\mathcal{O}_K,L}, \mathbb{X}_{\mathcal{O}_K,L}]$.

In the applications pursued here, it turns out that either L is unimodular, or there is some prime p which is totally ramified in $\mathcal{O}_K$; $\mathcal{O}_K$ acts on $\mathrm{disc}(L)$ through its quotient $\mathbb{F}_p$; and $\mathrm{disc}(L)$, as an abelian group, is isomorphic to $(\mathbb{Z}/p)^{2\lfloor (r-1)/2 \rfloor}$. This shows up in the analysis by Kudla and Rapoport [28] of occult period maps. The only impact on the present study is that it shapes the structure of the polarization in the moduli-theoretic intepretation of $\mathsf{Sh}_{\mathcal{O}_K,L}$.

In any event, the Shimura stack $\mathsf{Sh}_{\mathcal{O}_K,L}$ admits a smooth integral model over $\mathcal{O}_K[1/\Delta(K)\Delta_L]$.

5.4. **Shimura varieties and K3 surfaces.** Let $L \hookrightarrow L_{\mathrm{K3}}$ be a primitive sublattice of signature $(1, r-1)$. Consistent with earlier notation, we set

$$\mathsf{Sh}^L = \mathsf{Sh}_{\mathbb{K}^L}[G^L, \mathbb{X}^L] = \mathsf{Sh}_{\mathbb{K}_{L^\perp}}[G_{L^\perp}, \mathbb{X}_{L^\perp}].$$

Now let $\underline{\chi} = (\boldsymbol{\mu}_n, \chi^\omega, \chi)$ determine an action of $\boldsymbol{\mu}_n$ on $L^\perp$ as in 3.2. Let $E(\underline{\chi}) = \mathbb{Q}(\zeta_n)$.

- If $n \geq 3$, let $\mathsf{Sh}^{(L,\underline{\chi})} = \mathsf{Sh}_{\mathcal{O}_{E(\underline{\chi})},L^\perp}$; then $\mathsf{Sh}^{(L,\underline{\chi})}(\mathbb{C})$ is an arithmetic quotient of a complex ball.
- If $n = 2$, let $\mathsf{Sh}^{(L,\underline{\chi})} = \mathsf{Sh}_{L^\perp}$; then $\mathsf{Sh}^{(L,\underline{\chi})}(\mathbb{C})$ is an arithmetic quotient of a Hermitian symmetric space of type IV.

With this notation, we have:

Lemma 5.3. *The periods of a structured K3 surface determine holomorphic open immersions*

$$\mathsf{R}_{L,\mathbb{C}} \xrightarrow{\tau_{L,\mathbb{C}}} \mathsf{Sh}^L_{\mathbb{C}}$$

$$\mathsf{R}_{(L,\underline{\chi}),\mathbb{C}} \xrightarrow{\tau_{(L,\underline{\chi}),\mathbb{C}}} \mathsf{Sh}^{(L,\underline{\chi})}.$$

Proof. The period domain of a family of structured K3 surfaces is computed in, e.g., [15] and [17]. The interpretation in terms of Shimura varieties is standard, and is drawn out (in some cases) in, for instance, [27, 32, 41]. □

Remark 5.4. In the case of a datum $(L, \underline{\chi})$, the complement of the image of $\tau_{(L,\underline{\chi})}$, when known, is often a ball quotient in its own right; see, e.g., [25] for a representative example. Kudla and Rapoport, in several cases, interpret this complement as a "special cycle". In particular, this complement is itself a Shimura variety. The author conjectures that this structure of the complement holds integrally, as well. In the special case of cubic surfaces, this is worked out in [2]; for now, it seems that the general case remains open.

6. Integral period maps

With the notation established above, the Torelli theorem for complex K3 sufaces asserts that there is an open immersion

$$\mathsf{R}_{2d,\mathbb{C}} \xhookrightarrow{\tau_{2d,\mathbb{C}}} \mathsf{Sh}^{\langle 2d\rangle}_{\mathbb{C}}$$

of stacks over $\mathbb{C}$. In fact, it is known that this map preserves arithmetic:

Proposition 6.1. *The period map $\tau_{2d,\mathbb{C}}$ descends to a morphism $\tau_{2d} : \mathsf{R}_{2d} \to \mathsf{Sh}^{\langle 2d\rangle}$ of stacks over $\mathbb{Q}$.*

Proof. Rizov has proved this for R°_{2d}, using an analogue of CM theory for K3 surfaces; see [42, p.14] and [46, Thm. 5]. The statement for R_{2d} follows from descent relative to $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Q}$, since R°_{2d} is dense in R_{2d}. □

Using Proposition 6.1 as a starting point, we will show that other period maps also descend to a natural field of definition and extend integrally. We start with an interlude on level structures, so that we can work with quasiprojective schemes and verify descent in an elementary fashion.

6.1. Level structures.

It is possible to define the notion of a lattice polarized K3 surface with $\mathbb{K}$ level structure for an essentially arbitrary open subgroup of $G^L(\widehat{\mathbb{Z}})$; but we will content ourselves here with a more limited notion which is adequate for our purposes. (See [46, §5.2] and [42] for more details in the case $L = \langle 2d\rangle$.)

Fix an integer $N > 2$ which is relatively prime to $2p\Delta_L$. Then $\mathrm{SO}^L(\mathbb{Z}/N)$ is admissble; any automorphism of $L^\perp \otimes \mathbb{Z}/N$ lifts uniquely to $L_{\mathrm{K3}} \otimes \mathbb{Z}/N$ as an element which fixes L.

If $(Z \to S, \alpha) \in \mathsf{R}_L(S)$, a full level N structure on $(Z \to S, \alpha)$ is an isomorphism of formed spaces $\beta : L_{\mathrm{K3}} \otimes \mathbb{Z}/N \xrightarrow{\sim} R^2 f_* \mathbb{Z}/N(1)$ such that the following diagram commutes:

$$\begin{tikzcd} L_{\mathrm{K3}} \otimes \mathbb{Z}/N \arrow[r, "\beta", "\sim"'] & R^2 f_* \mathbb{Z}/N(1) \\ L \arrow[u] \arrow[r, "\alpha"] & \mathrm{Pic}_{Z/S}(S) \arrow[u, "c_{1,N}"'] \end{tikzcd}$$

where the right-hand vertical map is the Chern class, and the left-hand map is induced by the fixed inclusion $L \hookrightarrow L_{\mathrm{K3}}$.

Since $N > 2$, $\mathsf{R}_{L,N}$ is representable by a smooth, quasiprojective scheme over $\mathbb{Z}_{(p)}$ (see, e.g., [42, Cor. 2.4.3] for the case $L = \langle 2d \rangle$). Moreover, because of the admissibility condition, $\mathsf{R}_{L,N} \to \mathsf{R}_L$ is Galois, with covering group isomorphic to $\{g \in \mathrm{SO}_{L_{\mathrm{K3}}}(\mathbb{Z}/N) : g|_{L^\perp \otimes \mathbb{Z}/N} = \mathrm{id}\}$.

As before, given L, choose a primitive embedding of lattices $\langle 2d \rangle \hookrightarrow L$. The forgetful maps yield a Cartesian diagram

$$\begin{tikzcd} \mathsf{R}_{L,N} \arrow[r, hook] \arrow[d] & \mathsf{R}_{\langle 2d \rangle, N} \cong \mathsf{R}_{2d,N} \arrow[d] \\ \mathsf{R}_L \arrow[r, hook] & \mathsf{R}_{\langle 2d \rangle} \end{tikzcd}$$

where the horizontal arrows are closed immersions, and the vertical arrows are quotients by suitable subgroups of $\mathrm{SO}_{L_{\mathrm{K3}}}(\mathbb{Z}/N)$.

6.2. **Descent to the reflex field.**

Lemma 6.2. (a) *Let $L \hookrightarrow L_{\mathrm{K3}}$ be a primitive lattice of signature $(1, r-1)$. Then the complex period map $\tau_{L,\mathbb{C}}$ descends to a morphism $\tau_L : \mathsf{R}_L \to \mathsf{Sh}^L$ of stacks over $\mathbb{Q}$.*

(b) *Let $(L, \underline{\chi})$ be as in 3.2. Then the complex period map $\tau_{(L,\underline{\chi}),\mathbb{C}}$ descends to a morphism $\tau_{(L,\underline{\chi})}$ of stacks over $E(\underline{\chi})$.*

Proof. We address part (a) in detail. Fix some $N > 2$. Since $\mathsf{R}_L = [\mathsf{R}_{L,N}/G^L(\mathbb{Z}/N)]$ and $\mathsf{Sh}^L = [\mathsf{Sh}^L_N/G^L(\mathbb{Z}/N)]$, it suffices to show that the complex period map with level N structure, $\tau_{L,N,\mathbb{C}} : \mathsf{R}_{L,N,\mathbb{C}} \to \mathsf{Sh}^L_{N,\mathbb{C}}$ descends to $\mathbb{Q}$. Choose a primitive embedding $\langle 2d \rangle \hookrightarrow L$. We have a commuting diagram of universally injective morphisms of complex reduced quasiprojective varieties

(6.2.1)
$$\begin{tikzcd} \mathsf{R}_{L,N,\mathbb{C}} \arrow[r, hook, "\tau_{L,N,\mathbb{C}}"] \arrow[d, hook, "\phi_{L,2d,\mathbb{C}}"] & \mathsf{Sh}^L_{N,\mathbb{C}} \arrow[d, hook, "\psi_{G^L, G^{\langle 2d \rangle}}"] \\ \mathsf{R}_{2d,N,\mathbb{C}} \arrow[r, hook] & \mathsf{Sh}^{\langle 2d \rangle}_{N,\mathbb{C}}. \end{tikzcd}$$

Since $\mathsf{R}_{L,N,\mathbb{C}} \to \mathsf{R}_{2d,N,\mathbb{C}}$ and $\mathsf{R}_{2d,N,\mathbb{C}} \to \mathsf{Sh}^{\langle 2d \rangle}_{N,\mathbb{C}}$ descend to $\mathbb{Q}$, so does $\psi_{G^L, G^{\langle 2d \rangle}} \circ \tau_{L,N,\mathbb{C}}$. Since $\psi_{G^L, G^{\langle 2d \rangle}}$ is universally injective (Lemma 5.2), $\tau_{L,N,\mathbb{C}}$ is $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-equivariant on $\mathbb{C}$-points, and thus descends to $\mathbb{Q}$ as well.

The proof of (b) is exactly the same, except that the role of $G^L(\mathbb{Z}/N)$ is now played by the finite unitary group $G^{(L,\underline{\chi})}(\mathbb{Z}/N)$, and (6.2.1) is replaced with

$$\begin{tikzcd} \mathsf{R}_{(L,\underline{\chi}),N,\mathbb{C}} \arrow[r, hook, "\tau_{(L,\underline{\chi}),N,\mathbb{C}}"] \arrow[d, hook, "\phi_{(L,\underline{\chi}),L,\mathbb{C}}"] & \mathsf{Sh}^{(L,\underline{\chi})}_{N,\mathbb{C}} \arrow[d, hook, "\psi_{G^{(L,\underline{\chi})}, G^L}"] \\ \mathsf{R}_{L,N,\mathbb{C}} \arrow[r, hook] & \mathsf{Sh}^L_{N,\mathbb{C}}. \end{tikzcd}$$

□

6.3. **Integral extension.** Granting the existence of integral canonical models of Shimura varieties, it is not hard to see that τ_{2d} extends to a morphism of stacks over $\mathbb{Z}[1/6d]$; this is achieved in [41, Thm. 4.3.3]. We refer to [32, Cor. 5.15] for the difficult extension of this work to $\mathbb{Z}[1/2]$.

Remark 6.3. In fact, in [32], the proof naturally gives rise to a period map for a trivial double cover of R_{2d}. Taelman has observed [46] that by viewing the period map as measuring the primitive cohomology *twisted by the determinant*, the need for a double cover is eliminated.

Following Taelman's analysis [46, §5], let $\mathbb{K}_L^* = \{\gamma \in \widetilde{\mathrm{O}}_L(\widehat{\mathbb{Z}}) : \det(\gamma) \in \{\pm 1\}\}$. Then $\mathbb{K}_{L^\perp}$ acts on $L^\perp$ via the determinant; Taelman defines, for instance, a period map

$$\mathsf{R}_{2d}(\mathbb{C}) \to [\mathrm{SO}^{\langle 2d\rangle}(\mathbb{C})\backslash(\mathbb{X}^{\langle 2d\rangle} \times \mathrm{SO}^{2d}(\mathbb{A}_f)/(\mathbb{K}^{\langle 2d\rangle})^*].$$

The target space is isomorphic, as an analytic space, to our $\mathsf{Sh}_{\mathbb{K}^{\langle 2d\rangle}}[G^{\langle 2d\rangle}, \mathbb{X}^{\langle 2d\rangle}]$. However, this target naturally identifies a polarized Hodge structure (H, s) with $(H, -s)$. In this way, the effect of a choice of generator of $\langle 2d\rangle$ is erased.

More generally, by following Taelman's formulation, we can suppress the choice of a "positive light cone" in the definition of R_L in 3.2; two L-polarizations which agree up to sign are identified by the action of $\mathbb{K}_L^*$ through its determinant.

We now secure analogous results for other period maps.

Lemma 6.4. (a) *Let $L \hookrightarrow L_{\mathrm{K3}}$ be a primitive lattice of signature $(1, r-1)$. Then the period map extends to a morphism $\tau_L : \mathsf{R}_L \to \mathsf{Sh}^L$ of stacks over $\mathbb{Z}[1/2\Delta(L)]$.*

(b) *Let $(L, \underline{\chi})$ be as in 3.2. Then the period map extends to a morphism $\tau_{(L,\underline{\chi})} : \mathsf{R}_{(L,\underline{\chi})} \to \mathsf{Sh}^{(L,\underline{\chi})}$ of stacks over $\mathcal{O}_{E(\underline{\chi})}[1/2\Delta(L)]$.*

Proof. Since Sh^L is separated, it suffices to show that, for a fixed $p \nmid 2\Delta(L)$, τ_L extends to $\mathbb{Z}_{(p)}$. Let $N \geq 3$ be a natural number relatively prime to p. Since $\mathsf{R}_{L,N}$ is smooth (Proposition 3.8), the extension property of the integral canonical model implies that the morphism τ_L extends to $\mathbb{Z}_{(p)}$.

The proof of (b) is the same, except that the necessary smoothness is secured in Lemma 3.9. □

Remark 6.5. The generic fiber of the morphism $\mathsf{Sh}^L \hookrightarrow \mathsf{Sh}^{\langle 2d\rangle}$ is a closed immersion (Lemma 5.2). If it were known that $\psi_{L^\perp, \langle 2d\rangle^\perp}$ is a closed immersion of Shimura stacks, one could give an elementary proof of Lemma 6.4, as follows. Suppose $p \nmid \Delta(L)d(L)$; choose d with $p \nmid d$ such that there exists a primitive $\langle 2d\rangle \hookrightarrow L$. We start with a diagram as in (6.2.1), where all objects are defined over $\mathbb{Z}_{(p)}$, except that τ_L is only known to be defined over $\mathbb{Q}$:

$$\begin{tikzcd}
\mathsf{R}_{L,N} \arrow[r, dotted, "\tau_L"] \arrow[d, "\phi"] & \mathsf{Sh}_N^L \arrow[d, hook, "\psi"] \\
\mathsf{R}_{2d,N} \arrow[r, hook, "\tau_{2d}"] & \mathsf{Sh}_N^{\langle 2d\rangle}
\end{tikzcd}$$

We know that ϕ is a closed immersion and τ_{2d} is an open immersion, and thus the composition $\mathsf{R}_{L,N} \to \mathsf{Sh}_N^{\langle 2d\rangle}$ is a locally closed immersion. All schemes involved are Noetherian and $\mathsf{R}_{L,N}$ is reduced, so the image of $\mathsf{R}_{L,N}$ is an open subscheme of a closed subscheme of $\mathsf{Sh}_N^{\langle 2d\rangle}$ [45, Tag 03DQ]. We are operating under the hypothesis that ψ is a closed immersion (Lemma 5.1). Since Sh_N^L is reduced, ψ maps Sh_N^L isomorphically onto its image, a closed subscheme of $\mathsf{Sh}_N^{\langle 2d\rangle}$.

We have observed (§5.4) that $\tau_{2d} \circ \phi$ maps the characteristic zero fiber $\mathsf{R}_{L,N,\mathbb{Q}}$ into $\psi(\mathsf{Sh}^L_{N,\mathbb{Q}})$ inside $\mathsf{Sh}^{\langle 2d\rangle}_N$. Since $\psi(\mathsf{Sh}^L_N)$ is closed, $\mathsf{R}_{L,N,\mathbb{Q}}$ is dense in $\mathsf{R}_{L,N}$ (by flatness over $\mathbb{Z}_{(p)}$; see Proposition 3.3) and $\tau_{2d} \circ \phi(\mathsf{R}_{L,N})$ is locally closed, it follows that $\tau_{2d} \circ \phi_N(\mathsf{R}_{L,N}) \subseteq \psi(\mathsf{Sh}^L_N)$.

In particular, $\tau_{2d} \circ \phi$ factors through a locally closed immersion $\tau_L : \mathsf{R}_{L,N} \to \mathsf{Sh}^L_N$. We again invoke the fact that, for Noetherian schemes, a locally closed immersion factors as an open immersion followed by a closed immersion. The fact that $\dim \mathsf{R}_{L,N} = \dim \mathsf{Sh}^L_N$ (and the reducedness of $\mathsf{R}_{L,N}$) now implies that, in any such factorization, the closed immersion must be the identity map and therefore τ_{2d} is an open immersion.

7. From complete intersections to K3 surfaces

Thanks especially to works of Kondō, we know that sometimes one can associate a structured K3 surface to certain types of complete intersection varieties. Some of these constructions are reviewed here, with an eye towards making sense of these associations in families, and ultimately explaining the arithmetic origin of Kondō's analytic ball-quotient maps.

In an attempt to minimize repetition in the statement of our main results, we make the following definition:

Definition 7.1. Say that $(\mathsf{R}^\circ, \mathsf{N}, \mathsf{S}, \kappa, \tau)$ satisfies $(\dagger)$ over $\mathcal{O}$ if there is a diagram

$$(\dagger) \qquad \begin{array}{ccc} \mathsf{R} & \xrightarrow{\kappa} & \mathsf{N} \\ \downarrow{\scriptstyle \tau} & & \\ \mathsf{S} & & \end{array}$$

of stacks over $\mathcal{O}$ where κ induces an isomorphism on coarse moduli spaces, and τ induces an open immersion $\mathsf{R}^\circ(\mathbb{C}) \hookrightarrow \mathsf{S}(\mathbb{C})$.

We should note that, in many of the examples studied here (§7.2, 7.3, 7.7), Kudla and Rapoport have already shown that a transcendentally-defined occult period map descends to a natural cyclotomic field of definition [28, §9]. Their method of proof goes back (at least) to Deligne [11, Thm. 2.12]. Roughly speaking, one shows that a monodromy representation is so large that a certain abelian scheme admits no automorphisms, and thus descends. This strategy presumably also dispatches §7.6, perhaps with [14] providing the necessary monodromy calculation. Applications §7.4 and 7.5 don't literally fit within the framework of unitary Shimura varieties attached to quadratic imaginary fields, which may explain their omission from [28].

7.1. **Stacks of varieties with group action.**

In Kondō's constructions, the original variety is encoded in the fixed locus of the group action on the K3 surface. If a group scheme G/S acts on a scheme Z/S, one can define Z^G, the fixed point stack [43, Prop. 2.5].

Lemma 7.2. *Suppose $Z \to S$ is a K3 space and $G \subset \mathrm{Aut}_{Z/S}(S)$ is a nontrivial finite cyclic group.*

(a) *The fixed locus $Z^G \to S$ is a scheme.*
(b) *If $s \in S$, then Z^G_s is smooth, and has at most one component of dimension one and genus at least two.*
(c) *If S is irreducible with generic point η, and if $C_\eta \subset Z^G_\eta$ is a curve of genus at least two, then the closure C of C_η in Z^G is a smooth, proper relative curve over S.*

Proof. Since $Z \to S$ is an algebraic space, so is Z^G [43, Rem. 3.4(ii)]; since all components of all fibers are smooth (see below) of dimension at most one, Z^G is actually a scheme.

The smoothness assertion of (b) is proved in [7, Lemma 2.2] (in characteristic zero) and [22, Prop. 1.4] (in positive characteristic). The fact that there is at most one curve of general type is also in [7, Lemma 2.2]. (While the statement is only claimed for complex K3 surfaces, the argument relies on nothing more than the Hodge index theorem.)

Part (c) follows from the upper semicontinuity, on Z^G, of the function $z \mapsto \dim(Z^G_{\varpi(z)})$ [18, IV.13.1.3]. □

We will occasionally have cause to work with the stacks of smooth relative uniform cyclic covers of projective spaces, as in [5]. Recall that if $X \to S$ is a smooth scheme, then a smooth relative uniform cyclic cover of degree n consists of a morphism $f : Y \to X$ which commutes with an action of $\boldsymbol{\mu}_n$ on Y such that the branch divisor of f is smooth over S, and, Zariski-locally on X, Y is $\boldsymbol{\mu}_n$-equivariantly isomorphic to $\mathcal{O}_Y(U)[y]/(y^n - h)$. With a slight adjustment of the notation of [5], let $\mathsf{H}(n, m, d)$ be the stack of smooth relative uniform cyclic covers $f : Y \to P \to S$ of degree n, where $P \to S$ is a Brauer-Severi scheme of dimension m, and the branch divisor of f has degree d. Thus, for example, $\mathsf{H}(2, 1, 2g + 2)$ is the moduli stack of hyperelliptic curves of genus g. (A Brauer-Severi scheme $P \to S$ of dimension m is an S-scheme which, étale-locally on S, is isomorphic to the projective space of dimension m.)

In the special case where $m = 1$, let $\widetilde{\mathsf{H}}(n, 1, d)$ be the stack of smooth relative uniform cyclic covers of Brauer-Severi curves equipped with a labelling of the branch locus; there is a forgetful map $\widetilde{\mathsf{H}}(n, 1, d) \to \mathsf{H}(n, 1, d)$, with fiber a torsor under the symmetric group S_d on d letters. (Since a Brauer-Severi scheme with a section is trivial, the underlying scheme of an object in $\widetilde{\mathsf{H}}(n, 1, d)$ is actually a family of projective lines, rather than merely étale-locally a family of projective lines.)

Let $\widetilde{\mathsf{M}}_{0,d}$ be the moduli space of d distinct, labelled points in $\mathbb{P}^1$. By sending a labelled branched cover of the projective line to its branch locus, we obtain a morphism $\widetilde{\mathsf{H}}(n, 1, d) \to \widetilde{\mathsf{M}}_{0,d}$. In fact, this morphism is the rigidification along $\boldsymbol{\mu}_n$; it factors as $\widetilde{\mathsf{H}}(n, 1, d) \to \widetilde{\mathsf{H}}(n, 1, d) \mathbin{/\!\!/} \boldsymbol{\mu}_n \xrightarrow{\sim} \widetilde{\mathsf{M}}_{0,d}$, and in particular induces an isomorphism on coarse moduli spaces. This morphism is S_d-equivariant, and we have $\mathsf{H}(n, 1, d) \to \mathsf{M}_{0,d}$.

Below, we will often have a morphism $\alpha : \mathsf{S} \to \mathsf{T}$ of smooth stacks. Then each stack is normal, and in particular has a normal coarse moduli space. If α induces a bijection on geometric points, then (by Zariski's main theorem) α induces an isomorphism of coarse moduli spaces.

7.2. **Curves of genus four.** Here we follow [25]. The argument given here is also a prototype for the remainder of this section.

Let C/k be a smooth, projective nonhyperelliptic curve of genus 4 with no vanishing theta constants, over an algebraically closed field in which 6 is invertible. Its canonical model is the (complete) intersection in $\mathbb{P}^3$ of quadric and cubic surfaces Q and S. Let $\varpi : Z \to Q$ be the triple cover of Q branched along C; then Z comes equipped with an action by $\boldsymbol{\mu}_3$. Let M_1 and M_2 be smooth lines on Q which represent the two rulings, and let $N_i = \varpi^{-1} M_i$. Then each N_i is an elliptic curve, and the two of them pair as $(N_1, N_2) = 3$. Moreover, N_1 and N_2 span a primitive lattice of $\mathrm{Pic}(Z)$, isomorphic to $L_4 := U(3)$. Let $L_4 \hookrightarrow L_{\mathrm{K3}}$ be a primitive embedding; the orthogonal complement of this copy of $U(3)$ is $L_4^\perp \cong U(3) \oplus U \oplus E_8(-1)^{\oplus 2}$ [25, p. 386]. In the notation of §2, $d(L_4) = 3$, and so there is a closed immersion $\mathsf{R}_L \hookrightarrow \mathsf{R}_{\langle 6 \rangle}$ of smooth stacks over $\mathbb{Z}[1/6]$.

The action of $\boldsymbol{\mu}_3$ on Z is nonsymplectic, in the sense that χ^ω, the character of the representation of $\boldsymbol{\mu}_3$ by which $\boldsymbol{\mu}_3$ acts on $H^0(Z, \Omega^2)$, is faithful. Kondō explicitly writes down a certain representation ρ of $\boldsymbol{\mu}_3$ on $L_4^\perp$. (Of course, $L_4^\perp$ is free over $\mathbb{Z}[\zeta_3]$, in accordance with [31, Lemma 1.1].) Let χ_4 be the character of $\rho \oplus \rho_{\mathrm{triv}}^{\oplus 2}$. In the case where $k = \mathbb{C}$, Kondō shows that Z is an element of $\mathsf{R}_{L_4, \underline{\chi}_4}(\mathbb{C})$. Let N_4 be the moduli space of nonhyperelliptic curves of genus 4. It is not hard to extend the work in [25] to show:

Lemma 7.3. *There is a morphism $\kappa_4 : \mathsf{R}^\circ_{L_4, \underline{\chi}_4} \to \mathsf{N}_4$ of stacks over $\mathbb{Z}[\zeta_3, 1/6]$ which is a bijection on geometric points.*

Proof. Suppose $(Z \to S, \alpha, \rho) \in \mathsf{R}^\circ_{L_4,\underline{\chi}_4}(S)$. In particular, $Z \to S$ is an algebraic space. Let $B = Z^{\boldsymbol{\mu}_3} \to S$ be the scheme of fixed points (Lemma 7.2).

We will show that every fiber of $B \to S$ is a smooth, projective, nonhyperelliptic curve of genus 4. Then B is a scheme over S, and the sought-for functor $\kappa_4 : \mathsf{R}^\circ_{L_4,\underline{\chi}_4} \to \mathsf{N}_4$ is then given by $(Z \to S, \alpha, \rho) \mapsto (Z^{\boldsymbol{\mu}_3} \to S)$.

So, let s be a point of S. If s has residue characteristic zero, the proof of [25, Thm. 1] shows that B_s is a smooth, projective, nonhyperelliptic curve of genus 4.

If s has positive characteristic $p \geq 5$, since $\mathsf{R}^\circ_{L_4,\underline{\chi}_4}$ is smooth over $\mathbb{Z}[\zeta_3, 1/6]$, s lifts to characteristic zero. More precisely, there exist a mixed characteristic discrete valuation ring A, with general and special fibers η and $\circ$, and a point $P \in \mathsf{R}^\circ_{L_4,\underline{\chi}_4}(\operatorname{Spec} A)$ with $P_\circ = s$. The characteristic zero result for B_η, combined with the specialization argument of Lemma 7.2(c), shows there is a (necessarily unique) smooth projective curve C_s of genus 4 in B_s.

Moreover, the quotient $Z_s/\boldsymbol{\mu}_4$ is a quadric (cone) [25, p. 389], and C_s maps isomorphically onto its image in the quotient. Insofar as C_s is a genus 4 curve lying on a quadric surface in $\mathbb{P}^3$, it is not hyperelliptic.

This defines the morphism $\mathsf{R}^\circ_{L_4,\underline{\chi}_4} \to \mathsf{N}_4$. Now let k be an algebraically closed field in which 6 is invertible. The construction at the beginning of this subsection – modified to take a minimal resolution, if necessary, to account for the impact of vanishing theta characteristics – gives a set-theoretic section to $\mathsf{R}^\circ_{L_4,\underline{\chi}_4}(k) \to \mathsf{N}_4(k)$. □

We can finally explain the arithmetic origin of Kondō's observation that $\mathsf{N}_4(\mathbb{C})$ is an arithmetic ball quotient.

Proposition 7.4. *The tuple* $(\mathsf{R}^\circ_{L_4,\underline{\chi}_4}, \mathsf{N}_4, \mathsf{Sh}^{(L_4,\underline{\chi}_4)}, \kappa_4, \tau_{L_4,\underline{\chi}_4})$ *satisfies* (†) *over* $\mathbb{Z}[\zeta_3, 1/6]$.

Proof. This simply summarizes the foregoing. Consider κ_4 from Lemma 7.3. Since it yields a bijection on geometric points, it induces an isomorphism of coarse moduli spaces. For $\tau_{L_4,\underline{\chi}_4}$, Kudla and Rapoport [28, Thm. 8.1] interpret Kondō's isomorphism [25, Thm. 1] map as a morphism $\mathsf{R}^\circ_{L_4,\underline{\chi}_4,\mathbb{C}} \to \mathsf{Sh}^{(L_4,\underline{\chi}_4)}_{\mathbb{C}}$ (see §5.4). Then Lemma 6.4 shows that this map descends and spreads to $\mathbb{Z}[\zeta_3, 1/6]$. □

Remark 7.5. In characteristic zero, Kudla and Rapoport use a transcendental construction, and the fact that $\mathsf{Sh}^{(L_4,\underline{\chi}_4)}$ is a moduli space for abelian varieties with action by $\mathbb{Z}[\zeta_3]$, to interpret Kondō's construction as a morphism of stacks $\mathsf{N}_{4,\mathbb{C}} \to \mathsf{Sh}^{(L_4,\underline{\chi}_r)}_{\mathbb{C}}$. They then use a monodromy argument [28, p.579] to show that this map descends to a morphism of stacks over $\mathbb{Q}(\zeta_3)$, and conjecture that it extends to a morphism over $\mathbb{Z}[\zeta_3]$.

Remark 7.6. Let $\mathsf{R}^\circ_{L_4,\underline{\chi}_4} /\!\!/ \boldsymbol{\mu}_3$ be the rigidification of $\mathsf{R}^\circ_{L_4,\underline{\chi}_4}$ along $\boldsymbol{\mu}_3$ ([1]; see also [43, §5]). Then $\mathsf{R}^\circ_{L_4,\underline{\chi}_4} /\!\!/ \boldsymbol{\mu}_3$ has the same coarse moduli space as $\mathsf{R}^\circ_{L_4,\underline{\chi}_4}$, and the morphism of Proposition 7.4 factors as

$$\mathsf{R}^\circ_{L_4,\underline{\chi}_4} \longrightarrow \mathsf{R}^\circ_{L_4,\underline{\chi}_4} /\!\!/ \boldsymbol{\mu}_3 \longrightarrow \mathsf{N}_4.$$

In this notation, Kudla and Rapoport conjecture [28, Rem. 7.2] that the second map is an isomorphism of stacks. In fact, this has recently been resolved using transcendental means by Zheng [47, Prop. 7.9], who goes on to show that, at least over $\mathbb{C}$, there is an open immersion of orbifolds $\mathsf{N}_{4,\mathbb{C}} \to \mathsf{Sh}^{(L_4,\underline{\chi}_4)}_{\mathbb{C}}$. Zheng also proves analogous statements in the situations of §7.3 and 7.7 below.

7.3. **Curves of genus three.** Kondō has given [24] a similar characterization of $\mathsf{N}_3(\mathbb{C})$, the set of complex nonhyperelliptic curves of genus 3. This construction also descends to arithmetic geometry, as follows.

Let C/k be a smooth, projective nonhyperelliptic curve of genus 3 over an algebraically closed field in which 2 is invertible. Its canonical model is a smooth, plane quartic curve; let $\varpi : Z \to \mathbb{P}^2$ be the cyclic quartic cover ramified along C. Then Z is a K3 surface, and inside $\mathrm{Pic}(Z)$ is a lattice $L_3 \cong A_1 \oplus A_1(-1)^{\oplus 7}$ [24, p. 222]. (Briefly, Z is a double cover of Y, itself a double cover of $\mathbb{P}^2$ branched along C. Then Y is a del Pezzo surface of degree two, and thus may also be obtained as the blowup of a projective plane at seven points. The seven copies of $A_1(-1)$ in $\mathrm{Pic}(Z)$ are obtained from lifts to Z of the seven exceptional divisors; the remaining element of modulus two is the pullback of the class of a line on the projective plane.) Then L_3 embeds primitively into L_{K3}, with orthogonal complement $L_3^\perp \cong U(2)^{\oplus 2} \oplus D_8(-1) \oplus A_1(-1)^{\oplus 2}$, and $d(L_3) = 2$.

By construction, Z comes equipped with an action by $\boldsymbol{\mu}_4$. Then $\boldsymbol{\mu}_4$ acts on the space of holomorphic two forms via a faithful character, χ^ω. The action ρ of $\boldsymbol{\mu}_4$ on $L_3^\perp$ is given explicitly in [24, §2], and we let χ_4 be the character of $\rho \oplus \rho_{\mathrm{triv}}^{\oplus 8}$.

Proposition 7.7. *There is a morphism $\kappa_3 : \mathsf{R}^\circ_{L_3,\underline{\chi}_3} \to \mathsf{N}_3$ so that $(\mathsf{R}^\circ_{L_3,\underline{\chi}_3}, \mathsf{N}_3, \mathsf{Sh}^{(L_3,\underline{\chi}_3)}, \kappa_3, \tau_{L_3,\underline{\chi}_3})$ satisfies (†) over $\mathbb{Z}[\sqrt{-1}, 1/2]$.*

Proof. As in Lemma 7.3, the map κ_3 is given by $(Z \to S, \alpha, \rho) \mapsto (Z^{\boldsymbol{\mu}_4} \to S)$; [24, p. 225] and the étale Lefschetz fixed point theorem [SGA 5.III.(4.11.3)] provide the necessary geometric input to show that $Z^{\boldsymbol{\mu}_4}$ is a relative nonhyperelliptic curve of genus 3. The fact that κ_3 gives a bijection on geometric points is established by the construction at the beginning of this section. The asserted behavior of $\tau_{L_3,\underline{\chi}_3}$ is a special case of Lemma 6.4. □

Remark 7.8. See [28, §7] for earlier results over $\mathbb{Q}(\sqrt{-1})$.

7.4. **Curves of genus six.** Following [6], let N_6 denote the moduli stack (over $\mathbb{Z}[1/2]$) of non-special curves of genus 6. (Thus, a curve is represented by a point in N_6 if it is smooth, projective and irreducible of genus 6, and neither hyperelliptic, trigonal, bielliptic, nor smooth quintic and planar.)

In fact, let C/k be a non-special curve over an algebraically closed field. The canonical embedding of C is a quadric section of a unique quintic del Pezzo surface Y in $\mathbb{P}^5$. Let Z be the double cover of Y branched along C. (If C has fewer than five g^2_6's, then one must actually take the minimal resolution of this cover.) Then Z is a K3 surface with an action by $\boldsymbol{\mu}_2 = \{\pm 1\}$, and this action fixes a lattice in $\mathrm{Pic}(Z)$ isomorphic to $L_6 := A_1 \oplus A_1(-1)^4$ [6, §2.1]. (Note that $d(L_6) = 2$.)

Proposition 7.9. *There is a morphism $\kappa_6 : \mathsf{R}^\circ_{L_6,\underline{\chi}_6} \to \mathsf{N}_6$ such that $(\mathsf{R}^\circ_{L_6,\underline{\chi}_6}, \mathsf{N}_6, \mathsf{Sh}^{(L_6,\underline{\chi}_6)}, \kappa_6, \tau_{L_6,\underline{\chi}_6})$ satisfies (†) over $\mathbb{Z}[1/6]$.*

Proof. As before, κ_6 is given by sending $(Z \to S, \alpha, \rho) \in \mathsf{R}^\circ_{L_6,\underline{\chi}_2}(S)$ to its fixed locus $Z^{\boldsymbol{\mu}_2} \to S$ (see [6, p. 1452] for the argument, valid in any characteristic, that each geometric fiber $Z_{\bar{s}}^{\boldsymbol{\mu}_2}$ is a non-special curve of genus 6). The construction described above gives a set-theoretic section on geometric points, and $\tau_{L_6,\underline{\chi}_6}$ is supplied by Lemma 6.4. □

7.5. **Five points on a line.** Kondō has also explained how, in favorable cases, one can associate a structured K3 surface to certain configuration spaces of points.

For instance, as in [26], consider the moduli space $\widetilde{\mathsf{M}}_{0,5}$ of five distinct, ordered points in $\mathbb{P}^1$. (In fact, our discussion extends to the case of *stable* configurations of points.)

Fix an embedding $\beta : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as a coordinate line, and let $Q_\infty \in \mathbb{P}^2$ denote a point "at infinity" which is not contained in $\beta(\mathbb{P}^1)$.

Initially, let k be an algebraically closed field, and let $(P_1, \cdots, P_5) \in \widetilde{\mathsf{M}}_{0,5}(k)$ be an ordered 5-tuple of distinct points. Following [26, Sec. 3.1-3.2], let C be the cyclic degree five cover of $\mathbb{P}^1$ ramified exactly at $P_1, \cdots, P_5$. It naturally admits a model as a plane curve inside $\mathbb{P}^2$, intersecting

$\beta(\mathbb{P}^1)$ exactly at $Q_i := \beta(P_i)$ for $i = 1, \cdots, 5$. Let L_i denote the line connecting Q_i and Q_∞; and let $E_0 = \beta(\mathbb{P}^1)$.

Next, let X be the minimal resolution of the double cover of $\mathbb{P}^2$ branched along the sextic plane curve $C + E_0$, with covering involution τ. The degree five automorphism of C also induces a degree five automorphism σ of X. Moreover, because of our construction, we can identfy certain divisor (classes) on X which are fixed by τ.

Indeed, for $1 \le i \le 5$, there is an exceptional curve E_i of the minimal resolution of singularities corresponding to $Q_i \in C \cap E_0$. Moreover, the inverse image of L_i in X is the union of two smooth rational curves $F_{i,-}$ and $F_{i,+}$, which pass (respectively) through the two points R_- and R_+ of X lying over Q_∞. The involution τ exchanges $F_{i,-}$ and $F_{i,+}$, and σ stabilizes each of $F_{i,-}$ and $F_{i,+}$.

Finally, there is a sixteenth tautological cycle on X, namely, the inverse image $E_{0,X}$ of E_0 in X. It is stable under σ and τ.

Let L_5 be the lattice generated by these sixteen divisors, equipped with the intersection pairing. Then $L_5 \cong V \oplus A_4(-1) \oplus A_4(-1)$ [26, Lemma 4.2], and $d(L_5) = 2$. The labelling of the original points, combined with the labelling of the points in X over Q_∞, yields an inclusion $L_5 \hookrightarrow \mathrm{Pic}(X)$.

There is an embedding $L_5 \hookrightarrow L_{\mathrm{K3}}$. The orthogonal complement of L_5 is computed in [26, §4.3], and a structure ρ of $L_5^\perp$ as a $\boldsymbol{\mu}_5$-representation is described in [26, §5.2]. Let χ_5 be the character of $\rho \oplus \rho_{\mathrm{triv}}^{\oplus 10}$. Then X is represented by a k-point of $\mathsf{R}^\circ_{L_5,\chi_5}$. Conversely, we have:

Lemma 7.10. *There is a morphism* $\kappa_5 : \mathsf{R}^\circ_{L_5,\underline{\chi}_5} \to \widetilde{\mathsf{M}}_{0,5}$ *so that* $(\mathsf{R}^\circ_{L_5,\underline{\chi}_5}, \widetilde{\mathsf{M}}_{0,5}, \mathsf{Sh}^{(L_5,\underline{\chi}_5)}, \kappa_5, \tau_{L_5,\underline{\chi}_5})$ *satisfies* (†) *over* $\mathbb{Z}[\zeta_5, 1/10]$.

Proof. As usual, it suffices to describe κ_5. The previous construction gives the desired inverse on geometric points, and thus we have an induced isomorphism of coarse moduli spaces.

Suppose $(Z \to S, \iota, \alpha) \in \mathsf{R}^\circ_{L_5,\underline{\chi}_5}(S)$. Then there is also an involution $\beta \in \mathrm{Aut}_{Z/S}(S)$. (In characteristic zero, this is described in the last paragraph of the proof of [26, Lemma 5.7]; in positive characteristic, this then follows from a specialization argument.) The fixed locus Z^β is the disjoint union of a curve $C \to S$ with each fiber smooth and projective of genus 6 (use *loc. cit.* and Lemma 7.2(c)) and a relative rational curve. Moreover, the action of $\boldsymbol{\mu}_5$ on Z restricts to an action of $\boldsymbol{\mu}_5$ on C. The lattice polarization, in particular the numbering of the cycles $E_1, \cdots, E_5$, labels the fixed sections $C^{\boldsymbol{\mu}_5} \to S$. The quotient curve $C/\boldsymbol{\mu}_5 \to S$ has fibers of genus zero, and the sought-for configuration is $(C^{\boldsymbol{\mu}_5} \subset C/\boldsymbol{\mu}_5) \in \widetilde{\mathsf{M}}_{0,5}(S)$. □

In this case, diagram (†) is part of a larger diagram of moduli stacks. By its construction, the map $\kappa_5 : \mathsf{R}^\circ_{L_5,\underline{\chi}_5} \to \widetilde{\mathsf{M}}_{0,5}$ factors through $\widetilde{\mathsf{H}}(5,1,5)$. Now, if $(C_S \to \mathbb{P}^1_S) \in \mathsf{H}(5,1,5)(S)$, then $\mathrm{Pic}^0_{C/S}$ has an action by $\mathbb{Z}[\zeta_5]$, of signature (5.3.1) $\Sigma = \{(2,1),(0,3)\}$. Inside A_6 we have $\mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma}$, the locus of principally polarized abelian 6-folds with an action by $\mathbb{Z}[\zeta_5]$ of signature Σ. Consider the classical Torelli map $\tau_6 : \mathsf{M}_6 \to \mathsf{A}_6$. The image of the restriction to $\mathsf{H}(5,1,5)$ of τ_6 is open in $\mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma}$.

Of course, $\mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma}$ is a Shimura variety in its own right. The complex-analytic uniformization of $\mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma}$ is worked out in detail in [44, Case (5)]. Let $G = G_{\mathbb{Z}[\zeta_5],L_5^\perp}$, and let $\mathbb{X}_G$ be the corresponding Hermitian symmetric domain; it is isomorphic to the unit 2-ball $\mathbb{B}^2$. There is a compact open subgroup $\mathbb{K}_0 \subset G(\mathbb{A}_f)$ such that $\mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} \cong \mathsf{Sh}_{\mathbb{K}_0}[G, \mathbb{X}_G]$.

(Briefly, let $M = \mathbb{Z}[\zeta_5]^{\oplus 3}$, endowed with the Hermitian form h represented by $\mathrm{diag}(1, 1, \frac{1-\sqrt{5}}{2})$. The unitary group of (M, h) is an integral form of G, and $\mathbb{K}_0$ is the stabilizer of the lattice M. Conversely, $\mathbb{K}^{(L_5,\underline{\chi}_5)}$ can be recovered from $\mathbb{K}_0$ as those group elements which act trivially on the discriminant group of L.)

Then $\mathbb{K}_0 \supset \mathbb{K}^{(L_5,\underline{\chi}_5)} := \mathbb{K}_{\mathbb{Z}[\zeta_5],L_6^\perp}$, with quotient group $\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)} \cong \mathrm{O}(\mathrm{disc}(L^\perp)) \cong \mathrm{O}_3(\mathbb{F}_5) \cong \{\pm 1\} \times S_5$.

We summarize this discussion in:

Proposition 7.11. *There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$:*

$$\begin{tikzcd}
\mathsf{R}^\circ_{L_5,\underline{\chi}_5} \arrow[r] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] \arrow[rr, bend left, "\kappa_5"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}$$

where an arrow is labelled $[\Gamma]$ if it is a quotient by the finite group Γ; the given factorization of κ_5 is, on coarse moduli spaces, a composition of isomorphisms; and $\tau_{L_5,\underline{\chi}_5,\mathbb{C}}$ is an open immersion.

Proof. Since the canonical models of both $\mathsf{Sh}^{(L_5,\underline{\chi}_5)}$ and $\mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma}$ receive maps from $\mathsf{R}_{L_5,\underline{\chi}_5}$ over $\mathbb{Z}[\zeta_5, 1/10]$ with dense image, it suffices to observe that the quotient map $\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \to \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma}$ is defined on the canonical models. $\square$

7.6. **Six points on a line.** In [17, §12], Dolgachev and Kondō show that the configuration space of six labelled points on the (complex) projective line is an arithmetic quotient of $\mathbb{B}^4$.

The pointwise construction of *loc. cit.* works over an arbitrary algebraically closed field k. Let $(P_1, \cdots, P_6) \in \widetilde{\mathsf{M}}_{0,6}$ be an ordered 6-tuple of distinct points. Let C be the cyclic degree three cover of $\mathbb{P}^1$ ramified exactly at the P_i, and let Z' be the cyclic triple cover of the ambient weighted projective space $\mathbb{P}(1,1,2)$ ramified along C. (Explicitly, let $f(X_0, X_1)$ be a homogeneous form of degree 6 vanishing at the P_i; then Z' is given by the equation $X_3^3 + X_2^3 + f(X_0, X_1) = 0$ in $\mathbb{P}(1,1,2,2)$.) Then Z' comes with an action by $\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$; we single out the action of $\boldsymbol{\mu}_3$ on Z' via the diagonal embedding $\boldsymbol{\mu}_3 \hookrightarrow \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3 \hookrightarrow \mathrm{Aut}_k(Z')$. The variety Z' has three ordinary nodes; its minimal resolution, Z, is a K3 surface, and the $\boldsymbol{\mu}_3$ action lifts to Z. One finds that, by construction, $\mathrm{Pic}_{Z/k}(k)$ comes equipped with a primitive inclusion of the lattice $L'_6 := U \oplus E_6(-1) \oplus A_2(-1)^{\oplus 3}$, with orthogonal complement $A_2(1) \oplus A_2(-1)^{\oplus 3}$, and that $d(L'_6) = 2$. As a $\mathbb{Z}[\zeta_3]$-module, $(L'_6)^\perp$ is free of rank 4, and comes equipped with a Hermitian form of signature $(3,1)$. Let ρ be the corresponding $\boldsymbol{\mu}_3$-representation, and let χ'_6 be the character of $\rho \oplus \rho_{\mathrm{triv}}^{\oplus 14}$. As usual, we have (†) for $\mathsf{R}^\circ_{(L'_6,\underline{\chi}'_6)}$, $\widetilde{\mathsf{M}}_{0,6}$ and $\mathsf{Sh}^{(L'_6,\underline{\chi}'_6)}$ over $\mathbb{Z}[\zeta_3, 1/6]$.

Alternatively, we could use the strategy of §7.5, and compute the periods of C directly. If $(C \to S \to \mathbb{P}^1_S) \in \mathsf{H}(3,1,6)$, then C/S is a family of curves of genus 4, and $\mathrm{Pic}^0_{C/S}$ has an action by $\mathbb{Z}[\zeta_3]$ of signature $(1,3)$. (This is case (2) of [44].) The moduli space $\mathsf{A}_{\mathbb{Z}[\zeta_3],(1,3)}$ of principally polarized abelian fourfolds with action by $\mathbb{Z}[\zeta_3]$ of signature $(1,3)$ is isomorphic to $\mathsf{Sh}_{\mathbb{K}_0}[G, \mathbb{X}_G]$, where $G = G_{\mathbb{Z}[\zeta_3],(L'_6)^\perp}$ and $\mathbb{K}_0$ is the stabilizer of the lattice $(L'_6)^\perp$. There is a surjection $\mathsf{Sh}^{(L'_6,\underline{\chi}'_6)} \to \mathsf{Sh}_{\mathbb{K}_0}[G, \mathbb{X}_G]$ with covering map $\mathbb{K}_0/\mathbb{K}^{(L'_6,\underline{\chi}_6)} \cong \mathrm{O}(\mathrm{disc}((L'_6)^\perp)) \cong \boldsymbol{\mu}_2 \times S_6$ (it seems that, in the third displayed equation of [17, p.93], the authors may have neglected to account for the discriminant kernel) and we obtain:

Proposition 7.12. *There is a diagram of stacks over $\mathbb{Z}[\zeta_3, 1/6]$:*

$$\begin{tikzcd}
\mathsf{R}^{\circ}_{L'_6,\underline{\chi}'_6} \arrow[r] \arrow[rr, bend left, "\kappa'_6"] \arrow[dd, "\tau_{L'_6,\underline{\chi}'_6}"] & \widetilde{\mathsf{H}}(3,1,6) \arrow[r] \arrow[d, "{[S_6]}"] & \widetilde{\mathsf{M}}_{0,6} \arrow[d, "{[S_6]}"] \\
 & \mathsf{H}(3,1,6) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,6} \\
\mathsf{Sh}^{(L'_6,\underline{\chi}'_6)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L'_6,\underline{\chi}'_6)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_3],(1,3)} &
\end{tikzcd}$$

where an arrow is labelled $[\Gamma]$ if it is a quotient by the finite group Γ; the given factorization of κ'_6 is, on coarse moduli spaces, a composition of isomorphisms; and $\tau_{L_5,\underline{\chi}_5,\mathbb{C}}$ is an open immersion.

7.7. **Cubic surfaces.** Let Cub_2 be the moduli space of cubic surfaces. If $V/\mathbb{C}$ is a complex cubic surface, then either by associating a cubic threefold [4] or a K3 surface [16] to it and measuring its periods, one obtains an open immersion $\mathsf{Cub}_2(\mathbb{C}) \hookrightarrow \mathbb{B}^4/\Gamma$. The arithmetic nature of this map is explored in [2]. Unfortunately, there is a stack-theoretic mistake there. While Cub_2 and $\mathsf{H}(3,3,3)$ have isomorphic coarse moduli spaces, they are not literally the same stack; $\mathsf{H}(3,3,3) \to \mathsf{Cub}_2$ is the rigidification along the $\boldsymbol{\mu}_3$-action. We take the present opportunity to correct this oversight, and recast the main result in the framework developed here.

Let Cub_3 be the moduli space of smooth projective cubic *threefolds*. If $T \in \mathsf{Cub}_3(\mathbb{C})$, then its intermediate Jacobian is a principally polarized abelian fivefold. This gives a period map $\mathsf{Cub}_3(\mathbb{C}) \to \mathsf{A}_5(\mathbb{C})$, which is known to be an embedding. By using either monodromy considerations [11, Thm. 2.12] or the arithmetic nature of intermediate Jacobians [3, Thm. 6.1], one can show that this period map descends to a morphism $\mathsf{Cub}_3 \to \mathsf{A}_5$ of stacks over $\mathbb{Q}$. Using the algebro-geometric construction of the intermediate Jacobian as a Prym, one can actually spread out this out to achieve a morphism $\mathsf{Cub}_3 \to \mathsf{A}_5$ of stacks over $\mathbb{Z}[1/2]$ [2, Cor. 3.5].

Now, points of $H(3,3,3)$ correspond to cyclic triple covers of $\mathbb{P}^3$ branched along a cubic surface; as such, they are smooth projective threefolds in their own right. The $\boldsymbol{\mu}_3$ action on the threefold induces an action of $\mathbb{Z}[\zeta_3]$ on the corresponding intermediate Jacobian, with signature $(1,4)$. Ultimately, one obtains

Proposition 7.13. *There is a diagram of stacks over $\mathbb{Z}[\zeta_3, 1/6]$*

$$\begin{tikzcd}
\mathsf{Cub}_3 \arrow[d] & \mathsf{H}(3,3,3) \arrow[l, hook'] \arrow[r, "\kappa"] \arrow[d, hook, "\tau"] & \mathsf{Cub}_2 \\
\mathsf{A}_5 & \mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4)} \arrow[l, hook'] &
\end{tikzcd}$$

in which τ is an open immersion, and κ induces an isomorphism of coarse moduli spaces.

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Email address: `j.achter@colostate.edu`

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80523
URL: `https://www.math.colostate.edu/~achter`